Positive Eigenvectors of Wedge Maps (*).

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Summary. – *In this paper we investigate the existence of positive eigenvalues and corresponding eigenvectors of nonlinear and noncompact maps defined in a wedge W of a Banach space E. The results are established using the theory of 0-epi maps introduced by Furi-Martelli-Vignoli* [10]. *We prove a conjecture of I. Massabo - C. Stuart* [23] *and we obtain a nonlinear version of the celebrated Krein-Rutman* [18] *theorem, which brings about the different role of the two properties*

$$f(tx) = tf(x) \quad \textit{and} \quad f(x+y) = f(x) + f(y)\,.$$

0. – Introduction.

This paper has three main purposes. The first is the study of the fundamental properties of a very large class of maps acting on cones or, more generally, on wedges in Banach spaces. This class includes compact vector fields and other vector fields whose non-linear part is not compact. This study follows and expands the ideas of a previous paper by M. FURI, M. MARTELLI, A. VIGNOLI [10] and of some extensions and generalizations of it (see [15], [14], [11], [12], [13]).

The second purpose is to prove a conjecture of I. MASSABO and C. STUART [23] along with other results which can be obtained from the theorems of the first part. We point out that degree and index theory techniques were used in [23], while our approach is considerably simpler, being essentially based on Brouwer's fixed point theorem and suitable continuation principles.

The third purpose is to obtain a theorem on eigenvectors of positively homogeneous, order preserving and non-compact maps acting on cones, which is as close as possible to the extension of the classical KREIN-RUTMAN [18] theorem to linear non-compact maps. The result brings about the key points where the consequences of the property

$$f(x+y) = f(x) + f(y) \tag{0.1}$$

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are crucial and should be substituted by suitable assumptions in the case where f is only positively homogeneous. As a consequence of our results we obtain more precise statements of already existing theorems. For example R. NUSSBAUM [25] recently proved that if $f: K \to K$, where K is a cone in a Banach space E, is an α-contraction and there exists $\|u\| = 1$, $u \in K$ such that

$$\{\|f^n(u)\|\} \tag{0.2}$$

is unbounded, then there exists $\|x_0\| = 1$ such that

$$f(x_0) = \lambda x_0 \tag{0.3}$$

for some $\lambda \geqslant 1$. Here we are able to give a better estimate of the value of λ appearing in (0.3) and, in some cases, we are able to provide its exact value.

A few more words should be said before we start. Our theorems of the second and third part are based on the theory developed in the first part. No degree theory or index theory is used. Nevertheless we are able to prove a conjecture and improve results which were formulated within the framework of degree and index theory. This fact seems to suggest that the skepticism of some researchers towards the ideas and methods presented by FURI-MARTELLI-VIGNOLI in [10] and later used and improved in [15], [14], [13], [12], [30], [11] were at least premature. Moreover the methodology aspect should not be neglected either. To build index theory for non-compact maps requires a lengthy construction based on a suitable limiting process [26]. This construction becomes more complex when multivalued maps are involved [8].

Our approach is definitely simpler, and it provides, at least in this case, better results. Moreover it can be extended to the multivalued case in a straightforward manner.

1. – Notations and definitions.

A cone K in a Banach space E is a closed subset of E such that

(i) $x, y \in K$, $a, b \geqslant 0$ implies $ax + by \in K$;

(ii) $x \in K$ and $-x \in K$ implies $x = 0$.

A wedge W in a Banach space E is a closed subset of E which satisfies (i).

A cone K or a wedge W are said to be *normal* if there exists a constant $\gamma > 0$ such that

$$\|x + y\| \geqslant \gamma \|x\| \tag{1.1}$$

for every $x, y \in K$ or $x, y \in W$. Notice that $\gamma \leqslant 1$.

A cone K or a wedge W are said to be *quasi-normal* [27] if there exists $x_0 \in K$ ($x_0 \in W$) and $\gamma > 0$ such that

$$\|x + \lambda x_0\| \geqslant \gamma \|x\| \tag{1.2}$$

for every $x \in K$ ($x \in W$) and $\lambda \geqslant 0$.

Every normal cone is obviously quasi-normal. The cone K of non-negative functions in $C[0, 1]$ or $L^p[0, 1]$ for $1 \leqslant p \leqslant \infty$ is normal. The same cone in $C^k[0, 1]$, $k \geqslant 1$ is not normal but it is quasi-normal. Every cone K in a Hilbert space H is quasi-normal with $\gamma = 1$ (see Proposition 3.1).

A cone K induces a partial ordering, $\leqslant$, in E by setting

$$x \leqslant y \quad \text{iff} \quad y - x \in K . \tag{1.3}$$

A map $f\colon K \to K$ is *order preserving* if

$$x \leqslant y \Rightarrow f(x) \leqslant f(y) . \tag{1.4}$$

If $f = L$, a linear operator, then it is order preserving. In fact if $x \leqslant y$ or $y - x \geqslant 0$ then $L(y - x) \in K$ and

$$L(y - x) = Ly - Lx \in K \qquad \text{or } Lx \leqslant Ly . \tag{1.5}$$

Let $\mathcal{B}$ be the family of all bounded subsets of a cone K. A generalized measure of non-compactness on $\mathcal{B}$ is a function $\theta\colon \mathcal{B} \to [0, +\infty)$ such that

(1) $\theta(A) = 0$ if and only if $\bar{A}$ is compact;

(2) $\theta(\overline{\text{co}}\, A) = \theta(A)$, where $\overline{\text{co}}\, A$ denotes the convex closure of A;

(3) $\theta(A \cup B) = \max\,(\theta(A), \theta(B))$;

(4) $\theta(tA) = t\theta(A)$ for every $t \geqslant 0$;

(5) $\theta(A + B) \leqslant \theta(A) + \theta(B)$.

The function $\alpha\colon \mathcal{B} \to [0, +\infty)$ defined by

$$\alpha(A) = \inf\{\varepsilon > 0 \text{ such that } A \text{ can be covered by a finite family of sets of diameter } \leqslant \varepsilon\}$$

was first proposed by KURATOWSKI [19] and it satisfies (1)-(5) (see G. DARBO [4] for the first proof of (2)).

If the word « sets » is replaced by « spheres » we obtain another measure of non-compactness, which is frequently denoted by γ. It is easy to see that

$$\gamma(A) \leqslant \alpha(A) \leqslant 2\gamma(A) . \tag{1.6}$$

M. FURI and A. VIGNOLI [16] proved that if $S = \{x \in E: \|x\| = 1\}$ then $\alpha(S) = 2$.

In this paper we shall always use the Kuratowski measure of non-compactness unless otherwise stated. The results can be easily generalized to arbitrary measure of non-compactness.

A function $f: K \to K$ is α-Lipschitz (this term was first used by A. AMBROSETTI [1]) with constant p if

$$\alpha(f(A)) \leqslant p\alpha(A) \tag{1.7}$$

for every $A \in \mathfrak{B}$. If $p = 1$ we say that f is α-nonexpansive. If $p < 1$ we say that f is an α-contraction. If

$$\alpha(f(A)) < \alpha(A) \tag{1.8}$$

for every $A \in \mathfrak{B}$, $\alpha(A) \neq 0$ then f is said to be *condensing*. The constant $q = \beta(f) \geqslant 0$ is defined as the sup of all non-negative real numbers r such that

$$r\alpha(A) \leqslant \alpha(f(A)) . \tag{1.9}$$

Let $f: K \to K$ be positively homogeneous, i.e., $f(tx) = tf(x)$ for every $t \geqslant 0$. Suppose f sends bounded sets into bounded sets. Then we can define

$$\|f\| = \sup \{\|f(x)\|: \|x\| = 1, \ x \in K\} . \tag{1.10}$$

We really should use $\|f\|_K$, but no confusion will be possible since we are working on cones. Therefore we shall drop the subscript.

Observe that $\|f^2\| \leqslant \|f\|^2$. Therefore we can define

$$r(f) = \limsup_{n \to +\infty} \|f^n\|^{1/n} . \tag{1.11}$$

and obviously $r(f) \leqslant \|f\|$.

It can be shown that if f is positively homogeneous than its restriction, $\bar{f}$, to the set

$$\partial K_1 = \{x \in K: \|x\| = 1\}$$

is α-Lipschitz with constant p iff f is α-Lipschitz with the same constant. Therefore we define

$$\alpha(f) = \sup \left\{ \frac{\alpha(\bar{f}(A))}{\alpha(A)} : \alpha(A) \neq 0, \ A \subset \partial K_1 \right\} . \tag{1.12}$$

Since

$$\alpha(f^2) \leqslant (\alpha(f))^2$$

we obtain

$$\omega(f) = \limsup_{n\to+\infty} [\alpha(f^n)]^{1/n} \leqslant \alpha(f) . \tag{1.13}$$

In the case where f is a linear operator, L, the more accurate symbols

$$r_K(L) , \quad \alpha_K(L) , \quad \omega_K(L) \tag{1.14}$$

will be used to avoid confusion with

$$r(L) , \quad \alpha(L) , \quad \omega(L) . \tag{1.15}$$

It is easy to verify that

$$r_K(L) = \lim_{n\to+\infty} \|L^n\|^{1/n} \tag{1.16}$$

$$\omega_K(L) = \lim_{n\to+\infty} [\alpha_K(L^n)]^{1/n} . \tag{1.17}$$

2. – Definition and properties of 0-epi maps.

Let K, $W \subset E$ be respectively a cone and a wedge in a Banach space E.

W_r, K_r and the like.

Let $r > 0$. Denote by

$$D_r = \{x \in E: \|x\| \leqslant r\}$$

$$S_r = \{x \in E: \|x\| = r\}$$

$$B_r = D_r \setminus S_r .$$

Set

$$\begin{aligned} W_r &= W \cap D_r , \quad \partial W_r = W \cap S_r \\ K_r &= K \cap D_r , \quad \partial K_r = K \cap S_r . \end{aligned} \tag{2.1}$$

A subset $\Omega \subset W$ is *relatively open* if there exists an open set $O \subset E$ such that

$$\Omega = O \cap W .$$

The *relative border* of Ω, denoted by $\partial\Omega$, is the set

$$\bar{\Omega} \cap \overline{\Omega^c}$$

where $\bar{\Omega}$ is the closure of Ω, and $\Omega^c = W \setminus \Omega$. *For the sake of simplicity we shall say that Ω is open and $\partial\Omega$ is its border.*

p-0-epi maps

DEFINITION. – Let $\Omega \subset W$ be an open and bounded set. Let $T: \bar{\Omega} \to E$ be continuous and such that $T(x) \neq 0$ for every $x \in \partial\Omega$. We say that T is p-0-epi in $\bar{\Omega}$ if for every continuous map $h: \bar{\Omega} \to W$ such that

(i) h is α-Lipschitz with constant $\leqslant p$;

(ii) $h(x) = 0$ for every $x \in \partial\Omega$,

the equation

$$T(x) = h(x \tag{2.2}$$

has at least one solution (see [30] for the case when h is compact).

If T is p-0-epi in $\bar{\Omega}$ then obviously T is q-0-epi for every $q \leqslant p$.

If T is 0-0-epi then we shall simply say that T is 0-epi.

Existence results

PROPOSITION 2.1. – *Let $T: \bar{\Omega} \to E$ be p-0-epi in $\bar{\Omega}$. Then the equation $T(x) = 0$ has a solution $x_0 \in \Omega$.*

PROOF. – The function $h: \bar{\Omega} \to W$ defined by $h(x) = 0$ for every $x \in \bar{\Omega}$ satisfies (i) and (ii). Q.E.D.

PROPOSITION 2.2. – *Let $0 \in \Omega$. Then the identity is $(1-\varepsilon)$-0-epi in $\bar{\Omega}$, for every $0 < \varepsilon \leqslant 1$.*

PROOF. – Let $h: \bar{\Omega} \to W$ be an α-contraction such that $h(x) = 0$ for every $x \in \partial\Omega$. Let $r > 0$ be such that

$$\begin{aligned} \|h(x)\| &\leqslant r \quad \text{for every } x \in \bar{\Omega} \\ \|x\| &\leqslant r \quad \text{for every } x \in \partial\Omega\,. \end{aligned}$$

Extend h to an α-contraction $h_1: W_r \to W_r$ by setting

$$h_1(x) = \begin{cases} h(x) & \text{if } x \in \bar{\Omega} \\ 0 & \text{if } x \notin \Omega\,. \end{cases}$$

Then h_1 has a fixed point [4], which is obviously a fixed point of h since $0 \in \Omega$. Q.E.D.

The following example shows that the identity is not 1-0-epi in $\bar{\Omega}$, $0 \in \Omega$.

EXAMPLE 2.1. – Let l^2 be the Hilbert space of square summable sequences of real numbers and let W be the wedge of sequences having non-negative entries. Let $\bar{\Omega} = W_2$. Define

$$h(x) = \begin{cases} (\sqrt{1 - \|x\|^2}, x_1, x_2, \ldots) & \text{if } \|x\| \leqslant 1 \\ (2 - \|x\|)(0, x_1, x_2, \ldots) & \text{if } 1 \leqslant \|x\| \leqslant 2 . \end{cases}$$

Then h is α-nonexpansive. Moreover if $\|x\| = 2$ then $h(x) = 0$.

The equation

$$x = h(x) \tag{2.3}$$

does not have a solution. In fact if $\|x\| < 1$ then $\|h(x)\| = 1$, and if $\|x\| > 1$ then $\|h(x)\| < 1$. Hence the only solutions can be in ∂W_1. But if $\|x\| = 1$ then

$$h(x) = (0, x_1, x_2, \ldots) = (x_1, x_2, \ldots)$$

only if $x_1 = x_2 = \ldots = 0$.

PROPOSITION 2.3. – *Let $x_0 \in E$ and $0 < \varepsilon \leqslant 1$. Then the map $T(x) = x - x_0$ is $(1-\varepsilon)$-0-epi in $\bar{\Omega} \subset W$ if $x_0 \in \Omega$ and is not 0-epi if $x_0 \notin \Omega$.*

PROOF. – If $x_0 \notin \bar{\Omega}$ then the equation $T(x) = 0$ does not have any solution in Ω and therefore T is not 0-epi in $\bar{\Omega}$.

If $x_0 \in \partial\Omega$ then T does not satisfy the requirement $T(x) \neq 0$ for every $x \in \partial\Omega$.

If $x_0 \in \Omega$ and $h: \bar{\Omega} \to W$ is an α-contraction such that $h(x) = 0$ for every $x \in \partial\Omega$, then we can choose $r > 0$ so that

$$r \geqslant \|h(x)\| + \|x_0\| \quad \text{for every } x \in \bar{\Omega}$$
$$r \geqslant \sup \{\|x\| : x \in \partial\Omega\} .$$

Then we can extend h to a map $h_1: W_r \to W_r$ as in Proposition 2.2, and we can consider the α-contraction $k: W_r \to W_r$ defined by

$$k(x) = h_1(x) + x_0 . \tag{2.4}$$

Let $\bar{x}$ be a fixed point of k. Then

$$\bar{x} = h_1(\bar{x}) + x_0 . \tag{2.5}$$

If $\bar{x} \notin \Omega$ then $h_1(\bar{x}) = 0$ and $\bar{x} = x_0$, which is impossible since $x_0 \in \Omega$. Hence $\bar{x} \in \Omega$, and

$$\bar{x} - x_0 = h_1(\bar{x}) = h(\bar{x}) . \quad \text{Q.E.D.} \tag{2.6}$$

Additivity

PROPOSITION 2.4. – *Let $\Omega \subset W$ be open and bounded. Let $T\colon \bar{\Omega} \to E$ be p-0-epi in $\bar{\Omega}$. Assume that V_1, V_2 are two open subsets of Ω such that $V_1 \cap V_2 = \emptyset$ and*

$$T^{-1}(0) \subset V_1 \cup V_2 .$$

Then T is p-0-epi in either $\bar{V}_1$ or $\bar{V}_2$.

PROOF. – Assume that T is not p-0-epi in $\bar{V}_1$. Then there exists $h_1\colon \bar{V}_1 \to W$ such that $h_1(x) = 0$ for every $x \in \partial V_1$, h is α-Lipschitz with constant p and

$$T(x) \neq h_1(x)$$

for every $x \in \bar{V}_1$. Similarly if T is not p-0-epi in $\bar{V}_2$ we can find h_2 with similar properties. Define

$$h(x) = \begin{cases} h_1(x) & \text{if } x \in V_1 \\ h_2(x) & \text{if } x \in V_2 \\ 0 & \text{otherwise} . \end{cases}$$

Notice that h is continuous and $\alpha(h) = p$. Hence the equation

$$T(x) = h(x)$$

has a solution. Since $T(x) \neq 0$ if $x \notin V_1 \cup V_2$ and $T(x) \neq h_i(x)$ if $x \in V_1 \cup V_2$ we reach a contradiction. Thus T is p-0-epi in either $\bar{V}_1$ or $\bar{V}_2$.

Continuation Principles

PROPOSITION 2.5. – *Let $\Omega \subset W$ be open and bounded and $T\colon \bar{\Omega} \to E$ be p-0-epi in $\bar{\Omega}$. Assume that $\beta(T) > 0$. Let $R\colon \bar{\Omega} \to E$ be such that*

(i) *$\alpha(R - T) = q < \beta(T) = s$ and $q \leqslant p$;*

(ii) *$R - T\colon \bar{\Omega} \to W$;*

(iii) *$T(x) + t(R(x) - T(x)) \neq 0$ for every $x \in \partial\Omega$ and $t \in (0, 1]$.*

Then R is at least r-0-epi, where $r < s - q$, $r \leqslant p - q$.

PROOF. – Let $h\colon \bar{\Omega} \to W$ be an α-Lipschitz map with constant r and such that $h(x) = 0$ for every $x \in \partial\Omega$. We must show that the equation

$$R(x) = h(x) \tag{2.7}$$

has a solution. Let $\Sigma = \{(x, \lambda) \in \bar{\Omega}\times[0, 1]\colon T(x) + t(R(x) - T(x)) = h(x)$ for some $t \in [0, 1]\}$. Then $\Sigma \neq \emptyset$ since for $t = 0$ the equation $T(x) = h(x)$ has a solution. Moreover Σ is closed.

Let us show now that Σ is compact. Put

$$A = \{x \in \bar{\Omega}\colon (x, t) \in \Sigma \text{ for some } t \in [0, 1]\}.$$

Then $-T(A) \subset [0, 1]\times(R - T)(A) - h(A)$. Hence

$$s\alpha(A) \leqslant \alpha(T(A)) \leqslant \alpha(R - T)(A) + \alpha(h(A)) \leqslant (q + r)\alpha(A).$$

Since $q + r < s$ we must have $\alpha(A) = 0$. Thus A is totally bounded. This implies that Σ is compact and A is compact. Moreover $A \cap \partial\Omega = \emptyset$.

Let $\phi\colon \bar{\Omega} \to [0, 1]$ be a Urysohn's function such that

$$\phi(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in \partial\Omega \end{cases}$$

and define

$$k(x) = -\phi(x)(R(x) - T(x)) + h(x).$$

Then $k(x) = 0$ if $x \in \partial\Omega$ and k is α-Lipschitz with constant $\leqslant r + q \leqslant p$. Hence the equation

$$T(x) = k(x) \tag{2.8}$$

has a solution $\bar{x} \in \Omega$. Since $\phi(\bar{x}) \in [0, 1]$ we see that $\bar{x} \in A$. Thus $\phi(\bar{x}) = 1$ and

$$R(\bar{x}) = h(\bar{x}). \quad \text{Q.E.D.} \tag{2.9}$$

PROPOSITION 2.6. – *Let $\Omega \subset W$ be open and bounded and $f\colon \bar{\Omega} \to W$ be compact. Let $x_0 \in W$ and $0 < \varepsilon \leqslant 1$. Assume that*

$$x - tf(x) - x_0 \neq 0 \tag{2.10}$$

for every $x \in \partial\Omega$ and $t \in [0, 1]$. Then $T = I - x_0 - f$ is not 0-epi if $x_0 \notin \Omega$ and it is $(1 - \varepsilon)$-0-epi if $x_0 \in \Omega$.

PROOF. – Let $x_0 \notin \Omega$. Obviously $x_0 \notin \bar{\Omega}$. Assume $I - x_0 - f$ is 0-epi. Then $S = \{x : x - tf(x) - x_0 = 0 \text{ for some } t \in [0, 1]\}$ is compact and $S \cap \partial\Omega = \emptyset$. Let $\phi: \bar{\Omega} \to [0, 1]$ be a Urysohn's function such that

$$\phi(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \in \partial\Omega \,. \end{cases}$$

Define $h(x) = -\phi(x) f(x)$. Then $h: \bar{\Omega} \to W$ is compact and $h(x) = 0$ for every $x \in \partial\Omega$. Therefore the equation

$$x - f(x) - x_0 = h(x) \,. \tag{2.10}$$

has a solution. Thus

$$x - (1 - \phi(x)) f(x) - x_0 = 0 \,. \tag{2.11}$$

Since $\phi(x) \in [0, 1]$ it follows that $x \in S$. Hence $\phi(x) = 1$ and

$$x = x_0 \tag{2.12}$$

which is impossible.

Now let $x_0 \in \Omega$. Then $x - x_0$ is $(1 - \varepsilon)$-0-epi. Let $h: \bar{\Omega} \to W$ be an α-contraction such that $h(x) = 0$ for every $x \in \partial\Omega$. Define

$$S = \{x \in \Omega : x - x_0 - tf(x) = h(x)\} \,.$$

Then $S \neq \emptyset$ and compact. Moreover $S \cap \partial\Omega = \emptyset$. Let ϕ be a Urysohn's function as before and consider the equation

$$x - x_0 = \phi(x) f(x) + h(x) \tag{2.13}$$

which has a solution. Then $\phi(x) = 1$ and $x - x_0 - f(x) = h(x)$. Q.E.D.

The following proposition extends the result of Proposition 2.6 to α-contractions.

PROPOSITION 2.7. – *Let $\Omega \subset W$ be open and bounded. Let $f: \bar{\Omega} \to W$ be an α-contraction with constant p. Let $x_0 \in W$ and $0 < \varepsilon \leqslant p$. Assume that*

$$x - x_0 - tf(x) \neq 0 \tag{2.14}$$

for every $x \in \partial\Omega$, $t \in [0, 1]$. Then $I - x_0 - f$ is $(1 - p - \varepsilon)$-0-epi if $x_0 \in \Omega$. If $x_0 \notin \Omega$ then $I - x_0 - f$ is not p-0-epi in $\bar{\Omega}$.

PROOF. – Let $x_0 \notin \Omega$. Then $x_0 \notin \bar{\Omega}$. Let

$$\delta = \inf\{\|x - x_0\| : x \in \partial\Omega\}$$

and let ϱ be such that

$$\varrho\|f(x)\| < \delta$$

for every $x \in \bar{\Omega}$. We may assume $\varrho < 1$.

Assume that $I - x_0 - f$ is p-0-epi. Then the set $S = \{x \in \Omega : x - x_0 - tf(x) = 0$ for some $t \in [0, 1]$ and $x \in \Omega\}$ is non empty and compact. Moreover $S \cap \partial\Omega = \emptyset$. Let ϕ be a Urysohn's function such that

$$\phi(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \in \partial\Omega \end{cases}$$

and define $k(x) = (1 - \varrho)\phi(x)f(x)$. Then the equation

$$x - x_0 - f(x) = -k(x) \tag{2.15}$$

has a solution since k is an α-contraction with constant $(1 - \varrho)p < p$ and $k(x) = 0$ for every $x \in \partial\Omega$. Thus

$$x - x_0 - \varrho f(x) = 0 \,. \tag{2.16}$$

Since $\varrho\|f(x)\| < \delta$ we get a contradiction. Hence $x - x_0 - f(x)$ is not p-0-epi.

The proof that $I - x_0 - f$ is $(1 - p - \varepsilon)$-0-epi when $x_0 \in \Omega$ follows the same pattern of the similar part in Proposition 2.6 and it will be omitted. **Q.E.D.**

3. – Quasi-normal wedges and eigenvectors.

In section 1 we said that a cone K (or a wedge W) is quasi-normal if there exist $x_0 \in K$ ($x_0 \in W$), $x_0 \neq 0$, and $\gamma > 0$ such that

$$\|x + \lambda x_0\| \geqslant \gamma\|x\|$$

for every $\lambda \geqslant 0$ and every $x \in K$.

The definition of quasi-normality encountered in the literature is apparently weaker, because it requires the existence of $x_0 \in K$, $x_0 \neq 0$, such that

$$\|x + x_0\| \geqslant \gamma\|x\|$$

for every $x \in K$.

But the two definitions are really equivalent, since given $x \in K$ and given $\lambda > 0$ there exists $y \in K$ such that $x = \lambda y$. From

$$\|y + x_0\| \geqslant \gamma \|y\|$$

we get

$$\|\lambda y + \lambda x_0\| \geqslant \gamma \|\lambda y\|$$

and therefore

$$\|x + \lambda x_0\| \geqslant \gamma \|x\| .$$

A cone K is said to be normal if there exists $\gamma > 0$ such that

$$\|x + y\| \geqslant \gamma \|x\|$$

for every $x, y \in K$.

The constant γ is called the normality or quasi-normality constant of the cone respectively.

A normal cone is obviously quasi-normal. The converse is false as the following example shows.

EXAMPLE 3.1. – Let $C^k[0, 1]$ be the Banach space of real functions of class C^k in $[0, 1]$ endowed with the norm

$$\|x\|_k = \sum_{i=0}^{k} \|x^{(i)}\|$$

where

$$\|x^{(i)}\| = \max\{|x^{(i)}(t)|: t \in [0, 1]\}$$

and $x^{(0)}(t) = x(t)$.

Consider the cone K of nonnegative functions. Then K is not normal.

In fact take $x(t) = t^n$ and $y(t) = 1 - t^n$. Then

$$\|x + y\|_k = 1 .$$

But

$$\|x\|_k > 1 + n .$$

The cone K is quasi-normal. In fact let $x_0(t) \equiv 1$. Then it is easy to see that

$$\|x + \lambda\|_k \geqslant \|x\|_k$$

and $\gamma = 1$.

PROPOSITION 3.1 ([20], [3], [5]). – *Let K be a cone in a Hilbert space H. Then K is quasi-normal, with $\gamma = 1$.*

PROOF. – We may assume, without loss of generality, that K is not contained in a proper subspace of H. There exists a continuous linear functional $\phi: H \to \mathbf{R}$ such that $\|\phi\| = 1$ and

$$\phi(x) \geqslant 0 \tag{3.3}$$

for every $x \in K$ [6].

Let $\|y\| = 1$ be such that

$$\langle y, x\rangle = \phi(x)$$

for every $x \in K$.

If y belongs to K, set $x_0 = y$. Then write

$$x = x_1 + \mu x_0$$

with $\mu = \langle x, x_0\rangle$. We obtain

$$\|x + \lambda x_0\|^2 = \|x_1 + (\lambda + \mu)x_0\|^2 = \|x_1\|^2 + (\lambda + \mu)^2\|x_0\|^2 \geqslant \|x_1\|^2 + \mu^2\|x_0\|^2 = \|x\|^2 .$$

Thus

$$\|x + \lambda x_0\| \geqslant \|x\| .$$

If $y \notin K$, let z be the point in K which is closest to y [17]. It is easy to see that $0 < \|z\| < 1$ since K is not contained in a proper subspace. Moreover it is known that

$$\langle y - z, x - z\rangle \leqslant 0 \tag{3.4}$$

for every $x \in K$, and $\langle z, y - z\rangle = 0$. We therefore obtain

$$\langle y, x\rangle \leqslant \langle z, x\rangle . \tag{3.5}$$

Set $x_0 = z/\|z\|$. Q.E.D.

We see that $\gamma = 1$. It is known that this fact characterizes Hilbert spaces ([3], [5]) if $\dim H \geqslant 3$ and that every cone in a Banach space E is quasi-normal [20].

PROPOSITION 3.2. – *Let K be a cone and $\Omega \subset K$ be an open and bounded set such that $0 \in \Omega$. Let $\varrho = \sup\{\|x\| : x \in \partial\Omega\}$. Let $f: \bar{\Omega} \to K$ be compact and such that*

$$\delta = \inf\{\|f(x)\| : x \in \partial\Omega\} > \varrho .$$

Then $I - f$ is not 0-epi in $\bar{\Omega}$.

Proof. – We may assume, without loss of generality, that $\|f(x)\| = \delta$ for every $x \in \partial\Omega$. Let

$$r = \sup\{\|x\| + \|f(x)\| : x \in \bar{\Omega}\}$$

and choose ε such that

$$\delta > \varrho + \varepsilon . \tag{3.6}$$

Let $y \neq 0$, $y \in K$, $\|y\| < \varepsilon$. Define $g(x) = f(x) + y$. Observe that $g(x) \neq 0$ for every $x \in \bar{\Omega}$. Let $\tau\colon K_r|\{0\} \to \partial K_r$ be defined by

$$\tau(x) = 2r\,\frac{x}{\|x\|} .$$

Consider the map $h(x) = \tau(g(x))$. Using (3.6) it can be easily shown that for every $t \in [0, 1]$ and $x \in \partial\Omega$ we have

$$x \neq (1-t)f(x) + th(x) = f(x) + t(h(x) - f(x))$$

and that $k = h - f$ maps $\bar{\Omega}$ into K. Let $S = \{x \in \bar{\Omega}\colon x = f(x) + t(h(x) - f(x))\}$. If $I - f$ is 0-epi then $S \neq \emptyset$ and it is compact. Moreover $S \cap \partial\Omega = \emptyset$.

Let $\varphi\colon \bar{\Omega} \to [0, 1]$ be a Urysohn's function such that

$$\varphi(x) = \begin{cases} 0 & \text{if } x \in \partial\Omega \\ 1 & \text{if } x \in S \end{cases}$$

Define

$$q(x) = \varphi(x)\,k(x) .$$

Then q is compact and $q(x) = 0$ for every $x \in \partial\Omega$. Thus there exists x_0 such that

$$x_0 - f(x_0) = q(x_0) = \varphi(x_0)\,k(x_0) .$$

Then $x_0 \in S$ and $\varphi(x_0) = 1$. Thus

$$x_0 - f(x_0) = h(x_0) - f(x_0) .$$

This implies

$$x_0 = h(x_0) .$$

But $\|x_0\| \leqslant \varrho < r < \|h(x_0)\|$. Hence $I - f$ is not 0-epi. Q.E.D.

We are now ready to obtain our first theorem on positive eigenvalues and eigenvectors.

THEOREM 3.1. – *Let $f\colon \bar{\Omega} \to K$ be compact and such that*

$$\inf\{\|f(x)\| : x \in \partial\Omega\} = \delta > 0 . \tag{3.7}$$

Let $\varrho = \sup\{\|x\| : x \in \partial\Omega\}$. *Then there exists $x \in \partial\Omega$ and $\lambda \geqslant \delta/\varrho$ such that*

$$f(x) = \lambda x . \tag{3.8}$$

PROOF. – Assume $f(x) \neq tx$ for every $x \in \partial\Omega$ and every $t \geqslant \delta/\varrho$. Then $I - f$ is 0-epi in $\bar{\Omega}$ via the homotopy

$$x = tf(x)$$

(see Proposition 2.6 with $x_0 = 0$). Define

$$g(x) = \frac{\varrho + \varepsilon}{\delta} f(x) .$$

Then $\inf\{\|g(x)\| : x \in \partial\Omega\} > \varrho$. Moreover

$$x - f(x) - t\big(g(x) - f(x)\big) \neq 0$$

for every $t \in [0, 1]$ and every $x \in \partial\Omega$. By Proposition 2.5 $x - g(x)$ is 0-epi. But by Proposition 3.2 it is not. Hence

$$f(x) = \lambda x \tag{3.8}$$

for some $\lambda \geqslant \delta/\varrho$ and $x \in \partial\Omega$. Q.E.D.

We now take up the noncompact case. Recall that a cone $K \subset E$ is always quasi-normal.

PROPOSITION 3.3. – *Let $W \subset E$ be a quasi-normal wedge or a cone and let $\Omega \subset W$ be an open and bounded neighborhood of 0. Let $f\colon \bar{\Omega} \to K$ be an α-contraction with constant p.*

Assume that

$$\delta = \inf\{\|f(x)\| : x \in \partial\Omega\} > \frac{\varrho}{\gamma} \tag{3.9}$$

where $\varrho = \sup\{\|x\| : x \in \partial\Omega\}$ *and k is the quasi-normality constant of the wedge.*

Then $I - f$ is not 0-epi in $\bar{\Omega}$.

PROOF. – Let $\|x_0\| = 1$, $x_0 \in K$ be such that

$$\|x + \lambda x_0\| \geqslant \gamma \|x\| \tag{3.10}$$

for every $x \in W$ and $\lambda \geqslant 0$. Let $\psi\colon E \to \boldsymbol{R}$ be a linear continuous functional such that

$$\psi(x_0) = 1\,, \qquad \|\psi\| = 1\,. \tag{3.11}$$

Define $f_0(x) = \psi(f(x))\,x_0$ and $f_1(x) = f(x) - f_0(x)$. Since $\operatorname{Im} f$ is bounded we can c oose $D > 0$ so that

$$D \geqslant |\psi(f(x))|\,, \qquad \forall x \in \bar{\Omega}$$

and

$$D > \sup\{\|x - f_1(x)\| : x \in \bar{\Omega}\}\,.$$

Let

$$h(x) = [D - \psi(f(x))]\,x_0\,.$$

Then h is compact and $\operatorname{Im} h \subset W$. Observe that

$$x - f(x) \neq th(x)$$

for every $t \in [0, 1]$ and every $x \in \partial\Omega$. In fact if $x \in \partial\Omega$ then $\|x\| \leqslant \varrho$. But

$$\|f(x) + th(x)\| \geqslant \gamma \|f(x)\| \geqslant \gamma\delta > \varrho$$

since $t\big(D - \psi(f(x))\big) = \lambda \geqslant 0$.

Define $T(x) = x - f(x)$, $R(x) = x - f_1(x) - Dx_0$. Then

$$x - f(x) - th(x) = T(x) + t\big(R(x) - T(x)\big)\,.$$

Moreover $\alpha(R - T) = \alpha(h) = 0 < \beta(T)$ since $\beta(T) \geqslant 1 - p$.

Hence, by Proposition 2.5, R is 0-epi. In particular there exists $\bar{x}$ such that

$$\bar{x} - f_1(\bar{x}) = Dx_0\,.$$

But

$$\|\bar{x} - f_1(\bar{x})\| < D = D\|x_0\|\,.$$

This contradiction shows that $I - f$ is not 0-epi. Q.E.D.

In [23] I. MASSABO and C. STUART proved that if K is a normal cone with normality constant γ and if $f\colon \Omega \to K$ is an α-contraction with constant p such that

$$\delta = \inf\{\|f(x)\| : x \in \partial\Omega\} > \frac{p\varrho}{\gamma}$$

where Ω, ϱ, γ are as in Proposition 3.3, then there exists $x \in \partial\Omega$ and $\lambda \geqslant \delta/\varrho$ such that

$$f(x) = \lambda x .$$

They conjectured that the normality of the cone was unnecessary.

The following theorem shows that their conjecture is true.

THEOREM 3.2. – *Let W be a quasi-normal wedge or a cone and $\Omega \subset W$ be a bounded and open neighborhood of 0. Let $f\colon \bar{\Omega} \to W$ be α-Lipschitz with constant p. Assume that*

$$\delta = \inf\{\|f(x)\| : x \in \partial\Omega\} > \frac{p\varrho}{\gamma} \tag{3.13}$$

with ϱ, γ as in Proposition 3.3. Then there exists $x \in \partial\Omega$ and $\lambda \geqslant \delta/\varrho$ such that

$$f(x) = \lambda x . \tag{3.14}$$

PROOF. – Assume $f(x) \neq \lambda x$ for every $\lambda \geqslant \delta/\varrho$ and $x \in \partial\Omega$. Then obviously

$$f(x) \neq \lambda x$$

for every $\lambda \geqslant 0$ and $x \in \partial\Omega$, since for $\lambda < \delta/\varrho$ we cannot have

$$\delta \leqslant \|f(x)\| = \lambda\|x\| \leqslant \lambda\varrho .$$

Define $g(x) = (\varrho/\gamma\delta)f(x)$. Then

$$\alpha(g(A)) \leqslant \frac{\varrho}{\gamma\delta}\alpha(f(A)) \leqslant \frac{\varrho p}{\gamma\delta}\alpha(A) . \tag{3.15}$$

Hence g is an α-contraction since $\varrho p/\gamma\delta = r < 1$. Moreover $\inf\{\|g(x)\| : x \in \partial\Omega\} > \varrho/\gamma$.

Now observe that

$$x \neq \operatorname{tg}(x)$$

for every $t \in [0, 1]$ and $x \in \partial\Omega$. Therefore by Proposition 2.7 with $x_0 = 0$ we have that $I - g$ is 0-epi. But according to Proposition 3.3 $I - g$ is not 0-epi. This contradiction shows that there exists $x_0 \in \partial\Omega$ and $\lambda \geqslant \delta/\varrho$ such that

$$f(x_0) = \lambda x_0 . \quad \text{Q.E.D.} \tag{3.16}$$

Corollary 3.1 (I. MASSABO-C. STUART [23]).

Let K be a normal cone in a Banach space E, $\Omega \subset K$ be an open neighborhood of 0. *Let $f\colon K \to K$ be an α-contraction with constant p. Suppose that*

$$\delta = \inf\{\|f(x)\|\colon x \in \partial\Omega\} > \frac{p\varrho}{\gamma} \tag{3.17}$$

where $\varrho = \sup\{\|x\|\colon x \in \partial\Omega\}$, and γ is the normality constant of K. Then there exists $t > 0$ and $x \in \partial\Omega$ such that $f(x) = tx$.

The following two results do not derive from the theory of 0-epi maps, but we shall include them here for completeness.

The first is a theorem which can be proved using either the approach of M. MARTELLI [21], or P. MASSAT [24], or G. FOURNIER and M. MARTELLI [7].

THEOREM 3.3. – *Let K be a cone in a Banach space E and let $f\colon \partial K_r \to \partial K_r$ be an α-contraction or a condensing map. Then f has a fixed point.*

As a consequence of the above result we get the following.

THEOREM 3.4. – *Let $\Omega \subset K_r$ be an open set such that for every $\|x\| = r$ there exists a unique $t_x \in [0, 1]$ with the property that $t_x x \in \partial\Omega$. Let $f\colon \Omega \to K$ be α-Lipschitz with constant p (or condensing) such that*

$$\delta = \inf\{\|f(x)\|\colon x \in \partial\Omega\} > pr . \tag{3.18}$$

Then there exists $x_0 \in \partial\Omega$ and $\lambda \geqslant \delta/r$ such that

$$f(x_0) = \lambda x_0 . \tag{3.19}$$

PROOF. – Let $f_r\colon \partial K_r \to \partial K_r$ be defined by

$$f_r(x) = \frac{rf(t_x x)}{\|f(t_x x)\|} . \tag{3.20}$$

Then f_r is continuous [25].

For every $A \subset \partial K_r$ define

$$A_0 = \{t_x x\colon x \in A\} .$$

Then $\alpha(A_0) \leqslant \alpha(A)$. Moreover

$$\alpha(f_r(A)) \leqslant \frac{r}{\delta}\alpha(f(A_0)) \leqslant \frac{rp}{\delta}\alpha(A_0) \leqslant \frac{rp}{\delta}\alpha(A) . \tag{3.21}$$

Since $rp/\delta < 1$ we see that f_r is an α-contraction. Thus there exists $\bar{x} \in \partial K_r$ such that

$$f_r(\bar{x}) = \bar{x}$$

or

$$f(x_0) = \lambda x_0 \tag{3.22}$$

where $x_0 = t_{\bar{x}}\bar{x}$ and $\lambda \geqslant \delta/r$. If f is condensing and

$$\delta = \inf \{\|f(x)\| : x \in \partial\Omega\} > r \tag{3.23}$$

we can use the same proof to obtain again an eigenvector corresponding to an eigenvalue $\lambda \geqslant \delta/r$. Q.E.D.

4. – Krein-Rutman type theorems.

We are now ready to give a non-linear version of the Krein-Rutman theorem, where the full potential of the homogeneity property is used.

PROPOSITION 4.1. – *Let $f\colon W \times [0, +\infty) \to W$ be compact. Assume that $f(x, 0) = 0$ for every $x \in W$. Let $\Sigma = \{(x, \lambda) \in K \times [0, +\infty) : x = f(x, \lambda)\}$. Then there is an unbounded component of Σ containing* $(0, 0)$.

PROOF. – Observe that $H = W \times [0, +\infty)$ is a wedge in $E \times \mathbf{R}$, endowed with the norm $\|(x, \lambda)\| = \max\{\|x\|, |\lambda|\}$. Define $H_n = \{(x, \lambda) \in H : \|(x, \lambda)\| \leqslant n\}$ and set

$$\partial H_n = (W_n \times \{n\}) \cup (\partial W_n \times (0, n]) . \tag{4.1}$$

Let C be the connected component of Σ which contains $(0, 0)$. We want to show that C intersects ∂H_n.

In fact if this is not the case then we can find an open set $U \subset H_n = W_n \times [0, n]$ such that $\partial U \cap \Sigma = \emptyset$ and $C \subset U$ [29]. Let $\varphi\colon H_n \to [0, 1]$ be a Urysohn's function such that

$$\varphi(x, \lambda) = \begin{cases} 0 & \text{if } (x, \lambda) \notin U \\ 1 & \text{if } (x, \lambda) \in \Sigma \cap U , \end{cases} \tag{4.2}$$

Define $h(x, \lambda) = \big(f(x, \varphi(x, \lambda)n), \varphi(x, \lambda)n\big)$. Then h is compact and $h(x, \lambda) = 0$ if $(x, \lambda) \in \partial H_n$. Therefore the equation

$$(x, \lambda) = \big(f(x, \varphi(x, \lambda)n), \varphi(x, \lambda)n\big) \tag{4.3}$$

has a solution.

If $(x, \lambda) \notin U$ then $\varphi(x, \lambda) = 0$. Thus $\lambda = 0$, and we get

$$x = f(x, 0) = 0 .$$

But then $(0, 0) \in C$ and $\varphi(0, 0) = 1$. A contradiction. If $(x, \lambda) \in U$, then

$$\lambda = \varphi(x, \lambda)n$$

or $\lambda_n \in [0, n]$ and

$$x = f(x, \lambda_n) .$$

Thus $(x, \lambda_n) \in \Sigma$ and $\varphi(x, \lambda_n) = 1$. Hence $\lambda_n = n$ and

$$x = f(x, n) .$$

But $(x, n) \in \partial H_n$ and $\varphi(x, n) = 0$. Again a contradiction. Then C intersects ∂H_n. Since this is true for every n it follows that C is unbounded. Q.E.D.

For a different proof of Proposition 4.1 using degree theory see [28].

PROPOSITION 4.2. – *Let $f\colon W \to W$ be α-Lipschitz with constant p. Let $\Sigma = \{(x, \lambda) \in W \times [0, +\infty) = H$: such that $x = \lambda f(x)\}$. Then there is a connected component C of Σ which contains $(0, 0)$ and intersects ∂H_ϱ for every $\varrho < 1/p$.*

PROOF. – Observe that

$$\Sigma \cap H_\varrho$$

is compact. Therefore we can proceed as in the previous proposition. It is enough to observe that

$$h(x, \lambda) = (\varrho\varphi(x, \lambda)f(x), \varphi(x, \lambda))$$

is an α-contraction for every $\varrho < 1/p$. Q.E.D.

Let $f\colon W \to W$ be such that

$$f(tx) = tf(x) . \tag{4.4}$$

Observe that we can consider f as the radial extension of a map $g\colon \partial W_1 \to W$, by setting

$$f(x) = \begin{cases} \|x\| g\left(\dfrac{x}{\|x\|}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 . \end{cases}$$

Suppose $\|f\| = \sup\{\|f(x)\|\colon x = 1\} < +\infty$ and recall that

$$r(f) = \limsup_{n\to+\infty} \|f^n\|^{1/n} . \tag{4.6}$$

LEMMA 4.1. – *Let $f\colon W \to W$ be positively homogeneous. Assume that there exists $\|u\| = 1$ and $\delta \leqslant r(f)$ such that*

$$\limsup_{n\to+\infty} \frac{\|f^n(u)\|}{\delta^n} > 0 . \tag{4.7}$$

Let $\varrho < \delta$ and define $f_\varrho(x) = 1/\varrho f(x)$. Then

$$\|f_\varrho^n(u)\|$$

is unbounded.

PROOF. – We have

$$f\ (u) = \frac{1}{\varrho^n} f^n(u) .$$

Thus

$$\|f_\varrho^n(u)\| = \frac{\|f^n(u)\|}{\varrho^n} = \frac{\|f^n(u)\|}{\delta^n}\left(\frac{\delta}{\varrho}\right)^n .$$

Since $\delta/\varrho > 1$ and $\limsup\limits_{n\to+\infty} \|f^n(u)\|/\delta^n > 0$ we get that

$$\{\|f_\varrho^n(u)\|\}$$

is unbounded. Q.E.D.

LEMMA 4.2 [2]. – *Let $\{a_m\}$ be an unbounded sequence of positive numbers. Then there exists a subsequence $\{a_{m_p}\}$ such that for every $i \in N$ we have*

$$a_{m_{p-i}} \leqslant a_{m_p} \tag{4.8}$$

for all $m_p > i$.

PROOF. – Let $m_1 = 1$. Define m_2 as the smallest integer such that

$$a_{m_2} \geqslant a_j$$

for every $j \leqslant m_2$. Then define $m_3 > m_2$ as the smallest integer such that

$$a_{m_3} \geqslant a_j$$

for every $j \leqslant m_3$ etc. Q.E.D.

Let $f\colon W \to W$ be positively homogeneous. Denote by $\bar{f}$ the restriction of f to $\partial K_1 = K \cap D_1$.

LEMMA 4.3. – *Let $\bar{f}$ be α-Lipschitz with constant p. Then f is α-Lipschitz with the same constant.*

PROOF. – Obviously it is enough to show that f is α-Lipschitz in W_r for every r. If $r > 1$ we can define

$$\bar{f}_r\colon \partial W_r \to W$$

by

$$\bar{f}_r(x) = r\bar{f}\left(\frac{x}{r}\right).$$

Since $\bar{f}_r$ is α-Lipschitz with constant p it is enough to show that

$$f\colon W_1 \to W$$

is α-Lipschitz with the same constant. For a proof of this fact see [21] and [9]. Q.E.D.

Given a positively homogeneous function $f\colon W \to W$ recall that

$$\omega(f) = \limsup_{n \to +\infty} [\alpha(\bar{f}_n)]^{1/n} \leqslant \alpha(f)\,. \tag{4.9}$$

LEMMA 4.4. – *Let $f\colon W \to W$ be positively homogeneous. Assume that $\omega(f) < 1$ and there exists $\|u\| = 1$ such that $\{\|f^n(u)\|\}$ is unbounded. Then the sequence $\{v_n = f^n(u)/\|f^n(u)\|\}$ admits a convergent subsequence.*

PROOF. – Let $\{a_n = \|f^n(u)\|\}$. Then $\{a_n\}$ is an unbounded sequence of positive numbers. Let $\{a_{n_q}\}$ be the subsequence of Lemma 4.2 and let

$$A = \{v_{n_q}\}\,. \tag{4.10}$$

Let m be any positive integer such that $\alpha(f^m) = p_m < 1$. Define

$$A_m = \{v_{n_q}\colon n_q > m\}\,.$$

Obviously

$$\alpha(A) = \alpha(A_m)$$

for every m. Let $B_m = \{f^{n_q - m}(u)/\|f^{n_q}(u)\| : n_q > m\}$. Then $B_m \subset K \cap D_1$ and $f^m(B_m) = A_m$. Thus

$$\alpha(A) = \alpha(A_m) \leqslant p_m \alpha(B_m) \leqslant p_m \alpha(K_1) .$$

Take m^2. Then $\alpha(A) = \alpha(A_{m^2})$ and

$$\alpha(A) \leqslant p_m^2 \alpha(K_1) .$$

In general

$$\alpha(A) \leqslant p_m^q \alpha(K_1) . \tag{4.11}$$

Hence $\alpha(A) = 0$ and A admits a convergent subsequence (see [25] for a similar result). Q.E.D.

THEOREM 4.1. – *Let K be a cone and let $f\colon K \to K$ be positively homogeneous and order preserving. Assume that*

(i) $\alpha(f) < r(f)$;

(ii) *there exist* $\|u\| = 1$ *and* $\delta \in (\alpha(f), r(f)]$ *such that*

$$\limsup_{n \to +\infty} \frac{\|f^n(u)\|}{\delta^n} > 0 .$$

Then there exists $\|x_0\| = 1$ *such that* $f(x_0) = \mu x_0$, *for some* $\mu \in [\delta, r(f)]$.

PROOF. – Let

$$\alpha(f) < \varrho < \delta \tag{4.12}$$

and define

$$f_\varrho(x) = \frac{1}{\varrho} f(x) . \tag{4.13}$$

By Lemma 4.1 $\{\|f_\varrho^n(u)\|\}$ is unbounded. Moreover there exists a subsequence $\{n_i\}$ such that

$$\lim_{i \to +\infty} \frac{\|f^{n_i}(u)\|}{\delta^{n_i}} = \limsup_{n \to +\infty} \frac{\|f^n(u)\|}{\delta^n} > 0 .$$

By Lemma 4.4 we can find a subsequence of

$$\left\{ v_{n_i} = \frac{f^{n_i}(u)}{\|f^{n_i}(u)\|} \right\} \tag{4.14}$$

which converges to some vector $\|v\| = 1$. Let $\varepsilon > 0$ and consider the equation

$$x = \lambda[f_\varrho(x) + \varepsilon f_\varrho(u)]\,. \tag{4.15}$$

Since $\alpha(f_\varrho(A)) = (1/\varrho)\alpha(f(A)) \leqslant (\alpha(f)/\varrho)\,\alpha(A) = p\alpha(A)$, $p < 1$, there is, according to Proposition 4.2, a connected branch of solutions joining $(0, 0) \in K \times [0, +\infty)$ with ∂H_ε for every $1 < q < 1/p$.

From

$$x = \lambda[f_\varrho(x) + \varepsilon f_\varrho(u)]$$

we obtain

$$x > \lambda f_\varrho(x)\,, \qquad x > \lambda\varepsilon f_\varrho(u)\,. \tag{4.16}$$

By induction we find

$$x > \lambda^n \varepsilon f_\varrho^n(u)$$

or

$$\frac{x}{\|f_\varrho^n(u)\|} > \lambda^n \varepsilon \frac{f_\varrho^n(u)}{\|f_\varrho^n(u)\|}\,.$$

Using the convergent subsequence $\{m\}$ we get

$$\frac{\varrho^m}{\lambda^m \|f^m(u)\|}\, x > \varepsilon v_m\,. \tag{4.17}$$

Thus

$$\left[\frac{\varrho}{\lambda\delta}\right]^m \frac{\delta^m}{\|f^m(u)\|}\, x > \varepsilon v_m\,. \tag{4.18}$$

If $(\varrho/\lambda\delta)^m \to 0$ we reach the contradiction

$$\varepsilon v < 0\,.$$

Thus $\varrho/\lambda\delta \geqslant 1$ or $\lambda \leqslant \varrho/\delta < 1$. This implies the existence of $\|x_\varepsilon\| = 1$ such that

$$x_\varepsilon = \lambda_\varepsilon[f_\varrho(x_\varepsilon) + \varepsilon f_\varrho(u)]\,. \tag{4.19}$$

Set $\varepsilon_n = 1/n$ and get $\lambda_n \in (0, 1)$ such that

$$x_n = \lambda_n \left[f_\varrho(x_n) + \frac{1}{n} f_\varrho(u)\right]. \tag{4.20}$$

The sequence $\{x_n\}$ is obviously compact. Hence there exists $\|x_0\| = 1$ and λ_0 such that

$$x_0 = \lambda_0 f_\varrho(x_0)\,. \tag{4.21}$$

This implies

$$f(x_0) = \frac{\varrho}{\lambda_0} x_0 . \tag{4.22}$$

Since

$$\frac{\varrho}{\lambda_n} \geqslant \delta \quad \text{we obtain} \quad \frac{\varrho}{\lambda_0} \geqslant \delta .$$

Moreover

$$\lim_{n\to+\infty} \|f^n(x_0)\|^{1/n} = \frac{\varrho}{\lambda_0} \leqslant \limsup_{n\to+\infty} \|f^n\|^{1/n} = r(f) . \qquad \text{Q.E.D.}$$

COROLLARY 4.1. – *Let $f\colon K \to K$ be positively homogeneous and order preserving. Assume that f is an α-contraction and there exists $\|u\| = 1$ such that*

$$\{\|f^n(u)\|\} \quad \textit{is unbounded} .$$

Let $s = \limsup_{n\to+\infty} \|f^n(u)\|^{1/n}$. Then there exists $\|x_0\| = 1$ such that

$$f(x_0) = \lambda x_0 \tag{4.23}$$

for some $s \leqslant \lambda \leqslant r(f)$.

PROOF. – Since $\|f^n(u)\| \to +\infty$ we have

$$\limsup_{n\to+\infty} \|f^n(u)\|^{1/n} = s \geqslant 1 .$$

Hence $r(f) \geqslant \limsup_{n\to+\infty} \|f^n(u)\|^{1/n} = s \geqslant 1 > \alpha(f)$. Then

$$\limsup_{n\to+\infty} \frac{\|f^n(u)\|}{s^n} = 1$$

and, according to Theorem 4.1 there exists $\|x_0\| = 1$ such that

$$f(x_0) = \lambda x_0$$

for some $\lambda \in [s, r(f)]$. Q.E.D.

COROLLARY 4.2 [25]. – *Let $f\colon K \to K$ be positively homogeneous and order preserving. Assume that f is an α-contraction and there exists $\|u\| = 1$ such that*

$$\{\|f^n(u)\|\}$$

is unbounded. Then there exists $\|x_0\| = 1$ and $\lambda \geqslant 1$ such that

$$f(x_0) = \lambda x_0 .$$

PROOF. – See Corollary 4.1 where it is established that

$$\lambda \geqslant \limsup_{n\to+\infty} \|f^n(u)\|^{1/n} \geqslant 1 .$$

COROLLARY 4.3 [23]. – *Let $f\colon K \to K$ be α-Lipschitz with constant p, order preserving and positively homogeneous. Assume that there exists $\|u\| = 1$ such that*

$$f^n(u) \geqslant cu \tag{4.24}$$

for some $n \in N$ and some $c > p^n$. Then there exists $\|x_0\| = 1$ such that

$$f(x_0) = \lambda x_0 \tag{4.25}$$

for some $\lambda \in [c^{1/n}, r(f)]$.

PROOF. – Let $p^n < \varrho < c$, $d = \varrho^{1/n}$ and define

$$g(x) = \frac{1}{d} f(x) . \tag{4.26}$$

Then

$$\alpha\big(g(A)\big) = \frac{1}{d}\alpha\big(f(A)\big) \leqslant \frac{p}{d}\alpha(A) = q\alpha(A) , \qquad q < 1 .$$

Thus g is an α-contraction. Moreover for $m = kn$ we get

$$g^m(u) = \frac{1}{d^m} f^{kn}(u) \geqslant \left(\frac{c}{\varrho}\right)^k u$$

and

$$\left(\frac{\varrho}{c}\right)^k g^m(u) \geqslant u .$$

Since $(\varrho/c)^k \to 0$ as $k \to +\infty$ we must have that

$$\{\|g^m(u)\|\}$$

is unbounded.

More precisely

$$r(g) \geqslant \limsup_{m\to+\infty} \frac{\|g^m(u)\|}{\delta^m} \geqslant 1 \qquad \text{with} \qquad \delta = \left(\frac{c}{\varrho}\right)^{1/n} .$$

Thus there exists $\|x_0\| = 1$ and $\mu \in [(c/\varrho)^{1/n}, r(g)]$ such that

$$g(x_0) = \mu x_0 .$$

Hence

$$f(x_0) = dg(x_0) = \mu\, dx_0 = \lambda x_0$$

with $c^{1/n} \leqslant \lambda \leqslant r(f)$. Q.E.D.

The following example shows how crucial is the assumption

$$\text{there exists} \quad \|u\| = 1 \quad \text{such that} \quad \limsup_{n\to+\infty} \frac{\|f^n(u)\|}{\varrho^n} > 0$$

$$\text{for some} \quad \alpha(f) < \varrho < r(f)\,.$$

EXAMPLE 4.1. – We are looking for a non-trivial solution of the global « Cauchy » problem depending on a parameter

$$(*) \qquad x'(t) = \lambda \sqrt{x^2(t) + x^2(1-t)}\,, \qquad x(0) = 0$$

with $t \in [0, 1]$.

Changing the problem into an integral equation we study the existence of an eigenvalue and an eigenvector of the operator

$$T(x)(t) = \int\limits_0^t \sqrt{x^2(s) + x^2(1-s)}\, ds\,.$$

Observe that T is compact, positively homogeneous and sends the cone, K, of non-negative functions into itself. Moreover if

$$x \leqslant y \quad \text{then} \quad T(x) \leqslant T(y)\,.$$

Hence T is order preserving.

Some calculations show that

$$r = r(T) > 0 \quad \text{and} \quad \inf\{\|T(x)\|: \|x\| = 1\} = 0\,.$$

Moreover

$$\lim_{n\to+\infty} \frac{\|T^n(1)\|}{\varrho^n} > 0$$

with $\varrho = 1/\sqrt{2}$. Hence there exists $\mu \in [1/\sqrt{2}, r]$ and $\|x_0\| = 1$ such that

$$T(x_0) = \mu x_0\,.$$

A direct attempt at solving $(*)$ results in the discovery that no non-trivial solutions are possible for $\lambda \neq \sqrt{2}\ln(1+\sqrt{2})$ and that for $\lambda = \sqrt{2}\ln(1+\sqrt{2})$ there is one and only one solution defined by

$$x(t) = a\big(\sin h[2\ln(1+\sqrt{2})\,t + \ln(-1+\sqrt{2})] + 1\big)$$

where $a = x(\frac{1}{2})$.

Since $sx(t)$ is a solution of $(*)$ if and only if $x(t)$ is, we may assume $a = \frac{1}{2}$, in which case

$$\|x\| = 1\,.$$

Notice that

$$\sqrt{2}\ln(1+\sqrt{2}) < \sqrt{2}\,.$$

Hence the eigenvalue $\lambda = 1/\sqrt{2}\ln(1+\sqrt{2})$ belongs to the interval $(1/\sqrt{2}, r]$.

We are now interested in deriving a result similar to Theorem 4.1 for the case when the map f is a linear operator L. The two conditions

(i) $\alpha(L) < r_K(L)$;

(ii) there exists $\|u\| = 1$ such that $\limsup\limits_{n\to+\infty} \dfrac{\|L^n(u)\|}{\delta^n} > 0$

can be weakened somehow. The following two lemmas illustrate this fact and seem to show that the greater generality of the new conditions is essentially based on the property

$$L(x+y) = L(x) + L(y)\,.$$

LEMMA 4.5 [24]. – *Let $\alpha_K(L) < r_K(L)$ and let $\alpha_K(L) < \varrho < r_K(L)$. Then there exists a generalized measure of non compactness β such that*

$$\beta(L_\varrho(A)) \leqslant c\beta(A)$$

for every $A \subset K$, and with $c < 1$.

PROOF. – There exists m such that

$$[\alpha(L^m)]^{1/m} < \varrho$$

or

$$\alpha(L_\varrho^m) = \frac{\alpha(L^m)}{\varrho^m} < 1\,.$$

Define

$$\beta(A) = \sum_{i=0}^{m-1} \alpha(L_\varrho^i(A))\,.$$

It is easy to check that β is a measure of non-compactness and

$$\beta(L_\varrho(A)) \leqslant c\beta(A)$$

with $c < 1$. Q.E.D.

LEMMA 4.6. – *Let $r_K(L) > 0$. Then for every $\varrho < r_K(L)$ there exists $\|u\| = 1$ such that*

$$\{\|L_\varrho^n(u)\|\}$$

is unbounded.

PROOF. – If we assume that for every $\|x\| = 1$ there exists a constant M_x such that

$$\|L_\varrho^n(x)\| \leqslant M_x$$

then for every y we have

$$\|L_\varrho^n(y)\| = \|y\| \left\| L_\varrho^n\left(\frac{y}{\|y\|}\right) \right\| \leqslant \|y\| M_{y/\|y\|} .$$

By the Uniform Boundedness Principle there exists an open set $V \subset K$ and a constant M such that

$$\|L_\varrho^n(x)\| \leqslant M$$

for every $x \in V$ and every n.

Let $x_0 \in V$ and consider the open neighborhood of the origin in K

$$\Omega = \{w \in K \colon w = x_0 - x \text{ for some } x \in V\} .$$

We then have $r > 0$ such that $\|w\| = r$, $w \in K$ implies $w \in \Omega$. Thus

$$\|L_\varrho^n(w)\| \leqslant \|L_\varrho^n(x_0)\| + \|L_\varrho^n(x)\| \leqslant 2M .$$

Therefore

$$\|L_\varrho^n\|_K \leqslant \frac{2M}{r} .$$

But $\|L_\varrho^n\|_K = 1/\varrho^n \|L^n\|_K$ and $\|L_\varrho^n\|_K^{1/n} = 1/\varrho \|L^n\|_K^{1/n}$. Since

$$\lim_{n \to +\infty} \|L_K^{1/n}\|^n = r_K(L)$$

we see that

$$\lim_{n\to+\infty} \|L_\varrho^n\|^{1/n} > 1 .$$

This implies that $\{\|L_\varrho^n\|\}$ is unbounded. Therefore there exists $\|u\| = 1$ such that

$$\{\|L_\varrho^n(u)\|\}$$

is unbounded. Q.E.D.

THEOREM 4.2 [24]. – *Let $L\colon K \to K$ be such that*

(i) $\omega_K(L) < r_K(L)$.

Then there exists $\|x\| = 1$ such that

$$Lx = r_K(L)x . \tag{4.27}$$

PROOF. – Select $\varrho_n = r_K(L) - 1/n$. Then by Lemma 4.5 and 4.6 $L_n = L_{\varrho_n}$ satisfies the assumptions of Corollary 4.1. Thus there exists $\|x_n\| = 1$ such that

$$L_n(x_n) = s_n x_n \tag{4.28}$$

where $s_n \geqslant 1$. Hence

$$L(x_n) = \varrho_n s_n x_n$$

with

$$\varrho_n \leqslant \varrho_n s_n \leqslant r_K(L) .$$

Now consider the set

$$A = \{x_n\colon Lx_n = \varrho_n s_n x_n\} .$$

For every $x_n \in A$ we have

$$L(Lx_n) = (\varrho_n s_n)^2 x_n$$

and, after m steps, we obtain

$$x_n = \frac{1}{(\varrho_n s_n)^m} L^m(x_n) .$$

Choose m so that

$$[\alpha(L^m)]^{1/m} < r_K(L) .$$

Then for $n \geq n_0$ we have

$$[\alpha(L^m)]^{1/m} < \varrho_n$$

or

$$\alpha(L^m) < \varrho_n^m .$$

Define $A_{n_0} = \{x_n \in A : n \geq n_0\}$. Obviously

$$\alpha(A) = \alpha(A_{n_0}) \leq \frac{\alpha(L^m)}{\varrho_n^m} \alpha(A_{n_0})$$

and $\bar{A}_{n_0}$ is compact. Thus there exists $\|x_0\| = 1$ such that

$$Lx_0 = r_K(L)x_0 . \quad \text{Q.E.D.}$$

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