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POSITIVE DERIVATIONS ON f-RINGS

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Introduction

Throughout this paper A will denote an f-ring i.e. a lattice-ordered ring in the sense of Birkhoff and Pierce (1956) in which for all $x, y, z \in A$, $x \wedge y = 0$ implies $x \wedge zy = 0 = x \wedge yz$.

A group endomorphism $D: A \to A$ is *positive* if $D(x) \geqq 0$ whenever $x \geqq 0$ in A. A *derivation* on A is a group endomorphism $D: A \to A$ for which $D(xy) = xD(y) + D(x)y$ for all $x, y \in A$.

Our objective is to characterize algebraically the positive derivations on certain f-rings. Specifically, we show that if A is an archimedean f-ring then the positive derivations on A are precisely the positive endomorphisms of A with range contained in the nilpotents of A and vanishing on A^2.

Derivations on archimedean f-rings

We recall that A is *archimedean* if for some $x, y \in A$ we have $nx \leqq y$ for all natural numbers n, then $x \leqq 0$. Birkhoff and Pierce (1956) have shown that every archimedean f-ring is commutative.

We denoted by Rad (A) the set of nilpotent elements of A. Birkhoff and Pierce (1956) show that Rad (A) is a convex sublattice and a two-sided ideal of A (briefly, Rad (A) is an *l-ideal*) and that $A/\text{Rad}(A)$ is a reduced ring — that is, a ring with no non-zero nilpotents.

LEMMA 1. *If A is an archimedean f-ring then Rad (A) is a polar subset of A. In particular, $A/\text{Rad}(A)$ is an archimedean f-ring.*

PROOF. We denote by M the set of all $z \in A$ for which $|z| \leqq xy$ for some $x, y \in A$. Thus, M contains all products and is an l-ideal of A. We let $M = \{x \in A : |x| \wedge |z| = 0 \text{ for all } z \in M\}$ be the polar of M. Then M annihilates A for if $a \in M$ and $b \in A$ then $ab \in M$ so we have $|a| \wedge |ab| = 0$, and then $|ab| = |ab| \wedge |ab| = |a|\,|b| \wedge |ab| = 0$. Thus $M \subseteq \text{Rad}(A)$. On the other hand, in the proof of their theorem 3.11 Henriksen and Isbell (1962) show that $\mathrm{Rad}(A) \cap M = (0)$ holds if A is archimedean. Thus $\mathrm{Rad}(A) \subseteq M$ in this case, so $\mathrm{Rad}(A) = M$ is a polar subset and $A/\mathrm{Rad}(A)$ is archimedean by Bigard (1969).

LEMMA 2. *Let A be a commutative ring with characteristic 0 and $D: A \to A$ a derivation. If $a \in A$ is nilpotent then $D(a)$ is nilpotent.*

PROOF. Let a be nilpotent in the commutative ring A with characteristic 0 and let $D: A \to A$ be a derivation. We have $a^n = 0$, for some natural number n, so $na^{n-1}D(a) = 0$ and therefore $a^{n-1}D(a) = 0$. Now suppose that for some integer k, $1 \leqq k \leqq n$, we have $a^{n-k}D(a)^{2k-1} = 0$. By applying D to this expression and multiplying by $D(a)$ we get $a^{n-(k+1)}D(a)^{2(k+1)-1} = 0$. We can therefore continue until $D(a)^{2n-1} = 0$, so $D(a)$ is nilpotent.

An endomorphism T of the additive group of A is a *positive orthomorphism* if $x \wedge y = 0$ implies $x \wedge T(y) = 0$ in A.

THEOREM 3. (*Bigard and Keimel* (1969)). *A positive orthomorphism of A is a positive group endomorphism T for which $T(M) \subseteq M$ for each minimal prime subgroup M of A. If A is archimedean and reduced (that is, without proper nilpotents) then a positive orthomorphism $T: A \to A$ is generalized translation i.e. T satisfies $T(xy) = xT(y)$ for all $x, y \in A$.*

LEMMA 4. *If D is a positive derivation on an archimedean reduced f-ring A then $D = 0$.*

PROOF. We see firstly that D is a positive orthomorphism. Suppose that $x \wedge y = 0$ in A. We then have $xy = 0$ so that $xD(y) + D(x)y = 0$. Since $x, y \geqq 0$ and D is positive we have $xD(y) = 0 = D(x)y$, and therefore $x \wedge D(y) = 0$, since A is reduced. Now by theorem 3, D is a generalized translation. Thus for all $x, y \in A$ we have both $D(xy) = xD(y) + D(x)y$ and $D(xy) = xD(y)$. That is, for all $x, y \in A$ we have $D(x)y = 0$, so $D = 0$, since A is reduced.

We now prove the result mentioned in the introduction, algebraically characterizing positive derivations on archimedean f-rings. Notice that if $I \subseteq A$ is an ideal and $D: A \to A$ is a derivation then the map $\bar{D}: A/I \to A/I$ defined by $\bar{D}(a + I) = D(a) + I$ is a derivation.

THEOREM 5. *Suppose that A is an archimedean f-ring. Then the positive derivations on A are precisely the positive group endomorphisms $D: A \to A$ satisfying $D(A) \subseteq \mathrm{Rad}(A)$ and $D(A^2) = (0)$.*

PROOF. Let A be archimedean and $D: A \to A$ a positive homomorphism. If D is a derivation then $D(\mathrm{Rad}(A)) \subseteq \mathrm{Rad}(A)$ by lemma 2, since A

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is commutative, so we can define a positive derivation $\bar{D}$ of $A/\mathrm{Rad}(A)$ by $\bar{D}(x + \mathrm{Rad}(A)) = D(x) + \mathrm{Rad}(A)$. By Lemma 1 and lemma 4 we then have $\bar{D} = 0$. That is, $D(A) \subseteq \mathrm{Rad}(A)$. Since $\mathrm{Rad}(A)$ annihilates A, as we have noted in lemma 1, we have $D(xy) = xD(y) + D(x)y = 0$.

Conversely, suppose that $D(A) \subseteq \mathrm{Rad}(A)$ and $D(A^2) = (0)$. Then for all $x, y \in A$ we have $D(xy) = 0 = xD(y) + D(x)y$, so D is a derivation.

Bounded and almost-bounded elements

The results of the previous section show that we cannot expect a positive derivation on an f-ring to be too far from being zero. In this section we pursue the idea that the kernel of a positive derivation must be large.

If A has a multiplicative identity 1 then we say that $b \in A$ is *bounded* if $|b| \leqq n1$ for some natural number n. We note that if $D: A \to A$ is a positive derivation then $D(b) = 0$ for all bounded elements b of A since $D(1) = 0$.

A subset P of A is a *prime l-ideal* if P is a convex sublattice ideal of A for which the set $\{a \in A : a \geqq 0, a \notin P\}$ is closed under finite meet. A *minimal prime l-ideal* is a prime l-ideal minimal in the family of all prime l-ideals of A, ordered by inclusion. A family $\{P_\lambda : \lambda \in \Lambda\}$ of prime l-ideals of A is *dense* if $\cap \{P_\lambda : \lambda \in \Lambda\} = (0)$. Clearly if $\{P_\lambda : \lambda \in \Lambda\}$ is a dense family of prime l-ideals and $a + M \leqq b + M$ for all $\lambda \in \Lambda$ then $a \leqq b$. If A is a reduced f-ring then the family of all minimal prime l-ideals of A is dense.

LEMMA 6. *Let A be a reduced f-ring with identity* 1. *Then for an element $b > 0$ in A the following are equivalent*:

(i) $b = \vee\{b \wedge n1 : n$ *a natural number*$\}$

(ii) *b is the join of a family of bounded elements*

(iii) *there is a dense family $\{M_\lambda : \lambda \in \Lambda\}$ of minimal prime l-ideals of A such that $b + M_\lambda$ is bounded in A/M_λ, for all $\lambda \in \Lambda$.*

PROOF. The equivalence of (i) and (ii) is straightforward. Suppose that $b > 0$ in A and that $\mathcal{M}$ is the set of all minimal prime l-ideals M of A such that $b + M$ is bounded in A/M. In order to prove that (i) implies (iii) suppose that $I = \cap \mathcal{M} \neq (0)$. Then I contains an element x with $0 < x \leqq 1$. Clearly $b - x + M = b + M \geqq b \wedge n1 + M$ for all $M \in \mathcal{M}$. For every minimal prime l-ideal M not belonging to $\mathcal{M}$ the coset $b - x + M$ is unbounded in A/M; for if $b - x + M \leqq n1 + M$ for some natural number n, then $b + M \leqq x + n1 + M \leqq (n+1)1 + M$. Thus, $b - x + M > n1 + M \geqq b \wedge n1 + M$ for all minimal prime l-ideals M of A not belonging to $\mathcal{M}$, and every natural number n. Consequently, $b - x + M \geqq b \wedge n1 + M$ for every minimal prime l-ideal M of A. As the set of *all* minimal prime l-ideals is dense, we conclude that $b - x \geqq b \wedge n1$, and this for every natural number n. Thus, (i) does not hold.

In order to prove that (iii) implies (i) suppose that $b \neq \vee\{b \wedge n1 : n$ a natural number$\}$. Then there is an $x > 0$ in A such that $b - x \geqq b \wedge n1$ for all natural numbers n. For every $M \in \mathcal{M}$ we get $b - x + M \geqq b \wedge n1 + M = b + M$ for some n, so $-x + M \geqq 0$. As on the other hand $x + M \geqq 0$, we have $0 < x \in \cap \mathcal{M}$ which contradicts (iii).

We shall say that an element b of a reduced f-ring A with identity is almost-bounded if $|b|$ satisfies one of the equivalent conditions of lemma 6. We denote the set of *almost-bounded* elements of A by $\mathscr{E}(A)$, and from lemma 6 (iii) one readily deduces that $\mathscr{E}(A)$ is a convex sublattice and subring of A.

THEOREM 7. *Let A be a reduced f-ring with identity and let $D: A \to A$ be a positive derivation. Then $\mathscr{E}(A) \subseteq \operatorname{Ker} D$.*

PROOF. By theorem 3 every minimal prime l-ideal of A is invariant under D. Let b be an almost-bounded element of A, and let $\{M_\alpha\}$ be the set of minimal prime l-ideals of A for which b is bounded in A/M_α. Then, for each α, D defines a derivation D_α on A/M_α by $D_\alpha(x + M_\alpha) = D(x) + M_\alpha$, and since b is bounded in A/M_α we have $D_\alpha(b + M_\alpha) = 0$. That is, $D(b) \in \cap_\alpha M_\alpha = (0)$.

COROLLARY 8. *Let A be a reduced f-ring with identity. If $y \in A$ is such that uy is almost-bounded for some $u > 0$ with $u^\perp = (0)$, then $D(y) = 0$ for every positive derivation $D: A \to A$.*

PROOF. By theorem 7 we have $uD(|y|) + D(u)|y| = 0$ and therefore $uD(|y|) = 0$, for each positive derivation D on A. Since A is reduced we then have $u \wedge D(|y|) = 0$ and therefore $D(|y|) = 0$ since $u^\perp = (0)$. Consequently $D(y) = 0$.

COROLLARY 9. *If A is a reduced f-ring with identity 1 such that every $x > 1$ is invertible then the only positive derivation $D: A \to A$ is $D = 0$.*

PROOF. If A satisfies the assumptions then each $x > 1$ has the property that $(x^{-1})^\perp = (0)$ and $x^{-1}x = 1$ is bounded. Thus $D(x) = 0$ for all $x > 1$. Then $D(y) = 0$ for all $y \in A$, since $|y| \leqq |y| \vee 1$ for all $y \in A$.

COROLLARY 10. *If D is a positive derivation on a totally-ordered division ring then $D = 0$.*

We recall that a ring A is (von Neumann) *regular* if for each $a \in A$ there is an $x \in A$ for which $axa = a$ and $xax = x$. D. J. Johnson (1962) has shown that every regular f-ring A is strongly regular, that is, for each $a \in A$ there is an $x \in A$ for which $a^2x = 0$. In particular, every regular f-ring A is reduced and A/M is a totally-ordered division ring for each minimal prime l-ideal M.

THEOREM 11. *If* $D: A \to A$ *is a positive derivation on a regular* f*-ring not necessarily with identity) then* $D = 0$.

PROOF. Let M be a minimal prime l-ideal of A. By the remarks preceding this theorem and by theorem 3 we have $D(M) \subseteq M$. The derivation D defined on the totally-ordered division ring A/M by $\bar{D}(x + M) = D(x) + M$ then must be zero by corollary 10. Thus, $D(A) \subseteq \cap\{M: M$ is a minimal prime l-ideal$\} = (0)$, since A is reduced.

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