TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 151, October 1970

POSITIVE CLIFFORD SEMIGROUPS ON THE PLANE[1]

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Abstract. This work is devoted to a preliminary investigation of positive Clifford semigroups on the plane. A positive semigroup is a semigroup which has a copy of the nonnegative real numbers embedded as a closed subset in such a way that 0 is a zero and 1 is an identity. A positive Clifford semigroup is a positive semigroup which is the union of groups. In this work it is shown that if S is a positive Clifford semigroup on the plane, then each group in S is commutative. Also, a necessary and sufficient condition is given in order that S be commutative, and an example is given of such a semigroup which is, in fact, not commutative. In addition, both the number and the structure of the components of groups in S is determined. Finally, it is shown that S is the continuous isomorphic image of a semilattice of groups.

A topological semigroup is a Hausdorff space together with a continuous associative multiplication. A *real semigroup* has been defined by J. G. Horne, Jr. [4] to be a topological semigroup containing a subsemigroup R iseomorphic to multiplicative semigroup of real numbers, embedded as a closed subset of E^2 in such a way that 1 is an identity and 0 is a zero. Similarly, the author has defined a *positive semigroup* to be a topological semigroup containing a subsemigroup N iseomorphic to the multiplicative semigroup of nonnegative real numbers, embedded as a closed subset of E^2 so that 1 is an identity and 0 is a zero [2]. Relying heavily on the work done by Horne in [4] and [5], this work is devoted to a study of positive semigroups on E^2 with the additional requirement that these semigroups be the union of groups. Let us call such semigroups *positive Clifford semigroups* [3]. We will show that if S is a positive Clifford semigroup on E^2, then each group in S is commutative. Also, we will give a necessary and sufficient condition in order that a positive Clifford semigroup on E^2 be commutative, and we will give an example of a positive Clifford semigroup on E^2 which is, in fact, not commutative. We will show that each group in a positive Clifford semigroup S on E^2 has one, two, or four components, that each two dimensional group is $P\times P$, $P\times P\times\{1, -1\}$, or $P\times P\times F$, where F is the four group, and that each one dimensional group is P, $P\times\{1, -1\}$, or $P\times F$. Also, we will characterize S in terms of the sector of identity

Presented to the Society, November 8, 1968; received by the editors May 19, 1969 and, in revised form, January 8, 1970.

AMS 1969 *Subject Classifications*. Primary 2205.

Key Words and Phrases. Real semigroup, positive semigroup, Clifford semigroup, semilattice of groups.

(1) This paper contains part of a doctoral dissertation written under the direction of Professor D. R. Brown at the University of Tennessee.

components of these groups, and we will show that S is the continuous isomorphic image of a semilattice of groups.

2. **Preliminaries.** The closure of a subset A of a topological space is denoted $\bar{A}$. The set-theoretic difference of two sets A and B is denoted $A \backslash B$. An *iseomorphism* between two topological semigroups is a function which is both an algebraic iseomorphism and a homeomorphism. The inverse of an element s is denoted s^{-1}. The set $H(1)$ denotes the set of elements with inverses with respect to the identity element 1. In general, $H(e)$ denotes the maximal group having e as identity [1, p. 22]. Let G denote the component of the identity in $H(1)$. Throughout this work, E^2 will denote the Euclidean plane. We will use the terminology *two dimensional* to mean having an interior relative to E^2, and *one dimensional* to mean nontrivial but having no interior relative to E^2. Unless otherwise indicated, R will denote a semigroup iseomorphic to the multiplicative semigroup of real numbers. The set of all positive members of R is denoted P, and the set of all negative members by $-P$. The set of all nonnegative members of R, i.e. $P \cup \{0\}$, is denoted by N. The null set is denoted by $\square$. For additional terminology the reader is referred to [4].

3. **Components of maximal groups.** Henceforth S will denote a positive Clifford semigroup on E^2. We intend to imply that a fixed iseomorphic copy of the non-negative real numbers has been chosen. The proof of the first lemma is not difficult.

LEMMA 1. *The decomposition space* $[E^2 \backslash \{0\}]/\Phi$ *of* $E^2 \backslash \{0\}$ *is homeomorphic to a topological circle.*

LEMMA 2. *If* $H(e)$, $e \neq 1$, *is a two dimensional group in* S, *a positive Clifford semigroup on* E^2, *and* D_0 *is the component of* $H(e)$ *containing* e, *then* $\bar{D}_0$ *is iseomorphic to* $N \times N$.

Proof. It follows from [7] that D_0, as well as G, is a Lie group which is an open subset of E^2. Just as in the case of G, D_0 is iseomorphic to the nonzero complex numbers, to the two dimensional vector group, or to the group of affine transformations of the line [8, pp. 283, 257, 258]. Under the assumptions of this lemma, since $0 \in \bar{D}_0$, $\bar{D}_0 \neq E^2$ and, consequently, D_0 cannot be iseomorphic to the nonzero complex numbers. Hence, D_0 must be one of the latter two possibilities. The results of [6] in which Mostert determined the possibilities for the boundary L of G are also applicable in the present case. If L' is the boundary of D_0, the only possibility in which S is the union of groups is that $L' = A \cup B \cup \{0\}$, where A and B are groups having the property that $AB = \{0\}$. Now, Pe is an open ray meeting D_0 in e, and since Pe cannot meet A, B, or $\{0\}$, Pe is contained in D_0. By [4, p. 983], Pe is homeomorphic to P. Also, if $p_1, p_2 \in P$, $(p_1p_2)e = p_1(p_2e) = p_1(ep_2e) = (p_1e)(p_2e)$, so that Pe is, in fact, iseomorphic to P. In other words, Pe is a *one-parameter subgroup*. Thus, Horne's argument in [4] can be adapted to show that $\bar{D}_0$ is iseomorphic to $N \times N$.

LEMMA 3. *If P is the copy of the multiplicative positive real numbers contained in G, the identity component of $H(1)$, then for each nonzero group element $y \in H(e)$, $Py \subset H(e)$.*

Proof. First, $Py = P(ey) = (Pe)y$. Then, since Pe is a one-parameter subgroup contained in $H(e)$ (see the proof of Lemma 2), $Py = (Pe)y \subset H(e)$.

The proof of the next lemma is straightforward.

LEMMA 4. *Let D_0 be the identity component of $H(e)$, a group in S, and let D_1 be a component of $H(e)$ distinct from D_0. Then, $D_1 = xD_0 = D_0x$ for each $x \in D_1$.*

LEMMA 5. *Let D_1, D_2 be components of $H(e)$, a group in S. Then D_1D_2 is a component of $H(e)$.*

Proof. From the previous lemma, $D_1 = xD_0 = D_0x$, and $D_2 = yD_0 = D_0y$, where $x \in D_1$, $y \in D_2$, and D_0 is the identity component of $H(e)$. We also know that D_0 is a subgroup of $H(e)$. Thus, $D_1D_2 = (xD_0)(D_0y) = x(D_0D_0)y = x(D_0y) = x(yD_0) = (xy)D_0$, which is the component of $H(e)$ containing xy.

In view of Lemma 2, the following result is analogous to that in [5, p. 19].

LEMMA 6. *If $H(e)$ is a two dimensional group in S, then $H(e)$ has only a finite number of components.*

LEMMA 7. *The number of two dimensional maximal groups in S is finite.*

Proof. Suppose that S has infinitely many two dimensional maximal groups. Then, there are infinitely many identity components of two dimensional groups, and each identity component is iseomorphic to $P \times P$. If this is the case, there must be a sequence $\{C_n\}$ of these identity components whose bounding rays converge (in the limit superior, limit inferior sense) to a ray, say P_t. Let P_n and P_{n+1} be the bounding rays of C_n. Then, $\limsup \{P_{n+1}\} = \liminf \{P_{n+1}\} = P_t$, and $\limsup \{P_n\} = \liminf \{P_n\} = P_t$. Let T be a simple closed curve with the origin in the bounded component of its complement, and let T' be a second simple closed curve which contains T in the bounded component of its complement. By the Jordan Curve Theorem each of these simple closed curves separates E^2 into two components, one of which is bounded and the other unbounded. So, each of the rays P_n and P_{n+1} are cut by both T and T', since a ray is unbounded but meets the bounded component of E^2 created by T and the one created by T'. Now, it is not difficult to show that the decomposition space is both upper and lower semicontinuous. Consequently, we can pick a sequence $\{x_{n+1}\}$ such that $x_{n+1} \in P_{n+1}$ and x_{n+1} lies in the annular region between T and T', and such that $\{x_{n+1}\}$ converges to x, where $x \in P_t$ and x lies in the annular region between T and T'. Similarly, we can pick a sequence $\{x_n\}$ where $x_n \in P_n$ and lies in the annular region between T and T', and such that $\{x_n\}$ converges to x. Then, $\{x_{n+1}x_n\}$ converges to x^2. But, $x_{n+1}x_n = 0$, for each n, and $x^2 \neq 0$, since $x \neq 0$ and x is an element of a group. Thus, we have a contradiction.

The following lemma is a generalization of that in [4, p. 989].

Lemma 8. *Let S be a positive Clifford semigroup on E^2, and let e be the identity of a group in S. Let $D_0, D_1, \ldots, D_n$ be the components of $H(e)$, and let χ be the squaring function defined by $\chi(x)=x^2$. Then $\chi(\bigcup_{i=0}^n D_i)\subset\bigcup_{i=0}^n D_i$, $\chi(D_0)=D_0$, and $\chi(D_i)\cap D_i=\square$ if $i>0$.*

Lemma 9. *Let D_i, D_j be components of $H(e)$, a two dimensional group contained in S. Let M and M' be the two bounding rays of D_i. If $\chi(D_i)=D_j$, then $\chi(M)$ and $\chi(M')$ are the bounding rays of D_j.*

Proof. By Lemma 4, $D_i=xD_0=D_0x$, where $x\in D_i$ and D_0 is the identity component of $H(e)$. Let A and B be the bounding rays of D_0. By Lemma 2, we know that $AB=\{0\}$. Since e is an identity for $\bar{D}_0$, multiplication by x is a homeomorphism on $\bar{D}_0$. Hence, the boundary of D_i must be $x(A\cup B)\cup\{0\}=(A\cup B)x\cup\{0\}$. But, $x(A\cup B)=(A\cup B)x$, and we have $xA=Ax$, and $xB=Bx$. For, suppose that for some $a\in A$, $b\in B$ we have $xa=bx$. Then, $(xa)(xa)=(xa)(bx)=x(ab)x=x0x=0$, whence xa is a nilpotent element, and we have a contradiction [5, p. 19]. So, $xA=Ax$ and $xB=Bx$ are the bounding rays of D_i. Since xA and xB are contained in groups distinct from $H(e)$, but $\chi(xA)$ and $\chi(xB)\subset\bar{D}_j$, $\chi(xA)=(xA)(xA)=(xA)(Ax)=x(Ax)=x^2A$, and $\chi(xB)=x^2B$ are distinct (since multiplication by x^2 is a homeomorphism on $\bar{D}_0$) bounding rays of $\chi(D_i)=D_j$.

Let us pause for a moment for some additional terminology. If D_i and D_j are components of groups in S such that there is a ray Px with the property that $Px\subset\bar{D}_i\cap\bar{D}_j$, we will say that the two components share a bounding ray.

Lemma 10. *Let $D_0, D_1, D_2, \ldots$ be components of a two dimensional group $H(e)$ contained in S, with D_0 denoting the identity component. Then, if D_0 and D_1 share a bounding ray Px, and D_1 and D_2 share a bounding ray Pa, then $H(e)=H(1)$, $H(e)$ has exactly four components,* Cl $[H(e)]=E^2$, *and* Cl $[H(e)]$ *is iseomorphic to $R\times R$.*

Proof. Since $\chi(D_0)=D_0$ and $\chi(D_1)=D_i$ for some $i\neq 1$, we see that $\chi(D_1)=D_0$, whence, by the preceding lemma, Pa^2 is the bounding ray of D_0 distinct from the bounding ray Px. Now, Pa^2 must also be a bounding ray of $\chi(D_2)$. So, either $\chi(D_2)$ is a component D_3 sharing the bounding ray Pa^2 with D_0, or $\chi(D_2)=D_0$. Observe that $D_0D_i=D_iD_0=D_i$. For if $x\in D_i$, $D_0(D_0x)=(D_0D_0)x=D_0x=D_i$, and $(xD_0)D_0=x(D_0D_0)=x(D_0)=D_i$. Suppose that $\chi(D_2)=D_3$. Let $\{x_n\}$ be a sequence in D_1 and $\{y_n\}$ be a sequence in D_2 such that $\{x_n\}$ converges to a, and $\{y_n\}$ converges to a, where a is on the bounding ray Pa shared by D_1 and D_2. Then, $\{x_ny_n\}$ converges to $a^2\in Pa^2$, the bounding ray shared by D_0 and D_3. Since D_1D_2 must be a component of $H(e)$, this implies that $D_1D_2=D_0$, or $D_1D_2=D_3$. If $D_1D_2=D_0$, $D_1(D_1D_2)=D_1D_0$, $(D_1D_1)D_2=D_1D_0=D_1$, $D_0D_2=D_1$, and $D_2=D_1$, which is a contradiction. So, we must assume that $D_1D_2=D_3$. Then, $(D_1D_2)D_2=D_3D_2$, $D_1(D_2D_2)=D_3D_2$, and $D_1D_3=D_3D_2$. Also, $D_1(D_1D_2)=D_1D_3$, $(D_1D_1)D_2=D_1D_3$, $D_0D_2=D_1D_3$, and $D_2=D_1D_3$. So, $D_2=D_1D_3=D_3D_2$. Then, $D_2D_2=(D_3D_2)D_2$,

$D_2D_2=D_3(D_2D_2)$, $D_3=D_3D_3$, and $D_3=D_0$, which is a contradiction. So, we must have $\chi(D_2)=D_0$. If $\chi(D_2)=D_0$, $Pb^2=Px$, where Pb is the bounding ray of D_2 distinct from Pa. In this case any component sharing the bounding ray Pb with D_2 must go onto D_0 or D_1 under the squaring map. Thus, any such component must be some D_k. Suppose D_k does not share a bounding ray with D_0. We saw above that $D_1D_2=D_3$, where D_3 shares the bounding ray Pa^2 with D_0. In a similar fashion, $D_2D_k=D_1$, or $D_2D_k=D_0$. If $D_2D_k=D_0$, $D_2(D_2D_k)=D_2D_0$, $(D_2D_2)D_k=D_2$, $D_0D_k=D_2$, and $D_k=D_2$, which is a contradiction. So, $D_2D_k=D_1$. Now, $D_1(D_2D_k)=D_1D_1=D_0=(D_1D_2)D_k=D_3D_k$. But, if $D_3D_k=D_0$, $D_3(D_3D_k)=D_3D_0$, $(D_3D_3)D_k=D_3$, $D_0D_k=D_3$, and $D_k=D_3$, which is a contradiction. So, we are forced to conclude that D_k does indeed share a bounding ray with D_0, in which case, $H(e)=H(1)$, $H(e)$ has exactly four components, and $\mathrm{Cl}\,[H(e)]=E^2$, which is $R\times R$ [5, p. 18].

Lemma 11. *Let C_0 be the identity component of a two dimensional group $H(e)$ contained in S. Suppose that C_0 shares bounding rays with each of C_1 and C_2, both components of the same group as C_0. Then $H(e)=H(1)$, $H(e)$ has exactly four components, and* $\mathrm{Cl}\,[H(e)]=E^2$.

Proof. Let us denote by M the bounding ray shared by C_0 and C_1, by N the bounding ray shared by C_0 and C_2, and by K the bounding ray of C_1 not belonging to $\bar{C}_0$. It follows from Lemmas 5 and 8 and from the continuity of the squaring map that only another component C_3 of $H(e)$ can share the ray K with C_1. For, $\chi(K)=N$, and if $\{x_n\}$ is a sequence none of whose elements belongs to $\bar{C}_1$, but such that $\{x_n\}$ converges to $k\in K$, then the sequence $\{x_n^2\}$ converges to $k^2\in N$. So, the sequence $\{x_n^2\}$ must eventually be in $\bar{C}_0$ or $\bar{C}_2$. The possibility that $x_n^2\in N$ for every n can be eliminated by a relatively elementary proof. Hence, the sequence $\{x_n\}$ is eventually in a component C_3 of $H(e)$ which shares the bounding ray K with C_1. The result now follows from Lemma 10.

Lemma 12. *Let C_0 be the identity component of a two dimensional group $H(e)$ contained in S, and let M be a bounding ray of C_0. Let D be a component of a two dimensional group such that M is also one of its bounding rays. Then, D cannot be a nonidentity component of a two dimensional group distinct from $H(e)$.*

Proof. Since $\chi(M)=M$, by continuity the boundary of $\chi(D)$ must contain M. But, since $\chi(D)\cap D=\square$ by Lemma 8, this is impossible.

Theorem 1. *The union of all the identity components of the nonzero groups in S forms a sector.*

Proof. If the hypotheses of Lemma 11 are satisfied, $\bar{G}\backslash\{0\}$ is a sector and the theorem is established. By Lemma 12 an identity component of a two dimensional group cannot share a bounding ray with a nonidentity component of another two dimensional group. However, S can contain sectors of one dimensional groups, by which we mean nontrivial sectors with the property that each element of the sector lies on a one dimensional group contained in the sector. Thus, an identity component of a two dimensional group could possibly share a bounding ray, say M, with a sector T of one dimensional groups. Now, since $\chi(M)=M$, by a continuity argument like the one used in the proof of Lemma 11 we can show that T must, in fact, be a sector of identity components of one dimensional groups. So, the identity components of groups in S cannot be separated by nonidentity components, and hence they form a sector.

LEMMA 13. *Suppose that $H(1)$ has exactly two components, with C_0 denoting the identity component and C_1 denoting the other component. Let T denote the sector of identity components of all the nonzero groups in S, with M and M' denoting the bounding rays of T. Then, there is an x in C_1 such that $x^2=1$, and right translation by x leaves M and M' pointwise fixed.*

Proof. Let us consider the decomposition circle, and let us label arcs in a clockwise fashion. Let (a, b) denote the arc $\Phi(C_0)$. It follows from Lemmas 5 and 8 that $\chi(C_1)=C_0$, and consequently that there is an element x in C_1 such that $x^2=1$. It has been shown by Horne [5, pp. 18–21] that x is, in fact, in the center of S. By Lemma 4, $xC_0=C_1$, and since translation by x is a homeomorphism, it follows that the arc $\Phi(C_1)$ on the decomposition circle is either (xa, xb) or (xb, xa). Suppose that this arc is (xa, xb). Then, the arc (xb, a) contains no points which are the image under Φ of elements of $H(1)$. Since the squaring map χ is continuous, $\chi[(xb, a)]$ must contain the arc $(a^2, x^2b^2)=(a, b)$ or the arc (b, a). Each of these arcs contains arcs $\Phi(C_i)$, where C_i is a component of $H(1)$. This implies that for some $z\in S\backslash H(1)$, $z^2\in H(1)$, which is impossible because S is the union of groups. So, the arc $\Phi(C_1)$ must be (xb, xa). Now let us consider the arc (xa, a). The x translate of (xa, a) must contain (a, xa) or (xa, a). If the x translate of (xa, a) contains (a, xa), then there is an element z in $S\backslash H(1)$ such that $xz\in H(1)$. This is impossible, since if $xz\in H(1)$ and $x\in H(1)$, then $x(xz)=(xx)z=1\cdot z=z\in H(1)$. So, $x\cdot[(xa, a)]$ must contain (xa, a), and for the same reason as just given must, in fact, be equal to (xa, a). Similarly, $x\cdot[(b, xb)]=(b, xb)$. Hence, there is a point p in (xa, a) and a point q in (b, xb) such that $xp=p$ and $xq=q$. Now, translation by x is obviously an involution, by which we mean a homeomorphism of period two. So, if translation by x leaves more than two points fixed on the decomposition circle, every point on the circle would have to be left fixed [5, p. 20]. However, since $x\cdot[(a, b)]\neq(a, b)$ not every point is left fixed. So, p and q are the only fixed points under this translation. Now, $p^2=pp=(xp)p=xp^2$, so that $p^2=p$ or $p^2=q$. But, p^2 cannot be q. For, if so, $\chi[(p, a)]$ must contain (a, b) or (xb, xa) which is impossible, because z^2 cannot be in $H(1)$ if $z\in S\backslash H(1)$. Similarly, $q^2=q$. Since $p^2=p$ and $q^2=q$, the arc $\Phi(T)$ on the decomposition circle must contain (p, q). Now, the points p and q must be the image under Φ of one dimensional groups. For, if $xe=e$, where e is the identity of a two dimensional group, the translation by x

leaves the identity component of the group, and hence an arc on the circle pointwise fixed. Suppose that there is a point z on the arc (xa, p) which is the image under Φ of a ray from an identity component of a group in S. Then, xz is in the arc (p, a) which contains only points which are images under Φ of rays in identity components. This implies that the arc (z, xz) is the image under Φ of the identity component of a group. But, $p \in (z, xz)$, and p is the image under Φ of a one dimensional group, which is a contradiction. So, the arc on the circle which is the image under Φ of the sector T must be (p, q). Since $xp=p=p^2$, and $xq=q=q^2$, translation by x must leave M and M' pointwise fixed.

THEOREM 2. *If $H(1)$ has exactly two components, then the two bounding rays of the sector T of identity components of groups of S are connected maximal groups, and every other nonzero group has exactly two components. Furthermore, every group in S is commutative.*

Proof. Using the notation of the previous lemma, let the arc (p,q) on the decomposition circle be the image under Φ of the sector T, where $xp=p$ and $xq=q$. Then, $x\cdot[(p,q)]$ must contain (p,q) or (q,p). Since $x\cdot[(a,b)]\subset(q,p)$, $x\cdot[(p,q)]$ cannot contain (p,q). So, $x\cdot[(p,q)]$ must contain (q,p). Thus, $S\backslash\{0\}=T\cup xT$. Now, let e be the identity of a group $H(e)$ in S, and let D_0 be the identity component of $H(e)$. Then $(xe)^2=x^2e^2=e$, so that $xe\in H(e)$. As seen in the previous lemma, $xe=e$ only if e is the identity of M or of M', the bounding rays of T.

Let us note that D_0 is uniquely divisible, by which we mean that each element of D_0 has a unique nth root. For, by Lemma 3, each nonzero group is a union of rays, and therefore each connected group is a sector. If the identity component of $H(e)$ has more than one ray, it has an interior, and hence must be open in S. Therefore, it is a Lie group, and by Lemma 2 its closure must be iseomorphic to $N\times N$, so that it is iseomorphic to $P\times P$ itself. If the identity component of $H(e)$ is a trivial sector, it must be a single ray sector, and hence is of the form Pe, where $e\in E$. Then, as in the proof of Lemma 2, Pe is iseomorphic to P.

Now, if $xe\in D_0$, $xe=e$, since D_0 is uniquely divisible. Otherwise, $xe\in D_1$, a component of $H(e)$ distinct from the identity component, in which case $D_1=(xe)D_0=xD_0$. Since $S\backslash\{0\}=T\cup xT$, every nonzero group in S, except M and M', therefore has exactly two components. The closure of the identity component, D_0, of $H(e)$ is either iseomorphic to $N\times N$ by Lemma 2 or to N as above and is therefore commutative. Let $y, z\in D_1$. Then, $y=xs$, $z=xt$, for some $s, t\in D_0$. So, $yz=(xs)(xt)=(xs)(tx)=x(st)x=(xt)(sx)=(xt)(xs)=zy$. Similarly, if $y\in D_1$ and $z\in D_0$, we have $yz=zy$.

LEMMA 14. *Suppose that $H(1)$ has exactly four components, with C_0 denoting the identity component, and C_1, C_2, C_3 denoting the other components. Let T denote the sector of identity components of all the nonzero groups in S, with N_1 and N_4 denoting* *the bounding rays of T. Then, there is an* $x_i \in C_i$, $i=1, 2, 3$ *such that* $x_i^2=1$. *Moreover, there is a pair of these elements, say* x_1 *and* x_3, *such that* $x_1N_1=N_1$ *and* $x_3N_4=N_4$.

Proof. Let us consider the decomposition circle, and let us label arcs in a clockwise fashion. Let (a, b) denote the arc $\Phi(C_0)$. Since $H(1)$ is the four group when there are four components, we know that $\chi(C_i)=C_0$, for $i=1, 2, 3$, and consequently there is an x_i in each C_i such that $x_i^2=1$. It has been shown by Horne [5, pp. 18–21] that each x_i is, in fact, in the center of S. By Lemma 4, $x_iC_0=C_i$, and since translation by x is a homeomorphism, it follows that the arc $\Phi(C_i)$ on the decomposition circle is either (x_ia, x_ib) or (x_ib, x_ia). We might as well assume that the endpoints of all these arcs are distinct. The other cases can be handled similarly. Let us call the first of these arcs going clockwise on the circle from (a, b), (x_3a, x_3b) or (x_3b, x_3a) whichever is correct. If the arc is (x_3a, x_3b), $x_3\cdot[(b, x_3a)]$ would have to contain an arc which is the image under Φ of a component of $H(1)$. This is a contradiction, since if $x_3z \in H(1)$, $x_3(x_3z)=x_3^2z=z \in H(1)$. So, this arc is correctly labeled (x_3b, x_3a). Let us label the next arc which is the image under Φ of a component of $H(1)$ going clockwise from (x_3b, x_3a), (x_2b, x_2a) or (x_2a, x_2b), and the final such arc (x_1a, x_1b) or (x_1b, x_1a), whichever is correct. By the same type of argument as above, we can show that the final arc is correctly labeled (x_1b, x_1a). Now, if the middle arc is (x_2b, x_2a), $x_1\cdot[(x_2b, x_2a)]$ must contain $(x_1x_2b, x_1x_2a)=(x_3b, x_3a)$ or (x_3a, x_3b). In either case, this implies that $x_1z \in H(1)$ for some $z \in S\backslash H(1)$, which is a contradiction. So, the middle arc is properly labeled (x_2a, x_2b). Now, $x_1\cdot[(x_1a, a)]$ must contain (a, x_1a) or (x_1a, a). The same argument as above shows that $x_1\cdot[(x_1a, a)]$ cannot contain (a, x_1a), and in fact, must be equal to (x_1a, a). So, there is a point p in (x_1a, a) such that $x_1p=p$. Similarly, there is a point q in (x_3a, x_2a) such that $x_1q=q$. Now, $x_1p^2=(x_1p)p=p^2$, and $x_1q^2=q^2$. Since translation by x_1 is an involution and does not leave every point fixed because $x_1C_0=C_1$, p and q are the only points left fixed under this translation. So, $p^2=p$ or $p^2=q$, and $q^2=p$ or $q^2=q$. If $p^2=q$, $\chi[(p, a)]$ contains (a, q) or (q, a). In either case, this implies that there is some element z in $S\backslash H(1)$ such that $z^2 \in H(1)$, which is a contradiction. If $q^2=q$, we get the same contradiction. So, $p^2=q^2=p$. In a similar fashion, we can show that there is a point r in (b, x_3b) and a point s in (x_2b, x_1b) such that $x_3r=r$, $x_3s=s$, and $s^2=r^2=r$. The argument that the arc (p, r) is the arc $\Phi(T)$ on the circle is exactly analogous to the argument in Lemma 13. Also, by the same argument as appealed to repeatedly in this proof, we can show that $x_1\cdot[(x_1a, a)]=(x_1a, a)$, $x_3\cdot[(b, x_3b)]=(b, x_3b)$, $x_2\cdot[(b, x_3b)]=x_1\cdot[(b, x_3b)]=(x_2b, x_1b)$, and $x_2\cdot[(x_1a, a)]=x_3\cdot[(x_1a, a)]=(x_3a, x_2a)$. Furthermore, let us consider $x_2p=x_2(x_1p)=(x_2x_1)p=x_3p$. We have $x_1(x_2p)=x_2(x_1p)=x_2p$, so that translation by x_1 leaves x_2p fixed. So, $x_2p=x_3p$ is either p or q. But, $x_2\cdot[(x_1a, a)]=(x_3a, x_2a)$, so that $x_2p=x_3p=q$. Similarly, $x_1r=x_2r=s$. For our conclusion, $x_1p=p$ and $x_3r=r$ implies that $x_1N_1=N_1$ and $x_3N_4=N_4$.

THEOREM 3. *If* $H(1)$ *has exactly four components, then the two bounding rays of* *the sector T of identity components of groups are identity components of one dimensional groups having exactly two components, and every other nonzero group has exactly four components. Further, every group in S is commutative.*

Proof. Using the notation of the previous lemma, the arc (p, r) on the decomposition circle is the image under Φ of the sector T, where $x_1p=p$, and $x_3r=r$. In the proof of the lemma we noted that $x_2p=x_3p=q$ and $x_1r=x_2r=s$. It now follows, using the same argument as in Lemma 14, that $x_1\cdot[(p,a)]=(x_1a,p)$, $x_3\cdot[(b,r)]=(r,x_3b)$, $x_3\cdot[(p,a)]=(x_3a,q)$, $x_2\cdot[(p,a)]=(q,x_2a)$, $x_2\cdot[(b,r)]=(x_2b,s)$, and $x_1\cdot[(b,r)]=(s,x_1b)$. Thus, $S\backslash\{0\}=(T\cup x_1T\cup x_2T\cup x_3T)$. Now, let e be the identity of a maximal group $H(e)$ in S. Let D_0 be the identity component of $H(e)$. Then, $(x_ie)^2=x_i^2e^2=e$, so that $x_ie\in H(e)$. As seen in the previous lemma, $x_ie=e$ only if e is the identity of P_1 or P_4, the bounding rays of T, and $i=1$ or 3 respectively. Also, if $x_ie\in D_0$, $x_ie=e$, since D_0 is uniquely divisible (see proof of Theorem 2). Otherwise, $x_ie\in D_i$, a component of $H(e)$ distinct from the identity component, in which case $D_i=(x_ie)D_0=x_iD_0$. If $i\neq j$, x_ie and x_je are in distinct components. Otherwise, $(x_ie)(x_je)=(x_ix_j)e=x_ke\in D_0$, which has been shown above to be a contradiction. Since $S\backslash\{0\}=(T\cup x_1T\cup x_2T\cup x_3T)$, every nonzero group in S, except N_1 and N_4 has exactly four components. The identity component, D_0, of $H(e)$ is either iseomorphic to $N\times N$ or to N, and is, in any case, commutative. The proof that $H(e)$ is commutative follows in the same fashion as in Theorem 2.

THEOREM 4. *Let S be a positive Clifford semigroup on E^2. Then, the following are equivalent:*

(i) *If e and f are arbitrary idempotent elements of S, then $ef=fe$.*

(ii) *Each idempotent element of S is in the center of S.*

(iii) *S is commutative.*

Proof. It is shown in [1, p. 127] that (i) implies (ii). Let e and f be arbitrary idempotent elements of S. If e and f are in the center of S, then translation by either of these elements is a homomorphism. So, $e[H(f)]$ is the continuous homomorphic image of the group $H(f)$ and must therefore be a group. Moreover $e[H(f)]$ meets the group $H(ef)$ in ef, so that $e[H(f)]\subset H(ef)$. Similarly, $[H(e)]f\subset H(ef)$. Thus,

$$\begin{aligned} H(e)\cdot H(f) &= ([H(e)]e)\cdot(f[H(f)]) = [H(e)](ef)[H(f)] = [H(e)](fe)[H(f)] \\ &= ([H(e)]f)\cdot(e[H(f)]) \subset H(ef). \end{aligned}$$

Now, let $x\in H(e)$ and $y\in H(f)$. Then, xy, yx, xf, and $ey\in H(ef)$ which we know is commutative. So, $xy=(xe)(fy)=x(ef)y=x(fe)y=(xf)(ey)=(ey)(xf)=e(yx)f=(yx)(ef)=yx$, and we have that (ii) implies (iii). It is immediate that (iii) implies (i), and we are done.

LEMMA 15. *Let S be a positive Clifford semigroup on E^2 such that $H(1)$ has exactly two components. Let T denote the sector of identity components of groups in S, and let the boundary of T be denoted $(P_1\cup P_4\cup\{0\})=I$. Then, $IT\subset I$ and $TI\subset I$.*

Proof. Let e be the idempotent element of P_1, let f be the idempotent element of P_4, and let $P_1 \cup \{0\} = N_1$ and $P_4 \cup \{0\} = N_4$. By Lemma 13 there is an element $x \in S \backslash T$ such that $x^2 = 1$, and x is in the center of S. Hence, $(xe)^2 = e$. If $H(1)$ has exactly two components, the group $H(e)$ (see proof of Lemma 2) is connected, whence $xe = e$. Now, let $y \in T$ and $z \in Pe$. Then, $yz = yze$, and $xyz = x(yze) = (yze)x = (yz)(ex) = (yz)(xe) = yze$. Thus, $yz = yze = xyz$. Suppose that $yz \in S \backslash T$. Then, $yz = xw$, for some $w \in T$. But $yz = x(yz) = x(xw) = x^2 w = w \in T$, which is a contradiction. If $yz \in T$, $yz = x(yz) \in xT$, so that $yz \in (T \cap xT) = I$. Similarly, we can show that $zy \in I$. Thus, $T(Pe)$ and $(Pe)T$ are contained in I. Likewise, $T(Pf)$ and $(Pf)T$ are contained in I. Finally, $T \cdot 0 = 0 \cdot T = 0$, and we are done.

LEMMA 16. *Let S be a positive Clifford semigroup on E^2 such that $H(1)$ has exactly four components. Let T denote the sector of identity components of groups of S, and let the boundary of T be denoted $(P_1 \cup P_4 \cup \{0\}) = I$. Then, if $z \in P_1$ and $y \in T$, $zy \in (P_1 \cup \{0\})$ and $yz \in (P_1 \cup \{0\})$. Also, if $z \in P_4$ and $y \in T$, $zy \in (P_4 \cup \{0\})$ and $yz \in (P_4 \cup \{0\})$.*

Proof. Let p be the identity element of P_1 and let r be the identity element of P_4. Let $C_0 = P_1$ be the identity component of $H(p)$, and let C_1 be the remaining component. Then, using the notation of Theorem 3, $C_1 = x_2 C_0 = x_3 C_0$, as seen in the proof of Theorem 3. Also, $S \backslash \{0\} = (T \cup x_1 T \cup x_2 T \cup x_3 T)$. It must be shown that for $z \in C_0$ and $y \in T$, zy and $yz \in (C_0 \cup \{0\})$. It suffices to show that yp and $py \in (C_0 \cup \{0\})$. We might as well assume that $yp \neq 0$. Since $x_1 p = p$ and x_1 is in the center of S, $x_1 yp = yp$. If $yp \in T$, $yp = x_1 yp \in x_1 T$, so that $yp \in (x_1 T \cap T) = C_0$. Suppose $yp \in x_1 T$. Then, $yp = x_1 w$, for some $w \in T$. But, $yp = x_1 yp = x_1(x_1 w) = x_1^2 w = w \in T$, which is a contradiction. If $yp \in x_2 T$, $yp = x_2 w$, for some $w \in T$. Then, $yp = x_1(yp) = x_1(x_2 w) = (x_1 x_2)w = x_3 w \in x_3 T$. So, $yp \in (x_2 T \cap x_3 T) = C_1$. Similarly, if $yp \in x_3 T$, we can show that $yp \in C_1$. So, $yp \in H(p)$. Let $H(e)$ be the group containing y. Now, there exists an element k in $H(p)$ such that $(yp)k = p$. Then, $ep = e(ypk) = (ey)(pk) = ypk = p$. Let us now show that $yp \notin C_1$, if $y \neq e$. Suppose that $yp \in C_1$. Let q be the element in C_1 such that $q^2 = p$. By Lemma 4, $C_1 = qC_0 = C_0 q$. So, there exists $t \in C_0$ such that $yp = tq = qt$. Indeed, we can take $t = yq$. For, $yp = tq$ implies that $ypq = tqq$. Then $yq = ypq = tqq = tp = t$. Now, $y^2 p = y(yp) = y(qt) = (yq)t = (yq)(yq) = (yq)^2 = t^2 \in C_0$. Let $[y]$ be the one parameter subgroup in $H(e)$ generated by y. Let us consider the interval A from y to y^2 in $[y]$. We have shown that yp and $y^2 p \in H(p)$. If k is any element in A distinct from y and y^2 we can show that $kp \in H(p)$ by using the same type of argument as was used to show that if $y \in H(e)$ and $yp \neq 0$, then $yp \in H(p)$. So, the interval $(yp, y^2 p)$ is in $H(p)$, and this interval must be contained in Ap. Since yp is in C_1 and $y^2 p$ is in C_0, 0 must be in Ap. This implies that there exists an s in $[y]$ such that $sp = 0$. But, in this case, we have $s^{-1}(sp) = (s^{-1}s)p = ep = 0$, which is a contradiction. So, we must conclude that $yp \in (C_0 \cup \{0\}) = (P_1 \cup \{0\})$. The remaining conclusions of the lemma follow similarly.

THEOREM 5. *Let S be a positive Clifford semigroup on E^2, and let T denote the sector of identity components of nonzero groups in S. If the set E of idempotent elements of S is a subsemigroup, then T is a semigroup.*

Proof. Let $e, f \in E$. Let the identity component of the group $H(e)$ be denoted C_e. Let us consider $C_e \cdot C_f$, denoting $ef = g \neq 0$, $g \in E$. Then C_eC_f is a connected set meeting C_g in g. So, if $C_eC_f \cap (S\backslash T) \neq \square$, then $C_eC_f \cap I \neq \square$, where I is the boundary of T. The previous two lemmas show that $IT \subset I$ and $TI \subset I$. Suppose $0 \in C_eC_f$. Then, there is an x in C_e and a y in C_f such that $xy = 0$. But $x^{-1}(xy)y^{-1} = (x^{-1}x) \cdot (yy^{-1}) = ef = 0$, which is a contradiction. Suppose $C_eC_f \cap N_1 \neq \square$, where $N_1 = P_1 \cup \{0\}$ and P_1 is a bounding ray of T. Then, there exist $x \in C_e$, $y \in C_f$, $t \in N_1$ such that $xy = t \in N_1 \subset I$. So, $x^{-1}(xy)y^{-1} = (x^{-1}x)(yy^{-1}) = ef \in I$. Thus, $C_eC_f = (C_ee)(fC_f) = C_e(ef)C_f \subset I \subset T$.

THEOREM 6. *Let S be a positive Clifford semigroup on E^2 such that $H(1)$ has exactly two components. Let T denote the sector of identity components of groups in S. If T is a semigroup, then S is iseomorphic to $[(T \cup \{0\}) \times \{1, -1\}]/R$, for a suitable relation R.*

Proof. In Theorem 2 it was noted that the bounding rays P_1 and P_2 of T are each connected groups. Let $N_1 = P_1 \cup \{0\}$, and let $N_2 = P_2 \cup \{0\}$. Also, let $x \neq 1$ be a square root of 1. Since x is in the center of S, $(xN_1)^2 = N_1$ and $(xN_2)^2 = N_2$. Since P_1 is a connected group and S is the union of groups, it follows that $xN_1 \subset N_1$ and hence that $xN_1 = N_1$, because the translate by x of a ray is a ray. Similarly, $xN_2 = N_2$. By Lemma 13, $N_1 \cup N_2$ is an ideal in T. Now, with the usual topology and coordinatewise multiplication $[(T \cup \{0\}) \times \{1, -1\}]$ is a topological semigroup. Let us define a relation R on $[(T \cup \{0\}) \times \{1, -1\}]$ in the following manner. Let $[(a, 1), (b, 1)] \in R$ if and only if $a = b$. Let $[(a, -1), (b, -1)] \in R$ if and only if $a = b$. Let $[(a, 1), (b, -1)] \in R$ if and only if $a = b \in (N_1 \cup N_2)$. Let us require by definition that R be symmetric. Then, R is clearly an equivalence relation. Using the fact that $N_1 \cup N_2$ is an ideal in T, it also follows easily that R is a closed congruence, and consequently that $[(T \cup \{0\}) \times \{1, -1\}]/R$ is a topological semigroup on E^2. In the proof of Theorem 2 it was shown that $S = T \cup xT \cup \{0\}$. Let f be a function from $W = [(T \cup \{0\}) \times \{1, -1\}]$ onto S defined by $f[(a, -1)] = xa$ and $f[(a, 1)] = a$, where $x^2 = 1$ and $x \neq 1$. Let Φ be the natural map from W onto W/R. We see that f is one-to-one, except on elements of $N_1 \cup N_2$. The continuity of f and f^{-1} follows from the continuity of multiplication in T. It is easy to see that f is a homomorphism. Now, a function f^* is induced from W/R onto S, if we define $f^*(z) = f[\Phi^{-1}(z)]$. It follows easily that f^* is an iseomorphism.

It should be noted that, in view of the preceding theorem, if we have any positive semigroup on the closed half plane which is the union of connected groups, then we can easily construct a positive Clifford semigroup on E^2 in which each two dimensional group has exactly two components, and that *all* such entities in which T is a subsemigroup are formed in this manner.

THEOREM 7. *Let S be a positive Clifford semigroup on E^2 such that $H(1)$ has exactly four components. Let T denote the sector of identity components of nonzero groups in S. Then, if T is a semigroup, S is iseomorphic to $[(T \cup \{0\}) \times F]/R$ for a suitable relation R, where F is the four group.*

Proof. Let $F=\{x_1, x_2, x_3, 1\}$, where x_i, $i=1, 2, 3$ are as in Theorem 3. Then, F is the four group with $x_1x_2=x_2x_1=x_3$, $x_2x_3=x_3x_2=x_1$, $x_1x_3=x_3x_1=x_2$, and $x_1^2=x_2^2=x_3^2=1$. Now, with the usual topology and coordinatewise multiplication, $[(T \cup \{0\}) \times F]$ is a topological semigroup. Let us define a relation R on $[(T \cup \{0\}) \times F]$ in the following manner. Let $[(a, x_1), (b, 1)] \in R$ if and only if $a=b \in N_1$, where $N_1=P_1 \cup \{0\}$, and P_1 is a bounding ray of T. Let $[(a, x_1), (b, x_2)] \in R$ if and only if $a=b \in N_4$, where $N_4=P_4 \cup \{0\}$, and P_4 is a bounding ray of T. Let $[(a, x_2), (b, x_3)] \in R$ if and only if $a=b \in N_1$. Let $[(a, x_3), (b, 1)] \in R$ if and only if $a=b \in N_4$. Let $[(a, 1), (b, x_2)] \in R$ if and only if $a=b=0$, and let $[(a, 1), (b, x_3)] \in R$ if and only if $a=b=0$. Also, let us require that R be symmetric by definition, and let $[(a, x_i), (b, x_i)] \in R$ if and only if $a=b$, for $i=1, 2, 3$. Finally, let $[(a, 1), (b, 1)] \in R$ if and only if $a=b$. Then, R is clearly an equivalence relation, and using the fact that N_1 and N_4 are ideals in T and that $N_1N_4=\{0\}$ (since $N_1N_4 \subset (N_1 \cap N_4)=\{0\}$), it is easily seen that R is a closed congruence. Consequently, $[(T \cup \{0\}) \times F]/R$ is a topological semigroup on E^2. It was shown in the proof of Theorem 3 that $S\backslash\{0\} =(T \cup x_1T \cup x_2T \cup x_3T)$. Let f be a function from $W=[(T\cup \{0\}) \times F]$ onto S defined by $f[(a, 1)]=a$ and $f[(a, x_i)]=x_ia$, for $i=1, 2, 3$. Let Φ be the natural map from W onto W/R. Just as in Theorem 6, it is not difficult to show that f is a homomorphism, and that an iseomorphism f^* is induced from W/R onto S.

To conclude this section, let us construct an example of a noncommutative positive Clifford semigroup on E^2. In view of Theorem 4, the set E of idempotent elements must fail to be commutative. However, E does form a semigroup in the forthcoming example.

EXAMPLE 1. Let us consider five copies of $N \times N$. Let these copies be denoted by $J\times J$, $F\times F$, $N\times N$, $G\times G$, and $K\times K$. Let us now define a relation R on $T= [(J\times J) \cup (F\times F) \cup (N\times N) \cup (G\times G) \cup (K\times K)]$, by first requiring that $\Delta \subset R$. Also, let us define $[(a, b)_j, (c, d)_f] \in R$ if and only if $a=0=c$ and $b=d$, where $(a, b)_j \in (J\times J)$, and $(c, d)_f \in (F\times F)$. Continuing, let us define $[(a, b)_f, (c, d)_1] \in R$ if and only if $a=c$ and $b=0=d$, and $[(a, b)_1, (c, d)_g] \in R$ if and only if $a=0=d$ and $b=c$, where $(a, b)_1 \in (N\times N)$ and $(c, d)_g \in (G\times G)$. Also, let us define $[(a, b)_g, (c, d)_k] \in R$ if and only if $a=c$ and $b=0=d$, where $(c, d)_k \in (K\times K)$. Finally, let us require that R be symmetric. Now, let us define a multiplication on T in the following manner. Let multiplication be coordinatewise in each copy of $N\times N$. Let $(a, b)_j\cdot(c, d)_f=(c, d)_f\cdot(a, b)_j=(0, bd)_j$, $(a, b)_j\cdot(c, d)_1=(c, d)_1\cdot(a, b)_j=(acd, bcd)_j$, $(a, b)_j\cdot(c, d)_g=(bc, 0)_g$, $(a, b)_g\cdot(c, d)_j=(0, ad)_j$, $(a, b)_j\cdot(c, d)_k=(bc, 0)_k$, and $(a, b)_k \cdot(c, d)_j=(0, ad)_j$. Also, let $(a, b)_k\cdot(c, d)_g=(c, d)_g\cdot(a, b)_k=(ac, 0)_k$, $(a, b)_k\cdot(c, d)_1 =(c, d)_1\cdot(a, b)_k=(acd, bcd)_k$, $(a, b)_k\cdot(c, d)_f=(0, ad)_f$, and $(a, b)_f\cdot(c, d)_k=(bc, 0)_k$.

Finally, let $(a, b)_f \cdot (c, d)_1 = (c, d)_1 \cdot (a, b)_f = (ac, bcd)_f$, $(a, b)_g \cdot (c, d)_1 = (c, d)_1 \cdot (a, b)_g = (acd, bd)_g$, $(a, b)_f \cdot (c, d)_g = (bc, 0)_g$, and $(a, b)_g \cdot (c, d)_f = (0, ad)_f$. This multiplication is associative, though the cases to be checked are numerous, and its continuity follows from the continuity of real number multiplication. Moreover, the relation R, while obviously an equivalence relation, can easily be checked to be a closed congruence. The proof that T/R is Hausdorff is similar to that in [2, p. 29].

Thus, we have constructed an example of a noncommutative positive semigroup on a half plane. According to the comment following Theorem 6, we can now easily construct an example of a noncommutative positive Clifford semigroup on E^2 in which each two dimensional group has exactly four components. It is of interest to note that the semigroup on a half plane consisting of $T' = [(F \times F) \cup (N \times N) \cup (G \times G)]$, with points identified according to the relation R, is a subsemigroup of T/R. Moreover, T'/R can be used to construct an example of a noncommutative positive Clifford semigroup on E^2 in which each two dimensional group has exactly two components. However, since the bounding rays of T'/R are not individually ideals, T'/R cannot be used to construct by the method suggested in Theorem 7 such a semigroup in which each two dimensional group has exactly four components.

4. **Structure theorems.** In this section we will describe the structure of the maximal groups contained in S, a positive Clifford semigroup on E^2. We will also show that, under appropriate conditions, S is the continuous homomorphic image of the disjoint union of semigroups which are the closures of groups, and that S is iseomorphic to a semilattice of groups. By a *semilattice of groups* we will mean any isomorphic copy of a disjoint union of groups constructed in the following manner. First, let K be any semilattice, by which we mean a commutative idempotent semigroup. To each element α of K let us assign a group G_α such that G_α and G_β are disjoint if $\alpha \neq \beta$ in K. To each pair of elements α, β of K such that $\alpha > \beta$, let us assign a homomorphism $\Phi_{\alpha,\beta}$ of G_α into G_β such that if $\alpha > \beta > \gamma$ then $\Phi_{\alpha,\beta}\Phi_{\beta,\gamma} = \Phi_{\alpha,\gamma}$. Let $\Phi_{\alpha,\alpha}$ be the identity automorphism of G_α. Let A be the union of all the groups G_α ($\alpha \in K$), and let us define the product of any two elements a_α, b_β of A (a_α in G_α and b_β in G_β) by $a_\alpha b_\beta = (a_\alpha \Phi_{\alpha,\gamma})(b_\beta \Phi_{\beta,\gamma})$, where γ is the product $\alpha\beta$ in K. Then, we will call A a semilattice of groups [1, p. 128]. We will also need to use the notion of *disjoint union topology* which can be described in the following manner. If T is the disjoint union of sets S_β, $\beta \in \Omega$, then for T to have the disjoint union topology we define a set $\mathcal{O}$ to be open in T if and only if $\mathcal{O} \cap S_\beta$ is open in S_β, for each $\beta \in \Omega$. For the sake of simplification, throughout this section let us adopt the following notation. Let us denote the four group by F. Let

$$U = (N \times N \times \{1, -1\})/\alpha,$$

where α identifies $(0, 0, 1)$ and $(0, 0, -1)$. Let $V = (N \times N \times F)/\alpha$, where α identifies $(0, 0, 1)$, $(0, 0, x_1)$, $(0, 0, x_2)$, and $(0, 0, x_3)$. Let $W = (N \times R \times \{1, -1\})/\alpha$, where α

identifies $(0, 0, 1)$ and $(0, 0, -1)$. Finally let $Y=(N\times F)/\alpha$, where α identifies $(0, 1)$, $(0, x_1)$, $(0, x_2)$, and $(0, x_3)$.

THEOREM 8. *Let $H(e)$ be a two dimensional maximal group contained in S, a positive Clifford semigroup on E^2. Then* Cl $[H(e)]$ *is iseomorphic to the complex numbers, $N\times N$, $N\times R$, $R\times R$, U, V, or W.*

Proof. Let us first enumerate the cases involved in this theorem. First, $H(e)$ may be connected. In this case Cl $[H(e)]$ is iseomorphic to the multiplicative semigroup of complex numbers if S has only two idempotent elements [4, p. 987], and, by Lemma 2, Cl $[H(e)]$ is iseomorphic to $N\times N$ if S has more than two idempotent elements. Secondly, $H(e)$ may have exactly two components. In this case, the two components either share one bounding ray, two bounding rays, or their closures intersect only in $\{0\}$. We will show that Cl $[H(e)]$ is $N\times R$, $R\times R$, or U respectively in these cases. Finally, $H(e)$ may have exactly four components. If $H(e)$ is the only two dimensional group in S, by [5, p. 18] Cl $[H(e)]$ is iseomorphic to $R\times R$. There are two more subcases. First, the intersection of the closures of any two components of $H(e)$ may be $\{0\}$, in which case we will show that Cl $[H(e)]$ is iseomorphic to V. Also, if a nonidentity component of $H(e)$ shares a bounding ray with the identity component of $H(e)$, by arguments on the decomposition circle like those used in the proof of Lemma 14, it can be shown that the other two nonidentity components of $H(e)$ share a bounding ray. Then, if S has more than one two dimensional group, we will show that Cl $[H(e)]$ is iseomorphic to W.

Now, suppose $H(e)$ has exactly two components D_0 and D_1. By Lemma 2, $\bar{D}_0$ is iseomorphic to $N\times N$. Then, either these two components share a bounding ray, say Pe_1, where $e_1^2=e_1$, or $D_0\cap D_1=\{0\}$, or the two components share two bounding rays. Let us consider the first case. Let Pe_2, where $e_2^2=e_2$, be the other bounding ray of D_0, and let Px, where $x^2=e_2$ (see Lemma 9), be the other bounding ray of D_1. Also, let $G_1=\{x\in D_0 : xe_1=e_1\}$, and let $G_2=\{x\in D_0 : xe_2=e_2\}$. Now, $\chi(D_1)=D_0$, so that there is some element $a\in D_1$ such that $a^2=1$. Let us consider $\bar{G}_1\cup aG_1$. Since $\bar{G}_1$ is iseomorphic to $\bar{P}$, where P is the multiplicative group of positive real numbers [4, p. 987], the map f from $\bar{G}_1\cup aG_1$ onto R defined by $f(g_1)=g_1$ and $f(ag_1)=-g_1$, where $g_1\in\bar{G}_1$, is a homeomorphism. Now, let us consider the map $(x, y)\to xy$ from $(aG_1\cup\bar{G}_1)\times G_2$ to

$$\begin{aligned} aG_1\bar{G}_2\cup\bar{G}_1\bar{G}_2 &= [aG_1(G_2\cup\{e_2\})\cup(\bar{G}_1\bar{G}_2)] = (aG_1G_2\cup aG_1e_2\cup\bar{G}_1\bar{G}_2) \\ &= (D_1\cup Px\cup\bar{D}_0) = \text{Cl}\,[H(e)]. \end{aligned}$$

This map is one-to-one and onto $G_1\times G_2$, on $\{e_1\}\times G_2$, on $G_1\times\{e_1\}$, on $\{e_1, e_2\}$, on $aG_1\times G_1$, and on $aG_1\times\{e_1\}$, independently, and hence everywhere. It has been shown by Horne [4, pp. 987–988] that the map is a homeomorphism on $\bar{G}_1\times\bar{G}_2$. Since translation by the element a is a homeomorphism (recall that a is in the center of S), and consequently a set W is open in G_1 if and only if aW is open in

aG_1, it follows that the map $(x, y) \to xy$ is also open on $aG_1 \times \bar{G}_2$, and hence is a homeomorphism there. So, in this case, Cl $[H(e)]$ is iseomorphic to $N \times R$.

Let us now consider the case in which $H(e)$ has two components, D_0 and D_1, such that $\bar{D}_0 \cap \bar{D}_1 = \{0\}$. If D_0 is the identity component, we know that $\bar{D}_0$ is iseomorphic to $N \times N$. Furthermore, we know from Lemmas 4 and 13 that there is an element x in $H(1)$ such that $x^2 = 1$ and $x\bar{D}_0 = \bar{D}_1$. So, Cl $[H(e)] = \bar{D}_0 \cup x\bar{D}_0$, or Cl $[H(e)] = (N \times N) \cup x(N \times N)$. Let us define a function f from $[(N \times N) \times \{1, -1\}]$ onto Cl $[H(e)]$ in the following manner. Let $f[(a, b, 1)] = (a, b)$, and let $f[(a, b, -1)] = x(a, b)$. Then f is continuous and one-to-one, except that $f[(0, 0, 1)] = f[(0, 0, -1)]$. It is easily checked that f is an algebraic homomorphism. Now, if we define a relation α on $[(N \times N) \times \{1, -1\}]$ which contains the diagonal and which identifies $(0, 0, 1)$ and $(0, 0, -1)$, as in Theorem 6 an iseomorphism f^* is induced from U onto Cl $[H(e)]$.

The final case when $H(e)$ has two components is the one in which these components share two bounding rays. Here, Cl $[H(e)]$ is iseomorphic to $R \times R$ [4, p. 992].

The final two cases occur when $H(e)$ has exactly four components. For these remaining two cases, let D_i, $i = 0, 1, 2, 3$, denote the components of $H(e)$, with D_0 denoting the identity component. We know from Lemmas 4 and 14 that there are elements x_1, x_2, x_3 in $H(1)$ such that $x_1^2 = x_2^2 = x_3^2 = 1$, $x_1\bar{D}_0 = \bar{D}_1$, $x_2\bar{D}_0 = \bar{D}_2$, $x_3\bar{D}_0 = \bar{D}_3$, and $\{x_1, x_2, x_3, 1\}$ is the four group. We also know that $\bar{D}_0$ is iseomorphic to $N \times N$. Let us first consider the case in which $\bar{D}_i \cap \bar{D}_j = \{0\}$, for $i \neq j$. Since $H(e)/D_0$ is iseomorphic to the four group, it follows in a similar fashion to the case just done that $H(e)$ is iseomorphic to V. Here we define $f[(a, b, 1)] = (a, b)$ and $f[(a, b, x_i)] = x_i(a, b)$ for $i = 1, 2, 3$. Then, as before, an iseomorphism f^* is induced from V onto Cl $[H(e)]$.

In the final case, D_0 and D_1 share a bounding ray, and D_2 and D_3 share a bounding ray, but $(\bar{D}_0 \cup \bar{D}_1) \cap (\bar{D}_2 \cup \bar{D}_3) = \{0\}$. We know from an earlier case that $\bar{D}_0 \cup \bar{D}_1$ is iseomorphic to $N \times R$. Now, consider $x_2(\bar{D}_0 \cup \bar{D}_1) = x_2(\bar{D}_0 \cup x_1\bar{D}_0) = x_2\bar{D}_0 \cup x_2x_1\bar{D}_0 = x_2\bar{D}_0 \cup x_3\bar{D}_0 = \bar{D}_2 \cup \bar{D}_3$. Let us define a function from $[(N \times R) \times \{1, -1\}]$ onto Cl $[H(e)]$ in the following manner. Let $f[(a, b, 1)] = (a, b)$, and let $f[(a, b, -1)] = x_2(a, b)$. It follows in a similar fashion to the earlier case in which Cl $[H(e)]$ is iseomorphic to U that f is continuous and is an algebraic homomorphism. If we define a relation α on $(N \times R \times \{1, -1\})$ which contains the diagonal and which identifies $(0, 0, 1)$ and $(0, 0, -1)$, again an iseomorphism f^* is induced from W onto Cl $[H(e)]$.

THEOREM 9. *Let S be a positive commutative Clifford semigroup on E^2. Then, S is the continuous homomorphic image of the disjoint union of semigroups which are closures of groups and which are iseomorphic to the complex numbers, $N \times N$, $N \times R$, $R \times R$, U, V, W, N, R, or Y.*

Proof. We know from Theorem 8 that the closure of each two dimensional group in S is iseomorphic to one of the first seven possibilities given above. It is also not difficult to see that the closure of each one dimensional group is iseomorphic to one of the last three possibilities given above. Also, S is the union of such one and two dimensional groups, along with $\{0\}$. For each $e \in E$, let Ψ_e be an iseomorphism from $\mathrm{Cl}\,[H(e)]$ onto whichever of the ten possibilities above is appropriate. Let T be the disjoint union of $\{\Psi_e \,\mathrm{Cl}\,[H(e)] : e \in E\}$. Let us give T the disjoint union topology. Thus, we define a set $\mathcal{O}$ to be open in T if and only if $\mathcal{O} \cap \Psi_e \,\mathrm{Cl}\,[H(e)]$ is open in $\Psi_e \,\mathrm{Cl}\,[H(e)]$ for each e. Let us now proceed to define a multiplication on T. In S the idempotent element ef defines a continuous homomorphism Φ^e_{ef}: $H(e) \to H(ef)$ by $\Phi^e_{ef}(x) = xef = xf = fx$. Now, the following diagram is analytic and the continuous homomorphism Φ^{*e}_{ef} is induced.

$$\begin{tikzcd}
\Psi_e \,\mathrm{Cl}\,[H(e)] \arrow[r, "\Phi^{*e}_{ef}"] & \Psi_{ef} \,\mathrm{Cl}\,[H(ef)] \\
\mathrm{Cl}\,[H(e)] \arrow[u, "\Psi_e"] \arrow[r, "\Phi^e_{ef}"] & \mathrm{Cl}\,H(ef) \arrow[u, "\Psi_{ef}"']
\end{tikzcd}$$

Let x', y', $z' \in T$ such that $x' \in \Psi_e \,\mathrm{Cl}\,[H(e)]$, $y' \in \Psi_f \,\mathrm{Cl}\,[H(f)]$, and $z' \in \Psi_g \,\mathrm{Cl}\,[H(g)]$. Let us define a multiplication on T by defining $x'y' = \Phi^{*e}_{ef}(x) \cdot \Phi^{*f}_{ef}(y)$. Let $x = \Psi_e^{-1}(x')$, $y = \Psi_e^{-1}(y')$, and $z = \Psi_e^{-1}(z')$. Let us now show that T is a semigroup. We must first show that the multiplication is associative. We have

$$\begin{aligned}
(x'y')z' &= \Psi_{efg}[g(xy)] \cdot \Psi_{efg}[(ef)z] = \Psi_{efg}[g(xy)(ef)z] = \Psi_{efg}[xyz] \\
&= \Psi_{efg}[(fg)x] \cdot \Psi_{efg}[e(yz)] = x'(y'z').
\end{aligned}$$

Now, let $\{x_n'\}$ converge to $x' \in \Psi_e \,\mathrm{Cl}\,[H(e)]$ and $\{y_n'\}$ converge to $y' \in \Psi_f \,\mathrm{Cl}\,[H(f)]$. By the nature of the disjoint union topology $\{x_n'\}$ is eventually in $\Psi_e \,\mathrm{Cl}\,[H(e)]$, and $\{y_n'\}$ is eventually in $\Psi_f \,\mathrm{Cl}\,[H(f)]$. It now follows by the continuity of Φ^{*e}_{ef} and the continuity of multiplication in $\Psi_{ef} \,\mathrm{Cl}\,[H(ef)]$ that $\{x_n' y_n'\}$ converges to $x'y'$, and consequently that the multiplication is continuous. Since the disjoint union topology is obviously Hausdorff, we have that T is a topological semigroup. Finally, let us define α: $T \to S$ in the following way. If $x' \in T$, there is a unique e such that $x' \in \Psi_e \,\mathrm{Cl}\,[H(e)]$. So, let us define $\alpha(x') = \Psi_e^{-1}(x') = x$. Since Ψ_e is a homeomorphism, α is continuous. Suppose $\alpha(x') = x \in \mathrm{Cl}\,[H(e)]$ and $\alpha(y') = y \in \mathrm{Cl}\,[H(f)]$. Then, $x'y' = [\Psi_{ef}(fx)] \cdot [\Psi_{ef}(ey)] = \Psi_{ef}(efxy) = \Psi_{ef}(xy)$, so that $\alpha(x'y') = xy = \alpha(x) \cdot \alpha(y)$, and α is a homomorphism.

THEOREM 10. *Let S be a positive commutative Clifford semigroup on E^2. Then, there exists a semilattice of groups T which is a topological semigroup in the disjoint union topology, and there exists a continuous isomorphism from T onto S which, when restricted to each maximal group of T, is an iseomorphism.*

Proof. Let T be the disjoint union of the maximal groups in S. Since S is commutative, T is a semigroup under coordinatewise operations and is clearly isomorphic to S under the map $\Phi[(x, e)] = x$. Let us give T the disjoint union topology.

This is of course equivalent to thinking of T as a subset of $S \times E$, where S has its usual topology, but where E has the discrete topology. If we let j be the surjection of $S \times E$, where E has the discrete topology, onto $S \times E$, where E has the usual topology, and Π_1 be the projection in the first coordinate from $S \times E$, where E has the usual topology, into S, then $\Phi = \Pi_1 \circ j$. Since Π_1 and j are clearly continuous, it follows that Φ is a continuous isomorphism of T onto S. It now follows from the definition of the disjoint union topology that the restriction of Φ to each maximal group is an iseomorphism.

In conclusion we should note that due to the structure of the groups as described in this section, each maximal group $H(e)$ is a topological group.

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