POSITIVE FUNCTIONS ON C^*-ALGEBRAS

W. FORREST STINESPRING

1. **Introduction.** Let X be any set, let $\mathcal{S}$ be a Boolean σ-algebra of subsets of X, and let F be a function from $\mathcal{S}$ to non-negative operators on a Hilbert space $\mathcal{H}$ such that $F(X)=1$ and F is countably-additive in the weak operator topology. Neumark [2] has shown that there exists a Hilbert space $\mathcal{K}$ of which $\mathcal{H}$ is a subspace and a spectral measure E defined on $\mathcal{S}$ such that $F(S)P=PE(S)P$ for all S in $\mathcal{S}$, where P is projection of $\mathcal{K}$ on $\mathcal{H}$. Let us rephrase this situation so that we speak of algebras rather than Boolean algebras and linear functions rather than measures. Thus, we consider, instead of the Boolean σ-algebra $\mathcal{S}$, the C^*-algebra $\mathcal{A}$ of all bounded functions on X which are measurable with respect to $\mathcal{S}$. A C^*-algebra is defined as a complex Banach algebra with an involution $x \rightarrow x^*$ such that $\|xx^*\| = \|x\|^2$ for all x in the algebra. The measure F is supplanted by the linear function μ on $\mathcal{A}$

$$\mu(f) = \int f(\gamma)dF(\gamma), \qquad f \in \mathcal{A},$$

where the integral is to be taken in the weak sense. The theorem now asserts that $\mu(f)P = P\rho(f)P$, where

$$\rho(f) = \int f(\gamma)dE(\gamma), \qquad f \in \mathcal{A}.$$

In the original formulation, E was an improvement over F because E was a *spectral* measure; in the reformulation, ρ is an improvement over μ since ρ is a $*$-homomorphism. When the situation is phrased in this manner, the question naturally occurs: "Is it essential that the algebra $\mathcal{A}$ be commutative?" The present paper is devoted to a discussion of this point.

2. **The main theorem.** If $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras and μ is a linear function from $\mathcal{A}$ to $\mathcal{B}$, we shall say that μ is *positive* if $\mu(A) \geqq 0$ whenever $A \in \mathcal{A}$ and $A \geqq 0$. The algebra of $n \times n$ matrices with entries in $\mathcal{A}$ is also a C^*-algebra, which we shall denote by $\mathcal{A}^{(n)}$. By applying μ to each entry of an element of $\mathcal{A}^{(n)}$, we obtain an element of $\mathcal{B}^{(n)}$; this linear function from $\mathcal{A}^{(n)}$ to $\mathcal{B}^{(n)}$ will be denoted by $\mu^{(n)}$. We shall say that μ is *completely positive* if $\mu^{(n)}$ is positive for each positive integer n.

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THEOREM 1. *Let $\mathcal{A}$ be a C^*-algebra with a unit, let $\mathcal{H}$ be a Hilbert space, and let μ be a linear function from $\mathcal{A}$ to operators on $\mathcal{H}$. Then a necessary and sufficient condition that μ have the form*

$$\mu(A) = V^*\rho(A)V \qquad \text{for all } A \in \mathcal{A},$$

*where V is a bounded linear transformation from $\mathcal{H}$ to a Hilbert space $\mathcal{K}$ and ρ is a *-representation of $\mathcal{A}$ into operators on $\mathcal{K}$, is that μ be completely positive.*

PROOF OF NECESSITY. Suppose that $\mu(A) = V^*\rho(A)V$. Let $M = (A_{ij})$ be a non-negative matrix in $\mathcal{A}^{(n)}$. Then $\mu^{(n)}(M)$ is an operator on the direct sum of $\mathcal{H}$ with itself n times. What we have to check is that

$$\sum_{i,j} (\mu(A_{ij})x_j,\ x_i) \geqq 0$$

whenever $x_1, \cdots, x_n$ are vectors in $\mathcal{H}$. Since ρ is a *-representation, the matrix $(\rho(A_{ij}))$ is a non-negative operator on $\mathcal{K}\oplus \cdots \oplus\mathcal{K}$; and therefore,

$$\sum_{i,j} (\mu(A_{ij})x_j,\ x_i) = \sum_{i,j} (\rho(A_{ij})Vx_j,\ Vx_i) \geqq 0.$$

PROOF OF SUFFICIENCY. Suppose that μ is completely positive. Consider the vector space $\mathcal{A}\otimes\mathcal{H}$, the *algebraic* tensor product of $\mathcal{A}$ and $\mathcal{H}$. For $\xi = \sum_i A_i\otimes x_i$ and $\eta = \sum_j B_j\otimes y_j$ in $\mathcal{A}\otimes\mathcal{H}$ we define

$$(\xi,\ \eta) = \sum_{i,j} (\mu(B_j^*A_i)x_i,\ y_j).$$

Since μ was assumed to be completely positive, it follows that

$$(\xi,\ \xi) = \sum_{i,j} (\mu(A_j^*A_i)x_i,\ x_j) \geqq 0.$$

Hence $(\cdot\ ,\ \cdot)$ is a positive Hermitian bilinear form. There is a natural mapping ρ' from $\mathcal{A}$ to linear transformations on $\mathcal{A}\otimes\mathcal{H}$ given by

$$\rho'(A) \sum_i B_i \otimes y_i = \sum_i (AB_i) \otimes y_i.$$

We shall show that for all A in $\mathcal{A}$ and ξ in $\mathcal{A}\otimes\mathcal{H}$

$$(\rho'(A)\xi,\ \rho'(A)\xi) \leqq \|A\|^2(\xi,\ \xi). \tag{1}$$

If (1) were not universally true, we could find A in $\mathcal{A}$ and

$$\xi = \sum_i B_i \otimes x_i$$

in $\mathcal{A}\otimes\mathcal{H}$ such that $(\xi,\ \xi) \leqq 1$ and $\|A\| < 1$, but $(\rho'(A)\xi,\ \rho'(A)\xi) > 1$.

Then $(\rho'(A^*A)\xi, \xi) > 1$, which implies, by the Schwarz inequality, that $(\rho'(A^*A)\xi, \rho'(A^*A)\xi) > 1$. By continuing in this manner, we find that

$$(\rho'([A^*A]^{2^k})\xi, \xi) > 1 \qquad \text{for } k = 1, 2, \cdots .$$

Since μ is positive and $-\|C\|1 \leqq C \leqq \|C\|1$ whenever C is a self-adjoint element of $\mathcal{A}$, it follows that $-\|C\|\mu(1) \leqq \mu(C) \leqq \|C\|\mu(1)$ and hence that $\|\mu(C)\| \leqq \|C\|\|\mu(1)\|$; this inequality shows that μ is uniformly continuous on the self-adjoint elements of $\mathcal{A}$, and it is easy to see from this that μ must be uniformly continuous on all of $\mathcal{A}$. The uniform continuity of μ together with the fact that $\|A\| < 1$ shows that

$$(\rho'([A^*A]^{2^k})\xi, \xi) = \sum_{i,j} (\mu(B_j^*[A^*A]^{2^k}B_i)x_i, x_j)$$

converges to 0. This contradiction proves (1).

Let $\mathcal{N}$ be the set of all ξ in $\mathcal{A} \otimes \mathcal{H}$ such that $(\xi, \xi) = 0$. By the Schwarz inequality, $\mathcal{N}$ is a linear manifold and by (1), $\mathcal{N}$ is invariant under $\rho'(\mathcal{A})$. Therefore, the quotient space $\mathcal{A} \otimes \mathcal{H}/\mathcal{N}$ is a pre-Hilbert space, and each A in $\mathcal{A}$ naturally induces a bounded operator on the completion $\mathcal{K}$ of $\mathcal{A} \otimes \mathcal{H}/\mathcal{N}$. Let

$$Vx = 1 \otimes x + \mathcal{N} \qquad \text{for all } x \in \mathcal{H}.$$

Then $\|Vx\|^2 \leqq (\mu(1)x, x)$, so that V is a bounded linear transformation from $\mathcal{H}$ to $\mathcal{K}$. Now

$$\begin{aligned}(V^*\rho(A)Vx, y) &= (\rho(A)Vx, Vy) = (\rho'(A)1 \otimes x, 1 \otimes y) \\ &= (A \otimes x, 1 \otimes y) = (\mu(A)x, y)\end{aligned}$$

for all x and y in $\mathcal{H}$, and therefore $\mu(A) = V^*\rho(A)V$.

3. **Remarks.** The operator $\rho(1)$ is a projection. We can take $\rho(1)$ to be 1, for we can replace V by $\rho(1)V$ and then replace the space $\mathcal{K}$ by the subspace $\rho(1)\mathcal{K}$. Assuming that this has been done, we have $\mu(1) = V^*V$, so that if $\mu(1) = 1$, then V is an isometry. Since an isometry can be considered as an embedding, the Neumark theorem follows from Theorem 1, provided we can show that when $\mathcal{A}$ is commutative, positivity of μ implies complete positivity. This fact will be proved in the next section.

It might be thought possibly that positivity always implies complete positivity. We give a counter-example to show that this is not the case. Let $\mathcal{A}$ be the algebra of $n \times n$ matrices with complex entries. We denote by e_{ij} the matrix with 1 in the ith row and jth column and with zeros elsewhere. The e_{ij}'s are a basis for the vector space $\mathcal{A}$. We define a linear function μ from $\mathcal{A}$ to $\mathcal{A}$ by specifying the values of μ on this basis: $\mu(e_{ij})$ is to be a matrix whose (r, s)th entry is $(r-j)(s-i)$. Now μ is positive; for let $A=(a_{ij})\geqq 0$ be an $n\times n$ matrix and let

$$x = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

be any vector. Then

$$\begin{aligned}(\mu(A)x, x) &= \sum_{i,j,r,s} a_{ij}(r-j)(s-i)x_r\bar{x}_s \\ &= \sum_{i,j} a_{ij}\left(\sum_r (r-j)x_r\right)\left(\sum_s (s-i)\bar{x}_s\right) \geqq 0\end{aligned}$$

since $A \geqq 0$.

On the other hand, μ is not completely positive, for we shall show that $\mu^{(n)}$ is not positive, n being as above. Let E be the element of $\mathcal{A}^{(n)}$ whose (i, j)th entry is the matrix e_{ij}. It is easily seen that $E\geqq 0$. But $\mu^{(n)}(E)$ is not $\geqq 0$, for

$$\sum_{i,j,r,s} (r-j)(s-i)\delta_{ir}\delta_{js} = \sum_{i,j} (i-j)(j-i) < 0.$$

4. Conditions for complete positivity.

THEOREM 2. *If $\mathcal{A}$ is a W^*-algebra of finite type (see [1]), then the center-valued trace t is completely positive.*

PROOF. Suppose $M=(A_{ij})$ is a non-negative matrix in $\mathcal{A}^{(n)}$ and suppose $\epsilon>0$. It is shown in [1] that there exist a finite number $U_1, \cdots, U_r$ of unitary operators in $\mathcal{A}$ and non-negative numbers $\alpha_1, \cdots, \alpha_r$ with $\sum_k \alpha_k=1$ such that

$$\left\| t(A_{ij}) - \sum_k \alpha_k U_k^* A_{ij} U_k \right\| < \epsilon \qquad \text{for } i, j = 1, \cdots, n.$$

But then if $x_1, \cdots, x_n$ are any vectors,

$$\left| \sum_{i,j} (t(A_{ij})x_j, x_i) - \sum_{i,j,k} \alpha_k(U_k^* A_{ij} U_k x_j, x_i) \right| \leqq \epsilon \sum_{i,j} \left| (x_j, x_i) \right|$$

and it is clear that

$$\sum_{i,j,k} \alpha_k(U_k^* A_{ij} U_k x_j, x_i) = \sum_k \alpha_k \sum_{i,j} (A_{ij}U_k x_j, U_k x_i) \geqq 0$$

since $M\geqq 0$. It follows that $t^{(n)}(M)\geqq 0$.

THEOREM 3. *If $\mathcal{A}$ is a C^*-algebra and μ is a positive linear function with complex values, then μ is completely positive.*

PROOF. Suppose $M=(A_{ij})$ is a non-negative matrix in $\mathcal{A}^{(n)}$, and $\{\lambda_i\}$ are complex numbers. We wish to check that

$$\sum_{i,j} \mu(A_{ij})\lambda_j\bar{\lambda}_i \geqq 0.$$

But $\sum_{i,j} \mu(A_{ij})\lambda_j\bar{\lambda}_i = \mu(\sum_{i,j} A_{ij}\lambda_j\bar{\lambda}_i)$. Writing $M=N^*N$ with $N=(B_{ij})$, we see that

$$\sum_{i,j} A_{ij}\lambda_j\bar{\lambda}_i = \sum_{i,j} \lambda_j\bar{\lambda}_i \sum_k B_{ki}^* B_{kj}$$
$$= \sum_k \Big(\sum_i \lambda_i B_{ki}\Big)^* \Big(\sum_j \lambda_j B_{kj}\Big) \geqq 0$$

and so $\mu(\sum_{i,j} A_{ij}\lambda_j\bar{\lambda}_i) \geqq 0$ since μ is positive.

Theorem 3 together with Theorem 1 gives the known fact that a state of a C^*-algebra induces a representation [3].

THEOREM 4. *If $\mathcal{A}$ is a commutative C^*-algebra and μ is a positive operator-valued linear function on $\mathcal{A}$, then μ is completely positive.*

PROOF. We may take $\mathcal{A}$ as the algebra of all continuous complex-valued functions vanishing at ∞ on a locally compact Hausdorff space Γ. Let $M=(f_{ij})$ be a non-negative matrix in $\mathcal{A}^{(n)}$. If $x_1, \cdots, x_n$ are vectors in the Hilbert space, we wish to verify that

$$\sum_{i,j} (\mu(f_{ij})x_j, x_i) \geqq 0.$$

By the Riesz-Markoff theorem, there exists a regular measure m on Γ such that $\sum_i (\mu(f)x_i, x_i) = \int_\Gamma f dm$ for all $f \in \mathcal{A}$. Then by the Riesz-Markoff and Radon-Nikodym theorems, there exist measurable functions h_{ij} such that

$$(\mu(f)x_j, x_i) = \int_\Gamma f h_{ij} dm \qquad \text{for all } f \in \mathcal{A}.$$

Now the matrix $(h_{ij}(\gamma))$ is non-negative almost everywhere; for

$$\int_\Gamma f \sum_{i,j} h_{ij}\lambda_i\bar{\lambda}_j dm \geqq 0 \qquad \text{for all } f \geqq 0 \text{ in } \mathcal{A},$$

and hence $\sum_{i,j} h_{ij}(\gamma)\lambda_i\bar{\lambda}_j \geqq 0$ for all r-tuples $\lambda_1, \cdots, \lambda_r$ of complex numbers with rational real and imaginary parts and for all $\gamma \in \Gamma$ with the exception of $\gamma \in N$ where $m(N)=0$. Also the matrix $(f_{ij}(\gamma))$ is non-negative for all $\gamma \in \Gamma$ by Theorem 3. Therefore,

$$\sum_{i,j} f_{ij}(\gamma) h_{ij}(\gamma) \geqq 0 \qquad \text{almost everywhere}$$

and so

$$\sum_{i,j} (\mu(f_{ij})x_j, x_i) = \int_\Gamma \sum_{i,j} f_{ij}h_{ij}dm \geqq 0.$$

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University of Chicago