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Positive generalized white noise functionals

Yoshitaka YOKOI

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§1. Introduction

Let μ be the measure of Gaussian white noise (with variance 1) defined on the space E^* of all real tempered distributions. That is, the characteristic function $C(\xi)$ of μ is given by

$$C(\xi) \equiv \int_{E^*} e^{i\langle x, \xi\rangle} d\mu(x) = \exp\left[-\frac{1}{2}\int_{R} |\xi(u)|^2 du\right]$$

where ξ is an element of the real Schwartz space $E = \{$all real valued, C^∞- and rapidly decreasing functions on $\boldsymbol{R}\}$ ([3] and [21]). The elements of $(L^2) \equiv \{\varphi; \int_{E^*} |\varphi(x)|^2 d\mu(x) < \infty\}$ are called *Brownian functionals* ([7]). In [6] we can see the idea of generalized Brownian functionals constructed on the theories of Sobolev spaces and Gel'fand triplets. Many authors have greatly developed the analysis of not only Brownian functionals but also generalized Brownian ones, e.g., [6] ~ [8], [13] ~ [17], [19], [20], [22] and [25]. Some of them treated the problem of positive generalized Brownian functionals and pointed out that it would be important in the light of quantum field theory or relating to the Feynman path integral.

In this paper we also consider positive generalized functionals in the white noise calculus, which is developed in a somewhat more general situation. That is, we prepare a real separable Hilbert space E_0 and a self-adjoint operator D such that $D \geq 1$ and that D^{-h_0} is of Hilbert-Schmidt type for some $h_0 \geq 1$, construct a Gel'fand triplet $E \hookrightarrow E_0 \hookrightarrow E^*$ by equipping E_0 with increasing and compatible norms $\{\|D^p \cdot\|_{E_0}\}_{p=0}^{\infty}$, and observe a triplet $(\mathscr{S}) \hookrightarrow L^2(E^*, \mu) \hookrightarrow (\mathscr{S}')$ with the characteristic functional of μ:

$$C(\xi) \equiv \int_{E^*} e^{i\langle x, \xi\rangle} d\mu(x) = \exp\left[-\frac{1}{2}\|\xi\|_{E_0}^2\right], \qquad \xi \in E.$$

We show that every element of the space $(\mathscr{S})$ of test functionals has a unique continuous version on E^* and the product of every two elements of $(\mathscr{S})$ again belongs to $(\mathscr{S})$ (cf. [15] and [18]). Using these results we prove that any *positive* continuous linear functional on $(\mathscr{S})$ is the linear form given by the integration of the continuous version of the test function against a positive finite measure: *Let* $\Psi \in (\mathcal{S}')$ *be positive. Then, there exists a positive finite measure* ν_Ψ *on* $(E^*, \mathcal{B})$ *such that*

$$\langle \Psi, \varphi \rangle = \int_{E^*} \tilde{\varphi}(x) d\nu_\Psi(x) \qquad \textit{for each } \varphi \in (\mathcal{S})$$

where $\tilde{\varphi}$ *is the continuous version of* φ. This result is an improvement of Theorem 3.2 and its Remark in [22].

Moreover, since $(\mathcal{S})$ is very small, the space $(\mathcal{S}')$ of generalized functionals can include important functionals. In case $E_0 = L^2_{\mathrm{real}}(\boldsymbol{R})$ and $D = 1 + u^2 - (d/du)^2$, $(\mathcal{S}')$ includes the following generalized functionals, a kind of renormalizations: $:\dot{B}(t)^n:$, $:\exp[\lambda \dot{B}(t)]:$ and $:\exp[c \int_{\boldsymbol{R}} \dot{B}(u)^2 du]:$ for $c < 1/2$, λ and $t \in \boldsymbol{R}$. The latter two are positive.

We will give the definition of *positive* generalized functionals and several propositions which are equivalent to it.

We should mention Sugita's work [26] which was already done just before ours within the framework of Malliavin calculus. In the white noise analysis, we also have obtained the very similar results at least to outward seeming. However, we would like to note that we could not guarantee the existence of continuous versions for test functionals and that we could not grasp functionals like $:\exp[\lambda \dot{B}(t)]:$ for λ, $t \in \boldsymbol{R}$ or $:\exp[c \int_{\boldsymbol{R}} \dot{B}(u)^2 du]:$ for $c < 1/2$, if we rigged the space (L^2) (cf. [2]) by a compatible sequence of norms $\{\|(I - L)^{p/2} \cdot \|_{(L^2)}\}_{p=0}^\infty$, where L is the Ornstein-Uhlenbeck operator in the sense of [26]. In contrast with it, we make the rigged Hilbert space $(\mathcal{S}) \hookrightarrow (L^2) \hookrightarrow (\mathcal{S}')$ by operating $\sum_{n=0}^\infty \oplus D^p \otimes D^p \otimes \cdots \otimes D^p$ (= the sum of n-times tensor product of D^p's) in the space of the *Fock representation* of (L^2), where D is the ordinary differential operator $1 + u^2 - (d/du)^2$. We note further that, for n-ple Wiener integral $I_n(f_n)$ of smooth f_n, $(I - L)$ multiplies f_n by $(1 + n)$ but does not differentiate it.

Our main theorem is applied to the argument of Dirichlet forms in infinite dimension [9]. We hope that the results of this paper will find more applications in quantum field theory, in particular, with relation to the Feynman path integral.

The contents of this paper are organized as follows:

§2. The real rigged Hilbert space equipped with a sequence of norms induced by a self-adjoint operator.

§3. Summary of Gaussian measure and white noise functionals on E^*.

§4. The spaces of test functionals and generalized functionals.

§5. Positive generalized white noise functionals and main theorem.

§6. Concluding remarks and examples.

§2. The real rigged Hilbert space equipped with a sequence of norms induced by a self-adjoint operator

Let E_0 be a real separable Hilbert space with inner product $(\xi, \eta)_0$ and D a densely defined self-adjoint operator on E_0 such that

A) $D \geq 1$ and

B) D^{-h_0} is of Hilbert-Schmidt type for some integer $h_0 \geq 1$. Let E_p be the domain of D^p and $\|\xi\|_p = \|D^p\xi\|_0$ for $\xi \in E_p$ $(p = 0, 1, 2, \cdots)$. We note that $D^p = \int_1^\infty \lambda^p dF(\lambda)$ $(p \in \boldsymbol{Z})$, where $F(\cdot)$ is the spectral measure of D. We will use what these spectral decompositions afford us without saying explicitly. Since D is a closed operator, the norms $\|\cdot\|_p$ are compatible. We should also notice that the range of D is the whole space E_0 and that D^{-1} is a bounded operator of E_0 into E_0 with $\|D^{-1}\| \leq 1$. We readily obtain (Ref. Chap. I, [1]):

PROPOSITION 2.1. (a) *Let* $0 \leq p < q$ *and* $\rho = \|D^{-1}\|$. *Then,* $E_q \subset E_p$ *and* $\|\xi\|_p \leq \rho^{q-p}\|\xi\|_q$ *for* $\xi \in E_q$.

(b) E_p *is a Hilbert space with inner product* $(D^p\xi, D^p\eta)_0$ *for every* $p = 0, 1, 2, \dots$.

(c) *The inclusion mapping* $\iota_{p+h_0,p}$: $E_{p+h_0} \to E_p$, *is of Hilbert-Schmidt type with* $\delta \equiv \|\iota_{p+h_0,p}\|_{\text{H-S}} = \|D^{-h_0}\|_{\text{H-S}}$.

(d) $E \equiv \bigcap_{p=0}^\infty E_p$ *is a nuclear space.*

Now let us consider the dual space of E. Let E_{-p} be the completion of E_0 by the norm $\|x\|_{-p} = \|D^{-p}x\|_0$ for every $p = 0, 1, 2, \dots$. Then we easily see (Ref. 4.2, Chap. I, [2]):

PROPOSITION 2.2. (a) E_{-p} *is identified with the dual space of* E_p *for every* $p = 0, 1, 2, \dots$.

(b) *If* $0 \leq p < q$, *then* E_{-p} *is naturally identified with a linear subspace of* E_{-q}.

(c) *If* $0 \leq p < q$, *then* $\|x\|_{-q} \leq \rho^{q-p}\|x\|_{-p}$ *for every* $x \in E_{-p}$.

(d) *The inclusion mapping* $\iota_{-p,-p-h_0}$: $E_{-p} \to E_{-p-h_0}$ *is of Hilbert-Schmidt type with* $\delta \equiv \|\iota_{-p,-p-h_0}\|_{\text{H-S}} = \|D^{-h_0}\|_{\text{H-S}}$.

(e) $E^* \equiv \bigcup_{p=0}^\infty E_{-p}$ *is the dual space of the nuclear space* E.

There are various topologies for the space E^*, e.g., the weak, the strong and the inductive limit ones. Let us choose the inductive limit topology for E^*. By the inductive limit topology we mean the one which is locally convex; i.e., every fundamental neighbourhood of the origin in E^* under this topology has the form:

$$(2.1)\qquad \{\textstyle\sum_{p=0}^{\infty} \lambda_p y_p;\ \lambda_p = 0 \text{ except for a finite number of } p\text{'s},$$
$$\textstyle\sum_{p=0}^{\infty} |\lambda_p| \le 1 \quad \text{and} \quad \|y_p\|_{-p} < \varepsilon_p \quad \text{with} \quad y_p \in E_{-p}\}$$

where $\{\varepsilon_p\}_{p=0}^{\infty}$ is an arbitrarily given sequence of positive numbers. We call the set of the form (2.1) the convex envelope of the set

$$\bigcup_{p\ge 0}^{\infty} \{y \in E_{-p};\ \|y\|_{-p} < \varepsilon_p\}\,.$$

REMARK. *In the above case, the inductive limit topology and the strong topology coincide.*

The triplet $E \hookrightarrow E_0 \hookrightarrow E^*$ is called a rigged Hilbert space or a Gel'fand triplet.

Next, let us introduce some notations relating to the symmetric n-fold tensor product of a Hilbert space. Let X be a Hilbert space over $\boldsymbol{K}$ $(=\boldsymbol{R}$ or $\boldsymbol{C})$. We denote the symmetric n-fold tensor product of X by $X^{\hat{\otimes} n}$. If we treat the n-fold tensor product of a vector $x \in X$ in $X^{\hat{\otimes} n}$ we denote it by $x^{\hat{\otimes} n}$. Besides, $\hat{\bigotimes}_j x_j^{\hat{\otimes} n_j}$ with $\sum_j n_j = n$ is the symmetric n-fold tensor product of n vectors in all, consisting of n_j x_j's for each j. In particular, $\hat{\bigotimes}_{j=1}^n x_j$ or $x_1 \hat{\otimes} x_2 \hat{\otimes} \cdots \hat{\otimes} x_n$ is the symmetrization of $x_1 \otimes x_2 \otimes \cdots \otimes x_n$. In the case of $n = 0$, $X^{\hat{\otimes} n}$ denotes $\boldsymbol{K}$. Thus, if we denote the complexifications of E, E_p $(p \in \boldsymbol{Z})$ and E^* by H, H_p $(p \in \boldsymbol{Z})$ and H^* respectively, we can easily construct the rigged Hilbert space $H^{\hat{\otimes} n} \hookrightarrow H_0^{\hat{\otimes} n} \hookrightarrow H^{*\hat{\otimes} n}$ for every n. We will use the same symbols D and $\langle \cdot, \cdot \rangle$ for the complexifications of the operator D and the canonical bilinear form $\langle \cdot, \cdot \rangle$ between E^* and E.

§3. Summary of Gaussian measure and white noise functionals on E^*

We assume the following condition from now on:

$$(3.1)\qquad 0 < \rho \equiv \|D^{-1}\| < 1\,.$$

This condition is satisfied if, e.g., $1 < a \le D$ for some a. The ordinary differential operator $1 + u^2 - (d/du)^2$ is an important and typical example that satisfies this condition for $a = 2$. We will discuss the generalized functionals of the velocity process of a Brownian particle later on (in §6), using this operator.

The canonical bilinear form for $(x, \xi) \in E^* \times E$ is denoted by $\langle x, \xi \rangle$. Bilinear forms $\langle \cdot, \cdot \rangle$ define cylinder sets. For any $n \ge 1$, any $\xi_1 \in E$, $\cdots$, any $\xi_n \in E$ and any n-dimensional Borel set B_n, the set of the form $\{x \in E^*;\ (\langle x, \xi_1 \rangle, \cdots, \langle x, \xi_n \rangle) \in B_n\}$ is called a cylinder set. We denote the smallest σ-algebra containing all cylinder sets by $\mathscr{B}$.

By Bochner-Minlos' theorem ([21]), a probability measure μ on the measurable space $(E^*, \mathscr{B})$ is uniquely determined by the equation:

$$C(\xi) \equiv \exp\left[-\frac{1}{2}\|\xi\|_0^2\right] = \int_{E^*} \exp[i\langle x, \xi\rangle] d\mu(x) \qquad \text{for } \xi \in E. \tag{3.2}$$

Actually μ is already supported on E_{-h_0} by Gross-Sazonov's theorem ([3]) since D^{-h_0} is of Hilbert-Schmidt type.

We will denote the space $L^2(E^*, \mathscr{B}, \mu)$ briefly by (L^2) according to the notation in [7]. We call the elements of (L^2) *Gaussian white noise functionals* or simply *white noise functionals.* In case $E_0 = L^2_{real}(\boldsymbol{R})$, white noise functionals are called *Brownian functionals*, [7]. Note that (L^2) is a complex Hilbert space.

We summarize here the *Wiener-Itô decomposition* of (L^2):

$$(L^2) = \textstyle\sum_{n=0}^{\infty} \oplus \mathscr{H}_n \tag{3.3}$$

and the $\mathscr{T}$*-transform*:

$$(\mathscr{T}\varphi)(\xi) \equiv \int_{E^*} \varphi(x) \exp[i\langle x, \xi\rangle] d\mu(x) \qquad \text{for } \varphi \in (L^2) \text{ and } \xi \in E. \tag{3.4}$$

Let $H_k(u)$ be Hermite polynomials:

$$H_k(u) = (-1)^k \exp[u^2]\left(\frac{d}{du}\right)^k \exp[-u^2], \qquad k = 0, 1, 2, \cdots \tag{3.5}$$

and $\{\zeta_j\}_{j=0}^{\infty}$ be a complete orthonormal system of E_0. $\{\zeta_j\}_{j=0}^{\infty}$ can live in E since E is dense in E_0. Then, the polynomials of $\langle x, \zeta_j\rangle$'s of degree n,

$$\textstyle\prod_j H_{n_j}\left(\frac{\langle x, \zeta_j\rangle}{\sqrt{2}}\right) \qquad \text{with} \qquad \sum_j n_j = n, \tag{3.6}$$

are called Fourier-Hermite polynomials. Let $\mathscr{H}_n$ denote the subspace of (L^2) spanned by all Fourier-Hermite polynomials of degree n. We have

PROPOSITION 3.1. *Let $\sum_j n_j = n$, then*

(a) $\left\|\prod_j H_{n_j}\left(\frac{\langle x, \zeta_j\rangle}{\sqrt{2}}\right)\right\|_{(L^2)} = 2^{n/2}(\prod_j n_j!)^{1/2}$,

(b) $\left(\mathscr{T}\prod_j H_{n_j}\left(\frac{\langle x, \zeta_j\rangle}{\sqrt{2}}\right)\right)(\xi) = i^n C(\xi) \prod_j((\sqrt{2}\zeta_j)^{\hat{\otimes} n_j}, \xi^{\hat{\otimes} n_j})_{E_0^{\hat{\otimes} n_j}}$,

(c) $\|\hat{\bigotimes}_j(\sqrt{2}\zeta_j)^{\otimes n_j}\|_{E_0^{\hat{\otimes} n}} = 2^{n/2}(n!)^{-1/2}(\prod_j n_j!)^{1/2}$ *and*

(d) $(L^2) = \sum_{n=0}^{\infty} \oplus \mathscr{H}_n$ *(orthogonal sum).*

Proposition 3.1 defines the isomorphism between (L^2) and the Fock space $\Phi_0 = \sum_{n=0}^{\infty} \oplus \sqrt{n!} H_0^{\hat{\otimes} n}$ over H_0 with weights $\sqrt{n!}$. To describe more exactly, we define some notations. Let $\boldsymbol{k}$, $\boldsymbol{n}$, $\cdots$ be sequences of non-negative integers: $\boldsymbol{k}=(k_0, k_1, k_2, \cdots)$, $\boldsymbol{n}=(n_0, n_1, n_2, \cdots)$, $\cdots$. We write $|\boldsymbol{k}|=\sum_{j=0}^{\infty} k_j$, $\boldsymbol{k}!=\prod_j k_j!$, $\binom{\boldsymbol{n}}{\boldsymbol{k}}=\prod_j\binom{n_j}{k_j}$, $\boldsymbol{k}\le\boldsymbol{n}$ if $k_j\le n_j$ for every $j\ge 0$, and $\boldsymbol{n}\wedge\boldsymbol{k}=(n_0\wedge k_0, n_1\wedge k_1, \cdots, n_j\wedge k_j, \cdots)$.

We call $f_n\in H_0^{\hat{\otimes} n}$ the *representation kernel*, or simply, the *kernel* of $\varphi_n\in\mathscr{H}_n$ if $(\mathscr{T}\varphi_n)(\xi)=i^n C(\xi)(f_n, \xi^{\hat{\otimes} n})_{H_0^{\hat{\otimes} n}}$ for $\xi\in E$.

PROPOSITION 3.2. (a) *Let $\varphi\in(L^2)$ and suppose that $\varphi=\sum_{n=0}^{\infty}\varphi_n$ with $\varphi_n\in\mathscr{H}_n$ $(n\ge 0)$. Then for every $n\ge 0$ there exists a unique vector f_n in $H_0^{\hat{\otimes} n}$ such that*

$$(3.7)\qquad (\mathscr{T}\varphi_n)(\xi)=i^n C(\xi)(f_n, \xi^{\hat{\otimes} n})_{H_0^{\hat{\otimes} n}} \qquad \text{for } \xi\in E$$

and that

$$(3.8)\qquad \|\varphi_n\|_{(L^2)}=\sqrt{n!}\,\|f_n\|_{H_0^{\hat{\otimes} n}}\,.$$

(b) *(L^2) is isomorphic to the Fock space $\Phi_0=\sum_{n=0}^{\infty}\oplus\sqrt{n!}H_0^{\hat{\otimes} n}$ over H_0 with weights $\sqrt{n!}$. This isomorphism is given by the correspondence*

$$(3.9)\qquad \Gamma\colon (L^2)\ni\sum_{n=0}^{\infty}\varphi_n\mapsto(f_n)_{n=0}^{\infty}\in\sum_{n=0}^{\infty}\oplus\sqrt{n!}H_0^{\hat{\otimes} n}\,.$$

PROOF. (b) is clear if (a) is proved. So we prove (a). We can take $\{(2^n\boldsymbol{n}!)^{-1/2}\prod_j H_{n_j}(2^{-1/2}\langle x, \zeta_j\rangle);\ |\boldsymbol{n}|=n\}$ as an orthonormal base of $\mathscr{H}_n$ by using Proposition 3.1(a). Therefore every $\varphi_n\in\mathscr{H}_n$ has the expansion $\varphi_n=\sum_{|\boldsymbol{n}|=n}\beta_{\boldsymbol{n}}(2^n\boldsymbol{n}!)^{-1/2}\prod_j H_{n_j}(2^{-1/2}\langle x, \zeta_j\rangle)$ with $\|\varphi_n\|_{(L^2)}^2=\sum_{|\boldsymbol{n}|=n}|\beta_{\boldsymbol{n}}|^2$ in (L^2). Then it follows from Proposition 3.1(b) that the $\mathscr{T}$-transform of φ_n is given by

$$(3.10)\qquad (\mathscr{T}\varphi_n)(\xi)=i^n C(\xi)\Big(\sum_{|\boldsymbol{n}|=n}\beta_{\boldsymbol{n}}(\boldsymbol{n}!)^{-1/2}\,\hat{\otimes}_j\,\zeta_j^{\hat{\otimes} n_j}, \xi^{\hat{\otimes} n}\Big)_{H_0^{\hat{\otimes} n}}\,.$$

Consequently the kernel f_n of φ_n is given by

$$(3.11)\qquad f_n=\sum_{|\boldsymbol{n}|=n}\beta_{\boldsymbol{n}}(\boldsymbol{n}!)^{-1/2}\,\hat{\otimes}_j\,\zeta_j^{\hat{\otimes} n_j}\,.$$

By Proposition 3.1(c), $n!\|f_n\|_{H_0^{\hat{\otimes} n}}^2=\sum_{|\boldsymbol{n}|=n}|\beta_{\boldsymbol{n}}|^2=\|\varphi_n\|_{(L^2)}^2$ for every $n\ge 0$. □

REMARK. In the case where the basic space E_0 is $L^2_{real}(\boldsymbol{R})$, $\varphi_n(x)$ is the n-ple Wiener-Itô integral of f_n. (See [10].)

§4. The spaces of test functionals and generalized functionals

In this section, we firstly construct a Gel'fand triplet which has (L^2) as an intermediate space, i.e., $(\mathscr{S})\hookrightarrow(L^2)\hookrightarrow(\mathscr{S}')$. We will call $(\mathscr{S})$ and $(\mathscr{S}')$ the space of *test functionals* and the space of *generalized functionals* respectively. Secondly we state and prove, according to our case, two properties of the space

$(\mathscr{S})$ of test functionals: 1) every element of $(\mathscr{S})$ has a continuous version as a functional on the space E^* and 2) $(\mathscr{S})$ is an algebra, in particular $(\mathscr{S})$ is closed under the product of two elements of $(\mathscr{S})$.

For each $p \geq 0$, let Φ_p be the Fock space over H_p with weights $\sqrt{n!}$:

$$\Phi_p = \sum_{n=0}^{\infty} \oplus \sqrt{n!} H_p^{\hat{\otimes} n} . \tag{4.1}$$

The inner product $(\cdot, \cdot)_{\Phi_p}$ of Φ_p is defined as follows:

$$((f_n)_{n=0}^{\infty}, (g_n)_{n=0}^{\infty})_{\Phi_p} = \sum_{n=0}^{\infty} n!(f_n, g_n)_{H_p^{\hat{\otimes} n}} \qquad \text{for } (f_n)_{n=0}^{\infty} \text{ and } (g_n)_{n=0}^{\infty} \in \Phi_p . \tag{4.2}$$

Then the Fock spaces Φ_p form a descending sequence: $\cdots \subset \Phi_p \subset \cdots \subset \Phi_0$.

DEFINITION 4.1. Let $(\mathscr{S}_p)$ be the inverse image of Φ_p by the isomorphism Γ of (L^2) onto Φ_0. We write $(\mathscr{S}) = \bigcap_{p=0}^{\infty}(\mathscr{S}_p)$ and call $(\mathscr{S})$ *the space of test white noise functionals* or *of test functionals.*

We can identify the dual space of Φ_p with $\Phi_{-p} = \sum_{n=0}^{\infty} \oplus \sqrt{n!} H_{-p}^{\hat{\otimes} n}$ by using Riesz' Lemma. Therefore the dual space $(\mathscr{S}_p)'$ of $(\mathscr{S}_p)$ is isomorphic to Φ_{-p}. We see that the spaces $(\mathscr{S}_p)'$ can naturally form an increasing sequence: $(L^2)' = (\mathscr{S}_0)' \subset \cdots \subset (\mathscr{S}_p)' \subset \cdots$. Let us identify (L^2) and its dual space $(L^2)'$ under the bijective conjugate linear map from $(L^2)'$ onto (L^2) induced by Riesz' lemma.

DEFINITION 4.2. Let $(\mathscr{S}_{-p})$ denote the dual space of $(\mathscr{S}_p)$. We write $(\mathscr{S}') = \bigcup_{p=0}^{\infty}(\mathscr{S}_{-p})$ and call $(\mathscr{S}')$ *the space of generalized white noise functionals* or *of generalized functionals.*

$(\mathscr{S})$ is topologized by the projective limit topology of $\{(\mathscr{S}_p)\}_{p=0}^{\infty}$. It would be natural that $(\mathscr{S}')$ is topologized by the inductive limit topology of $\{(\mathscr{S}_{-p})\}_{p=0}^{\infty}$. From now on we adopt these topologies. Thus we have obtained a new Gel'fand triplet $(\mathscr{S}) \hookrightarrow (L^2) \hookrightarrow (\mathscr{S}')$ (Ref. [2]). More exactly we have

PROPOSITION 4.1. *If* $0 \leq p$, *then*

(a) *the inclusion mapping* $\iota_{(\mathscr{S}_p),(\mathscr{S}_{p+q+h_0})}: (\mathscr{S}_{p+q+h_0}) \to (\mathscr{S}_p)$ *is of Hilbert-Schmidt type for sufficiently large* $q \geq 1$ *and*

(b) $(\mathscr{S}) \equiv \bigcap_{p=0}^{\infty}(\mathscr{S}_p)$ *is a nuclear space.*

PROOF. These properties are clear from the fact that each Fock space Φ_p has an orthonormal base

$$\bigcup_{n=0}^{\infty} \{(\underset{\substack{\uparrow \\ \text{0-th}}}{0}, \cdots, 0, \underset{\substack{\uparrow \\ n\text{-th}}}{(\boldsymbol{n}!)^{-1/2} \hat{\otimes}_j \zeta_j^{(p)\otimes n_j}}, 0, 0, \cdots); |\boldsymbol{n}| = n\} , \tag{4.3}$$

where $\{\zeta_j^{(p)}\}_{j=0}^{\infty}$ is an orthonormal base of H_p such as $\{\zeta_j^{(p)}\}_{j=0}^{\infty} \subset E$. For

$$\begin{aligned}\|\iota_{(\mathscr{S}_p),(\mathscr{S}_{p+q+h_0})}\|_{\mathrm{H-S}}^2 &= \sum_{n=0}^{\infty} \sum_{|\boldsymbol{n}|=n} n!\,\|(\boldsymbol{n}!)^{-1/2}\,\hat{\otimes}_j\,\zeta_j^{(p+q+h_0)\hat{\otimes} n_j}\|_{H_p^{\hat{\otimes} n}}^2 \\ &= \sum_{n=0}^{\infty} \sum_{|\boldsymbol{n}|=n}^{\infty} \frac{n!}{\boldsymbol{n}!} \prod_j (\|\zeta_j^{(p+q+h_0)}\|_{H_p}^2)^{n_j} \\ &= \sum_{n=0}^{\infty} (\sum_{j=0}^{\infty} \|\xi_j^{(p+q+h_0)}\|_p^2)^n \\ &\le \sum_{n=0}^{\infty} (\sum_{j=0}^{\infty} \rho^{2q}\|\xi_j^{(p+q+h_0)}\|_{p+q}^2)^n \\ &\le \sum_{n=0}^{\infty} \rho^{2qn}\|D^{-h_0}\|_{\mathrm{H-S}}^n \\ &\le (1-\rho^{2q}\delta)^{-1}\end{aligned}$$

holds if $\rho^{2q}\delta < 1$ ($\delta = \|D^{-h_0}\|_{\mathrm{H-S}}$). □

To state that every element of $(\mathscr{S})$ has a continuous version and that $(\mathscr{S})$ is an algebra, we prepare two lemmas which are easily proved by elementary combinatorics ideas.

LEMMA 4.1. *Let $(X, (\cdot,\cdot))$ be a finite dimensional vector space with inner product $(\cdot,\cdot)$ and $\{\zeta_j\}_j$ be an orthonormal base of X. For any choice $\{\xi_i\}_{i=1}^n$ ($n \ge 2$) of n vectors from the base $\{\zeta_j\}_j$ and for $f_n = \xi_1 \otimes \cdots \otimes \xi_n \in X^{\otimes n}$, define* $f_{n|n-2k}$ by

$$f_{n|n-2k} = (\xi_{n-2k+1}, \xi_{n-2k+2}) \cdots (\xi_{n-1}, \xi_n)\xi_1 \otimes \cdots \otimes \xi_{n-2k}$$

and for general $f_n \in X^{\otimes n}$ define $f_{n|n-2k}$ linearly. Then, for the symmetric n-fold tensor product $f_n = \hat{\bigotimes}_j \zeta_j^{\hat{\otimes} n_j}$, we have

$$f_{n|n-2k} = \frac{(n-2k)!}{n!} \sum_{\substack{|\boldsymbol{k}|=k \\ 2\boldsymbol{k} \le \boldsymbol{n}}} \binom{\boldsymbol{n}}{2\boldsymbol{k}} (2\boldsymbol{k})! \frac{k!}{\boldsymbol{k}!} \hat{\otimes}_j \zeta_j^{\hat{\otimes}(n_j - 2k_j)}.$$

LEMMA 4.2. *Let X and $\{\zeta_j\}_j$ be the same as in Lemma 4.1. Let $f_m = \sum_{i_1} \cdots \sum_{i_m} \alpha_{i_1 \cdots i_m} \zeta_{i_1} \otimes \cdots \otimes \zeta_{i_m}$ and $g_n = \sum_{j_1} \cdots \sum_{j_n} \beta_{j_1 \cdots j_n} \zeta_{j_1} \otimes \cdots \otimes \zeta_{j_n}$ be any symmetric tensors in $X^{\hat{\otimes} m}$ and $X^{\hat{\otimes} n}$ respectively. Define $f_m \hat{\otimes}_k g_n$ by*

$$\begin{aligned} f_m \hat{\otimes}_k g_n = \frac{1}{(m+n-2k)!} &\sum_{\sigma \in \mathfrak{S}(m+n-2k)} \sum_{i_1} \cdots \sum_{i_m} \sum_{j_1} \cdots \sum_{j_n} \alpha_{i_1 \cdots i_m} \beta_{j_1 \cdots j_n} \\ &\times (\zeta_{i_{m-k+1}}, \zeta_{j_{n-k+1}}) \cdots (\zeta_{i_m}, \zeta_{j_n}) \\ &\times \zeta_{\sigma(i_1)} \otimes \cdots \otimes \zeta_{\sigma(i_{m-k})} \otimes \zeta_{\sigma(j_1)} \otimes \cdots \otimes \zeta_{\sigma(j_{n-k})} \qquad \text{for } k \le m \wedge n, \end{aligned}$$

where $\mathfrak{S}(m+n-2k)$ is the permutation group of symbols $\{i_1, \cdots, i_{m-k}, j_1, \cdots, j_{n-k}\}$. Then, for $f_m = \hat{\bigotimes}_j \zeta_j^{\hat{\otimes} m_j}$ and $g_n = \hat{\bigotimes}_j \zeta_j^{\hat{\otimes} n_j}$ with $|\boldsymbol{m}| = m$ and

$|\boldsymbol{n}| = n$, *we have*

$$f_m \hat{\otimes}_k g_n = \frac{(m-k)!(n-k)!}{m!n!} \sum_{k \le m \wedge n} \binom{m}{k}\binom{n}{k} k! \,\hat{\otimes}_j\, \zeta_j^{\hat{\otimes}(m_j+n_j-2k_j)} .$$

DEFINITION 4.3. Let the kernels f_m and g_n of $\varphi_m \in \mathscr{H}_m \cap (\mathscr{S})$ and $\psi_n \in \mathscr{H}_n \cap (\mathscr{S})$ be expressed in the following form, using orthonormal base $\{\zeta_j^{(p)}\}_{j=0}^{\infty}$ of E_p, respectively:

$$f_m = \sum_{i_1} \cdots \sum_{i_m} \alpha^{(p)}_{i_1 \cdots i_m} \zeta^{(p)}_{i_1} \otimes \cdots \otimes \zeta^{(p)}_{i_m} \quad \text{with} \quad \|f_m\|^2_{H_p^{\hat{\otimes} m}} = \sum_{i_1} \cdots \sum_{i_m} |\alpha^{(p)}_{i_1 \cdots i_m}|^2$$

and

$$g_n = \sum_{j_1} \cdots \sum_{j_n} \beta^{(p)}_{j_1 \cdots j_n} \zeta^{(p)}_{j_1} \otimes \cdots \otimes \zeta^{(p)}_{j_n} \quad \text{with} \quad \|g_n\|^2_{H_p^{\hat{\otimes} n}} = \sum_{j_1} \cdots \sum_{j_n} |\beta^{(p)}_{j_1 \cdots j_n}|^2 .$$

(1) For $2k \le n$, define $f_{n|n-2k}$ by

$$\sum_{i_1} \cdots \sum_{i_n} \alpha^{(p)}_{i_1 \cdots i_n} (\zeta^{(p)}_{i_{n-2k+1}}, \zeta^{(p)}_{i_{n-2k+2}})_0 \cdots (\zeta^{(p)}_{i_{n-1}}, \zeta^{(p)}_{i_n})_0 \zeta^{(p)}_{i_1} \otimes \cdots \otimes \zeta^{(p)}_{i_{n-2k}} .$$

(2) For $k \le m \wedge n$, define $f_m \hat{\otimes}_k g_n$ by

$$\begin{aligned} &\frac{1}{(m+n-2k)!} \sum_{\sigma \in \mathfrak{S}(m+n-2k)} \sum_{i_1} \cdots \sum_{i_m} \sum_{j_1} \cdots \sum_{j_n} \alpha^{(p)}_{i_1 \cdots i_m} \beta^{(p)}_{j_1 \cdots j_n} \\ &\times (\zeta^{(p)}_{i_{m-k+1}}, \zeta^{(p)}_{j_{n-k+1}})_0 \cdots (\zeta^{(p)}_{i_n}, \zeta^{(p)}_{j_m})_0 \\ &\times \zeta^{(p)}_{\sigma(i_1)} \otimes \cdots \otimes \zeta^{(p)}_{\sigma(i_{m-k})} \otimes \zeta^{(p)}_{\sigma(j_1)} \otimes \cdots \otimes \zeta^{(p)}_{\sigma(j_{n-k})} . \end{aligned}$$

Using Schwarz' inequality, we obtain the following estimations: for $p \ge h_0$,

$$\|f_{n|n-2k}\|^2_{H_p^{\hat{\otimes}(n-2k)}} \le \|f_n\|^2_{H_p^{\hat{\otimes} n}} \delta^{4k} \rho^{2k(p-h_0)} \tag{4.4}$$

and

$$\|f_m \hat{\otimes}_k g_n\|^2_{H_p^{\hat{\otimes}(m+n-2k)}} \le \frac{1}{(m+n-2k)!} \|f_m\|^2_{H_p^{\hat{\otimes} m}} \|g_n\|^2_{H_p^{\hat{\otimes} n}} \delta^{4k} \rho^{4k(p-h_0)} . \tag{4.5}$$

Further, these estimations tell us that $f_{n|n-2k}$ and $f_m \hat{\otimes}_k g_n$ do not depend on p. Clearly, $f_{n|n-2k} \in H_p^{\hat{\otimes}(n-2k)}$ and $f_m \hat{\otimes}_k g_n \in H_p^{\hat{\otimes}(m+n-2k)}$ hold for any $p \ge 0$.

PROPOSITION 4.2. *Suppose that* $\varphi_n \in \mathscr{H}_n \cap (\mathscr{S})$ $(n \ge 0)$ *and that* f_n *is the representation kernel of* φ_n $(n \ge 0)$. *For every* $x \in E^*$, *write*

$$\mathfrak{I}_n(f_n)(x) = \sum_{k=0}^{[n/2]} \frac{n!(-2)^{-k}}{(n-2k)!k!} \langle x^{\hat{\otimes}(n-2k)}, f_{n|n-2k} \rangle . \tag{4.6}$$

Then (1) $\varphi_n(x) = \mathfrak{I}_n(f_n)(x)$ μ-a.e. $x \in E^*$ *and* (2) $\mathfrak{I}_n(f_n)(x)$ *is continuous in* $x \in E^*$.

PROOF. (1) Let $\varphi_n = \sum_{|\boldsymbol{n}|=n} \alpha_{\boldsymbol{n}}^{(0)}(2^n\boldsymbol{n}!)^{-1/2}\prod_j H_{n_j}(2^{-1/2}\langle x, \zeta_j^{(0)}\rangle)$. Then, the kernel f_n of φ_n is given by $f_n = \sum_{|\boldsymbol{n}|=n}\alpha_{\boldsymbol{n}}^{(0)}(\boldsymbol{n}!)^{-1/2}\hat{\otimes}_j \zeta_j^{(0)\hat{\otimes} n_j}$. By the formula $H_n(u) = \sum_{k=0}^{[n/2]}(-1)^k n!(2u)^{n-2k}/(n-2k)!k!$, Lemma 4.1 and the estimation (4.5), we have

$$(4.7)\quad \varphi_n(x) = \sum_{|\boldsymbol{n}|=n}\alpha_{\boldsymbol{n}}^{(0)}\frac{1}{\sqrt{\boldsymbol{n}!}}\sum_{2\boldsymbol{k}\le \boldsymbol{n}}(-1)^{|\boldsymbol{k}|}\frac{\boldsymbol{n}!2^{-|\boldsymbol{k}|}}{(\boldsymbol{n}-2\boldsymbol{k})!\boldsymbol{k}!}\langle x^{\hat{\otimes}(n-2|\boldsymbol{k}|)}, \hat{\bigotimes}_j \zeta_j^{(0)\hat{\otimes}(n_j-2k_j)}\rangle$$

$$= \sum_{k=0}^{[n/2]}(-1)^k\frac{n!2^{-k}}{(n-2k)!k!}\langle x^{\hat{\otimes}(n-2k)}, f_{n|n-2k}\rangle = \mathfrak{I}_n(f_n)(x)\,.$$

(2) We have to show that $|\mathfrak{I}_n(f_n)(x+y) - \mathfrak{I}_n(f_n)(x)|$ is small if $y \in E^*$ is small with respect to the inductive limit topology of E^*. It is enough to show it for $|\langle (x+y)^{\hat{\otimes}m} - x^{\hat{\otimes}m}, f_m\rangle|$ $(n \ge m \ge 0)$. Let $x \in E_{-q}$ and let $y = \sum_{p=0}^N \lambda_p y_p$ with $\sum_{p=0}^N |\lambda_p| \le 1$, $y_p \in E_{-p}$ and $\|y_{-p}\| \le \delta_p$. We can assume that $\delta_p \le 1$ for any $p \ge 0$ without loss of generality. Then we have

$$(x+y)^{\hat{\otimes}m} - x^{\hat{\otimes}m} = m\sum_{l=0}^{m-1}\frac{1}{l+1}\binom{m-1}{l}x^{\hat{\otimes}(m-1-l)}\hat{\otimes}\left(\sum_{p=0}^N \lambda_p y_p\right)^{\hat{\otimes}(l+1)}.$$

By dividing the factor $(\sum_{p=0}^N \lambda_p y_p)^{\hat{\otimes}(l+1)}$ into two parts $\sum_{p_0\vee p_1\cdots\vee p_l\le q}\hat{\bigotimes}_{j=0}^q(\lambda_{p_j}y_{p_j})$ and $\sum_{p>q}^N\sum_{p_0\vee p_1\vee\cdots\vee p_l=p}\hat{\bigotimes}_{j=0}^q(\lambda_{p_j}y_{p_j})$, we can obtain the estimation of $|\langle (x+y)^{\hat{\otimes}m} - x^{\hat{\otimes}m}, f_m\rangle|$ $(n \ge m \ge 0)$:

$$(4.8)\quad |\langle (x+y)^{\hat{\otimes}m} - x^{\hat{\otimes}m}, f_m\rangle|$$

$$\le \sum_{l-1}^{N-1} m\frac{1}{l+1}\binom{m-1}{l}\|x\|_{-q}^{m-1-l}\sum_{p_0\vee p_1\cdots\vee p_l\le q}\|f_m\|_{H_q^{\hat{\otimes}m}}$$

$$\times\prod_{j=0}^N |\lambda_{p_j}|\|y_{p_j}\|_{-q}$$

$$+ \sum_{l-1}^{N-1} m\frac{1}{l+1}\binom{m-1}{l}\|x\|^{m-1-l}\sum_{p>q}^N\sum_{p_0\vee p_1\vee\cdots\vee p_l=q}\|f_m\|_{H_q^{\hat{\otimes}m}}$$

$$\times\prod_{j=0}^N |\lambda_{p_j}|\|y_{p_j}\|_{-p}$$

$$\le \sum_{p=0}^q m(1+\|x\|_{-q})^{m-1}\|f_m\|_{H_q^{\hat{\otimes}m}}|\lambda_p|\,\|y_p\|_{-q}$$

$$+ \sum_{p>q}^N m(1+\|x\|_{-p})^{m-1}\|f_m\|_{H_p^{\hat{\otimes}m}}|\lambda_p|\,\|y_p\|_{-p}\,.$$

Therefore, for a previously given $\varepsilon > 0$, if we choose δ_p for any $p \ge 0$:

$$(4.9)\qquad \delta_p = (m(1+\|x\|_{-q})^{m-1}\|f_m\|_{H_q^{\hat{\otimes}m}}\rho^{q-p})^{-1}\varepsilon \qquad \text{for}\quad p \le q$$

and

$$\delta_p = (m(1+\|x\|_{-p})^{m-1}\|f_m\|_{H_p^{\hat{\otimes}m}})^{-1}\varepsilon \qquad \text{for}\quad q < p < \infty\,,$$

we see that $|\langle (x+y)^{\hat{\otimes}m} - x^{\hat{\otimes}m}, f_m\rangle| \le \sum_{p=0}^q |\lambda_p|\varepsilon + \sum_{p>q}^N |\lambda_p|\varepsilon \le \varepsilon$. □

PROPOSITION 4.3. *Suppose that* $\varphi = \sum_{n=0}^{\infty} \varphi_n \in (\mathscr{S})$ *with* $\varphi_n \in \mathscr{H}_n \cap (\mathscr{S})$ *and that* f_n *is the kernel of* φ_n. *Then*

(1) $\sum_{n=0}^{\infty} \mathfrak{I}_n(f_n)(x)$ *converges absolutely for every* $x \in E^*$. *Moreover,*

(2) *if we write* $\tilde{\varphi}(x) = \sum_{n=0}^{\infty} \mathfrak{I}_n(f_n)(x)$ *for every* $x \in E^*$, $\tilde{\varphi}(x)$ *is continuous in* $x \in E^*$ *and* $\tilde{\varphi}(x) = \varphi(x)$, μ-a.e. $x \in E^*$. (*From now on, we denote this continuous version of* $\varphi \in (\mathscr{S})$ *by* $\tilde{\varphi}$.)

PROOF. (1) By the estimation analogous to (4.8), we have

$$
\begin{aligned}
(4.10)\qquad & \sum_{n=0}^{\infty} |\mathfrak{I}_n(f_n)(x)| \\
& \le \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{n!2^{-k}}{(n-2k)!k!} \|f_n\|_{H_p^{\hat{\otimes} n}} \delta^{2k} \rho^{2k(p-h_0)} \|x\|_{-p}^{n-2k} \\
& \le \left(\sum_{n=0}^{\infty} n! \|f_n\|^2_{H_p^{\hat{\otimes} n}}\right)^{1/2} \left(\sum_{n=0}^{\infty} (\delta\rho^{p-h_0} + \|x\|_{-p})^{2n}\right)^{1/2} \\
& = \|\varphi\|_{(\mathscr{S}_p)} (1 - (\delta\rho^{p-h_0} + \|x\|_{-p})^2)^{-1/2} < \infty
\end{aligned}
$$

if $\delta\rho^{p-h_0} + \|x\|_{-p} < 1$. While, the condition $\delta\rho^{p-h_0} + \|x\|_{-p} < 1$ is satisfied for sufficiently large p, because $\|x\|_{-p} \le \rho^{p-q}\|x\|_{-q}$ holds if x belongs to E_{-q} for some $q \ge 0$. Recall that $0 < \rho = \|D^{-1}\| < 1$.

(2) By the similar computation to the proof of Proposition 4.2,

$$
\begin{aligned}
(4.11)\quad & |\mathfrak{I}(f_n)(x+y) - \mathfrak{I}(f_n)(x)| \\
& \le \sum_{k=0}^{[n/2]} \frac{n!2^{-k}}{(n-2k-1)!k!} (1 + \|x\|_{-q})^{n-2k-1} \|f_n\|_{H_q^{\hat{\otimes} n}} \delta^{2k} \rho^{2k(q-h_0)} \sum_{p=0}^{q} |\lambda_p| \|y_p\|_{-q} \\
& \quad + \sum_{k=0}^{[n/2]} \frac{n!2^{-k}}{(n-2k-1)!k!} \sum_{p>q}^{N} \|f_n\|_{H_q^{\hat{\otimes} n}} \delta^{2k} \rho^{2k(q-h_0)} (1 + \|x\|_{-q})^{n-2k-1} |\lambda_p| \|y_p\|_{-p} \\
& \le \sum_{p=0}^{q} |\lambda_p| \sqrt{n} \sum_{k=0}^{n-1} \binom{n-1}{k} (1 + \|x\|_{-q})^{n-k-1} \sqrt{n!} \|f_n\|_{H_{q+r}^{\hat{\otimes} n}} \rho^{nr} (\delta\rho^{q-h_0})^k \|y_p\|_{-q} \\
& \quad + \sum_{p>q}^{N} |\lambda_p| \sqrt{n} \sum_{k=0}^{n-1} \binom{n-1}{k} (1 + \|x\|_{-p})^{n-k-1} \sqrt{n!} \|f_n\|_{H_{q+r}^{\hat{\otimes} n}} \rho^{nr} (\delta\rho^{q-h_0})^k \|y_p\|_{-p} \\
& \le \sum_{p=0}^{q} |\lambda_p| \sqrt{n} ((1 + \|x\|_{-q})\rho^r + \delta\rho^{q+r-h_0})^{n-1} \sqrt{n!} \|f_n\|_{H_{q+r}^{\hat{\otimes} n}} \|y_p\|_{-q} \rho^r \\
& \quad + \sum_{p>q}^{N} |\lambda_p| \sqrt{n} ((1 + \|x\|_{-p})\rho^r + \delta\rho^{p+r-h_0})^{n-1} \sqrt{n!} \|f_n\|_{H_{p+r}^{\hat{\otimes} n}} \|y_p\|_{-p} \rho^r .
\end{aligned}
$$

Thus, if r is sufficiently large such that $(1 + \|x\|_{-p} + \delta\rho^{p-h_0})\rho^r < 1$ for any $p \ge q$, we have

(4.12) $|\tilde{\varphi}(x+y)-\tilde{\varphi}(x)|$

$$= |\textstyle\sum_{n=1}^{\infty}\{\mathfrak{I}(f_n)(x+y)-\mathfrak{I}(f_n)(x)\}|$$

$$\le \textstyle\sum_{p=0}^{q}|\lambda_p|(\sum_{n=1}^{\infty}n!\|f_n\|^2_{H_{q+r}^{\hat{\otimes}n}})^{1/2}(\sum_{n=1}^{\infty}n((1+\|x\|_{-q}+\delta\rho^{q-h_0})\rho^r)^{2(n-1)})^{1/2}$$

$$\times\|y_p\|_{-q}\rho^r$$

$$+\textstyle\sum_{p>q}^{N}|\lambda_p|(\sum_{n=1}^{\infty}n!\|f_n\|^2_{H_{p+r}^{\hat{\otimes}n}})^{1/2}(\sum_{n=1}^{\infty}n((1+\|x\|_{-p}+\delta\rho^{p-h_0})\rho^r)^{2(n-1)})^{1/2}$$

$$\times\|y_p\|_{-q}\rho^r$$

$$\le\textstyle\sum_{p=0}^{q}|\lambda_p|\|\varphi\|_{(\mathscr{S}_{q+r})}(1-(1+\|x\|_{-q}+\delta\rho^{q-h_0})^2\rho^{2r})^{-1}\|y_p\|_{-q}\rho^r$$

$$+\textstyle\sum_{p>q}^{N}|\lambda_p|\|\varphi\|_{(\mathscr{S}_{p+r})}(1-(1+\|x\|_{-p}+\delta\rho^{p-h_0})^2\rho^{2r})^{-1}\|y_p\|_{-p}\rho^r\,.$$

Since r does not depend on y_p's, if we choose δ_p's as follows:

$$\begin{cases}\delta_p=\|\varphi\|^{-1}_{(\mathscr{S}_{q+r})}(1-(1+\|x\|_{-q}+\delta\rho^{q-h_0})^2\rho^{2r})\varepsilon & \text{for}\quad 0\le p\le q\,,\\ \delta_p=\|\varphi\|^{-1}_{(\mathscr{S}_{p+r})}(1-(1+\|x\|_{-p}+\delta\rho^{p-h_0})^2\rho^{2r})\varepsilon & \text{for}\quad q<p<\infty\,,\end{cases}$$

we obtain $|\tilde{\varphi}(x+y)-\tilde{\varphi}(x)|\le\sum_{p=0}^{N}|\lambda_p|\delta_p\rho^r<\varepsilon$ for $^\forall y\in U=$ the absolutely convex envelope of $\bigcup_{p=0}^{\infty}\{y_p\in E_{-p};\ \|y_p\|_{-p}<\delta_p\}$. □

Because of the estimation $|\tilde{\varphi}(x)|\le\|\varphi\|_{(\mathscr{S}_p)}(1-(\delta\rho^{p-h_0}+\|x\|_{-p})^2)^{-1/2}$ for every fixed $x\in E^*$ and sufficiently large p, we have the following corollaries:

COROLLARY 4.1. *For any $x\in E^*$, $\tilde{\varphi}(x)$ defines the functional δ_x of $\varphi\in(\mathscr{S})$, i.e., $\langle\delta_x(\cdot),\varphi(\cdot)\rangle=\tilde{\varphi}(x)$, and δ_x belongs to $(\mathscr{S}')$.*

COROLLARY 4.2. *Suppose that φ, $\varphi_n\in(\mathscr{S})$ $(n=1,2,\cdots)$ and that φ_n converges to φ in $(\mathscr{S})$. Then $\tilde{\varphi}_n(x)$ converges to $\tilde{\varphi}(x)$ at each $x\in E^*$.*

Now, let us show the second property of $(\mathscr{S})$:

PROPOSITION 4.4 (Cf. Th. 7.4 and Th. 7.5 of [15]). *Supppose φ and $\psi\in(\mathscr{S})$. Then $\bar{\varphi}$ and $\varphi\psi\in(\mathscr{S})$. Moreover,*

(a) $\|\varphi\|_{(\mathscr{S}_p)}=\|\bar{\varphi}\|_{(\mathscr{S}_p)}$ *for any $p\ge 0$,*

(b) *there exist a positive number $K>0$ and an integer $q>0$, both independent of φ, ψ and p, such that*

$$\|\varphi\psi\|_{(\mathscr{S}_p)}\le K\|\varphi\|_{(\mathscr{S}_{p+q})}\|\psi\|_{(\mathscr{S}_{p+q})}\,.$$

PROOF. We give an outline of the proof of $\varphi\psi\in(\mathscr{S})$. Suppose $\varphi=\sum_{m=0}^{\infty}\varphi_m$ and $\psi=\sum_{n=0}^{\infty}\psi_n$ with $\varphi_m\in\mathscr{H}_m$ and $\psi_n\in\mathscr{H}_n$. Let the kernels of φ_m and ψ_n be f_m and g_n respectively.

Then by Lemma 4.2, the estimation (4.5) and the formula

$$H_m(u)H_n(u) = \sum_{k=0}^{m\wedge n} 2^k k! \binom{m}{k}\binom{n}{k} H_{m+n-2k}(u),$$

we can obtain

$$\Im_m(g_m)\Im_n(f_n) = \sum_{k=0}^{m\wedge n} k! \binom{m}{k}\binom{n}{k} \Im_{m+n-2k}(g_m \hat{\otimes}_k f_n). \tag{4.13}$$

Thus

$$\begin{aligned}(4.14)\quad & \|\Im_m(g_m)\Im_n(f_n)\|_{(\mathscr{S}_p)} \\ & \le \sum_{k=0}^{m\wedge n} \left(\binom{m}{k}\binom{n}{k}\binom{m+n-2k}{n-k}\right)^{1/2} (\delta\rho^{(p-h_0)})^{2k} \|\varphi_m\|_{(\mathscr{S}_p)} \|\psi_n\|_{(\mathscr{S}_p)} \\ & \le (1+\delta^2\rho^{2(p-h_0)})^{(m+n)/2} \rho^{q(m+n)} \|\varphi_m\|_{(\mathscr{S}_{p+q})} \|\psi_n\|_{(\mathscr{S}_{p+q})}\end{aligned}$$

if $p \ge h_0$ and $q > 0$. Consequently

$$\begin{aligned}(4.15)\quad \|\varphi\psi\|_{(\mathscr{S}_p)} & \le \sum_{m,n=0}^{\infty} \Im_m(g_m)\Im_n(f_n)\|_{(\mathscr{S}_p)} \\ & \le (1-(1+\delta^2\rho^{2(p-h_0)})\rho^{2q})^{-1} \|\varphi\|_{(\mathscr{S}_{p+q})} \|\psi\|_{(\mathscr{S}_{p+q})} \\ & \le (1-(1+\delta^2\rho^{-2h_0})\rho^{2q})^{-1} \|\varphi\|_{(\mathscr{S}_{p+q})} \|\psi\|_{(\mathscr{S}_{p+q})}\end{aligned}$$

if q is large enough for $(1+\delta^2\rho^{-2h_0})\rho^{2q} < 1$. This shows that K and q are independent of φ, ψ and p. Therefore $\varphi\psi$ belongs to $(\mathscr{S})$. □

§5. Positive generalized white noise functionals and main theorem

In this section we prove our main theorem: *Every positive generalized white noise functional is expressed in terms of the integral given by some positive finite measure on* $(E^*, \mathscr{B})$. In proving this theorem, Minlos' theorem and the algebra A, linearly spanned by exponential functions, play important roles. Firstly, let us prepare two auxiliary propositions relating to exponential functions and the algebra A.

PROPOSITION 5.1. *For any* $\zeta \in E$, $\exp[i\langle x, \zeta\rangle]$ *is in* $(\mathscr{S})$ *and the mapping* $\zeta \mapsto \exp[i\langle x, \zeta\rangle]$ *from* E *into* $(\mathscr{S})$ *is continuous.*

PROOF. Suppose $\zeta \in E$ and consider the $\mathscr{T}$-transform of $\exp[i\langle x, \zeta\rangle]$. We see that for every $\xi \in E$

$$\{\mathscr{T}\exp[i\langle x, \zeta\rangle]\}(\xi) = C(\xi+\zeta) = \sum_{n=0}^{\infty} i^n C(\xi) \left\langle \frac{i^n}{n!} C(\zeta)\zeta^{\hat{\otimes} n}, \xi^{\hat{\otimes} n}\right\rangle. \tag{5.1}$$

Consequently we have

$$\|\exp[i\langle x,\zeta\rangle]\|_{(\mathscr{S}_p)} = C(\zeta)\exp[\tfrac{1}{2}\|\zeta\|_p^2] < \infty \qquad \text{for any} \quad p \geq 0\,. \tag{5.2}$$

Thus $\exp[i\langle x,\zeta\rangle] \in (\mathscr{S})$.

The continuity of $\exp[i\langle x,\zeta\rangle]$ in $\zeta \in E$ follows from the inequalities

$$\begin{aligned}(5.3)\quad & \|\exp[i\langle x,\zeta_0\rangle] - \exp[i\langle x,\zeta\rangle]\|^2_{(\mathscr{S}_p)} \\ & \leq 2\sum_{n=0}^{\infty}\frac{1}{n!}|C(\zeta_0)-C(\zeta)|^2\|\zeta_0\|_p^{2n} + 2C(\zeta)^2\sum_{n=0}^{\infty}\frac{1}{n!}\|\zeta_0^{\hat{\otimes}n}-\zeta^{\hat{\otimes}n}\|^2_{E_p^{\hat{\otimes}n}}\end{aligned}$$

and

$$\sum_{n=0}^{\infty}\frac{1}{n!}\|\zeta_0^{\hat{\otimes}n}-\zeta^{\hat{\otimes}n}\|^2_{E_p^{\hat{\otimes}n}} \leq \|\zeta_0-\zeta\|_p^2\exp[(\|\zeta\|_p+\|\zeta_0\|_p)^2] \tag{5.4}$$

□

PROPOSITION 5.2. *Let us define the algebra $A \subset (\mathscr{S})$ by*

$$A = \{\textstyle\sum_j \beta_j \exp[i\langle x,\xi_j\rangle];\ \text{finite sum},\ \beta_j \in \boldsymbol{C},\ \xi_j \in E\}\,.$$

Then A is dense in $(\boldsymbol{S})$.

PROOF. Since $(\mathscr{S})$, as a nuclear space, is topologized by the countable Hilbertian norms $\{\|\cdot\|_{(\mathscr{S}_p)}\}_{p=0}^{\infty}$, we have only to prove that A is dense in $(\mathscr{S}_p)$ for each $p \geq 0$. If we denote the isomorphism between $(\mathscr{S}_p)$ and the Fock space Φ_p by Γ, then we obtain

$$\Gamma\{\exp[i\langle\cdot,\zeta\rangle]\} = \left(C(\zeta)\frac{i^n}{n!}\zeta^{\hat{\otimes}n}\right)_{n=0}^{\infty}\,.$$

Let $g(t) = (i^n(n!)^{-1}(t\zeta)^{\hat{\otimes}n})_{n=0}^{\infty}$. We can easily see that $\Gamma^{-1}((g(t+h)-g(t))/h)$ belongs to A, that $g'(t) = \lim_{h\to 0}(g(t+h)-g(t))/h$ converges in Φ_p for $t \in \boldsymbol{R}$, and that $g'(0) = (0, i\zeta, 0, 0, \cdots)$ is in Φ_p. Inductively, we have that $g^{(n)}(0) = (0,\cdots,0,i^n\zeta^{\hat{\otimes}n},0,\cdots)$ is in Φ_p. It is well-known that the Fock space $\Phi_p = \sum_{n=0}^{\infty}\oplus\sqrt{n!}\mathscr{S}_p^{\hat{\otimes}n}$ is generated by elements of the form $(0,\cdots,0,\zeta^{\hat{\otimes}n},0,\cdots)$ with $\zeta \in E$ and $n \geq 0$. These facts show that A is dense in $(\mathscr{S}_p)$. (See [4].) □

Next, let us define positive generalized white noise functionals.

DEFINITION 5.1. A generalized white noise functional $\Psi \in (\mathscr{S}')$ is said to be *positive*, denoted by $\Psi \geq 0$, if $\langle\Psi,\psi\rangle \geq 0$ for all $\psi \in (\mathscr{S})$ such that $\tilde{\psi}(x) \geq 0$ at every point $x \in E^*$.

We note the following: If we recall the well-known fact that $\mu(U)$ is positive (>0) for every nonempty open subset U of E^*, the condition for $\psi \in (\mathscr{S})$ in Definition 5.1 can be replaced by the condition that $\psi(x) \geq 0$ for μ-a.e. $x \in E^*$.

Let $(\mathscr{S}')_>$ denote the set $\{\Psi; \Psi \in (\mathscr{S}'), \Psi \geq 0\}$ ([24]).

DEFINITION 5.2. A generalized white noise functional $\Psi \in (\mathscr{S}')$ is said to be *multiplicatively positive on* A if $\langle \Psi, \alpha\bar{\alpha} \rangle \geq 0$ for all $\alpha \in A$.

Notice that $\alpha \in A$ is continuous; i.e., $\tilde{\alpha} = \alpha$. Further we easily see that $\Psi \in (\mathscr{S}')$ is multiplicatively positive on A if and only if $\langle \Psi, \exp[i\langle \cdot, \xi \rangle] \rangle$ is a positive-definite function of $\xi \in E$.

Now we are ready to prove our main theorem:

THEOREM 5.1. *Suppose* $\Psi \in (\mathscr{S}')_>$. *Then there exists a unique finite measure* ν_Ψ *on* $(E^*, \mathscr{B})$ *which satisfies that*

$$\langle \Psi, \psi \rangle = \int_{E^*} \tilde{\psi}(x) d\nu_\Psi(x) \tag{5.5}$$

for all $\psi \in (\mathscr{S})$.

PROOF. Let $\Psi \in (\mathscr{S}')_>$. Since $\alpha(x)$ is itself continuous in x and $\tilde{\alpha}(x)\bar{\tilde{\alpha}}(x) \geq 0$ hold for $\alpha \in A$ and any $x \in E^*$, we see that $\langle \Psi, \alpha\bar{\alpha} \rangle \geq 0$ for $\alpha \in A$, i.e., $\langle \Psi, \exp[i\langle \cdot, \xi \rangle] \rangle$ is positive-definite in $\xi \in E$. Further, by Proposition 5.1, $\langle \Psi, \exp[i\langle \cdot, \xi \rangle] \rangle$ is continuous in $\xi \in E$. Therefore Bochner-Minlos' theorem defines a finite measure ν_Ψ on $(E^*, \mathscr{B})$ uniquely by the equation

$$\langle \Psi, e^{i\langle \cdot, \xi \rangle} \rangle = \int_{E^*} e^{i\langle x, \xi \rangle} d\nu_\Psi(x) \,. \tag{5.6}$$

It follows from the linearity in φ of $\langle \Psi, \varphi \rangle$ and $\int_{E^*} \varphi d\nu_\Psi$ that the following equality holds:

$$\langle \Psi, \alpha \rangle = \int_{E^*} \alpha(x) d\nu_\Psi(x) \qquad \text{for any} \quad \alpha \in A \,. \tag{5.7}$$

We want to extend (5.7) to the form (5.5).

Now let $\psi \in (\mathscr{S})$. Using Proposition 5.2, we can approximate ψ in $(\mathscr{S})$ with a sequence $\{\alpha_n\}_{n=1}^\infty \subset A$. Then $|\alpha_n - \psi|^2$ converges to 0 in $(\mathscr{S})$, because the inequality

$$\|\alpha_n - \psi)(\overline{\alpha_n - \psi})\|_{(\mathscr{S}_p)} \leq K \|\alpha_n - \psi\|^2_{(\mathscr{S}_{p+q})} \qquad \text{for any} \quad p \geq 0 \tag{5.8}$$

follows from Proposition 4.4. Next, to show that $\{\alpha_n\}_{n=0}^\infty$ is a Cauchy sequence in $L^2(E^*, \mathscr{B}, \nu_\Psi)$, we note that $|\alpha_n - \alpha_m|^2$ is in A and that

$$\|\alpha_n - \psi)(\overline{\alpha_m - \psi})\|_{(\mathscr{S}_p)} \le K \|\alpha_n - \psi\|_{(\mathscr{S}_{p+q})} \|\alpha_m - \psi\|_{(\mathscr{S}_{p+q})} . \tag{5.9}$$

We have

$$\begin{aligned} &\int_{E^*} |\alpha_n - \alpha_m|^2 \, d\nu_\Psi \\ &= \langle \Psi, |\alpha_n - \alpha_m|^2 \rangle \\ &= \langle \Psi, |\alpha_n - \psi|^2 - (\alpha_n - \psi)(\overline{\alpha_m - \psi}) - (\alpha_m - \psi)(\overline{\alpha_n - \psi}) + |\alpha_m - \psi|^2 \rangle \\ &= \langle \Psi, |\alpha_n - \psi|^2 \rangle - \langle \Psi, (\alpha_n - \psi)(\overline{\alpha_m - \psi}) \rangle \\ &\quad - \langle \Psi, (\alpha_m - \psi)(\overline{\alpha_n - \psi}) \rangle + \langle \Psi, |\alpha_m - \psi|^2 \rangle . \end{aligned} \tag{5.10}$$

The last four terms tend to 0 as $m, n \to \infty$ because of (5.8) and (5.9).

Since $L^2(E^*, \mathscr{B}, \nu_\Psi)$ is a Banach space, there exists an element χ of $L^2(E^*, \mathscr{B}, \nu_\Psi)$ such that $\alpha_n \to \chi$ $(n \to \infty)$ in $L^2(E^*, \mathscr{B}, \nu_\Psi)$. We can choose a subsequence $\{\alpha_{n_j}\}_{j=1}^\infty$ of $\{\alpha_n\}_{n=1}^\infty$ satisfying that

$$\alpha_{n_j}(x) \to \chi(x) \quad (j \to \infty) \qquad \nu_\Psi\text{-a.e.} \quad x \in E^* .$$

On the other hand, by Corollary 4.2, the continuous versions satisfy that

$$\tilde{\alpha}_{n_j}(x) \to \tilde{\psi} \quad (j \to \infty) \qquad \text{for any} \quad x \in E^* .$$

It follows that $\chi(x) = \tilde{\psi}(x)$ ν_Ψ-a.e. $x \in E^*$. Clearly $\int_{E^*} \alpha_{n_j} \, d\nu_\Psi \to \int_{E^*} \chi \, d\nu_\Psi$ $(j \to \infty)$. Therefore $\langle \Psi, \psi \rangle = \int_{E^*} \chi \, d\nu_\Psi = \int_{E^*} \tilde{\psi} \, d\nu_\Psi$. □

THEOREM 5.2. *Let $\Psi \in (\mathscr{S}')$. $\Psi \ge 0$ if and only if $\langle \Psi, e^{i\langle \cdot, \xi \rangle} \rangle$ is a positive-definite function of $\xi \in E$.*

PROOF. Suppose that $\Psi \in (\mathscr{S}')_>$. For $\alpha \in \mathbf{A}$, $\tilde{\alpha} = \alpha$ and $\tilde{\alpha}(x)(\tilde{\alpha})^-(x) \ge 0$ at every every $x \in E^*$. Thus $\langle \Psi, \alpha\bar{\alpha} \rangle \ge 0$.

Conversely, suppose that $\Psi \in (\mathscr{S}')$ and that $\langle \Psi, \exp[i\langle \cdot, \xi \rangle] \rangle$ is positive-definite in $\xi \in E$. Then the same argument as in the proof of Theorem 5.1 holds and so, there exists a unique measure ν_Ψ such that

$$\langle \Psi, \psi \rangle = \int_{E^*} \tilde{\psi} \, d\nu_\Psi \qquad \text{for all} \quad \psi \in (\mathscr{S}) .$$

In particular for $\psi \in (\mathscr{S})$ satisfying $\tilde{\psi}(x) \ge 0$ at each $x \in E^*$

$$\langle \Psi, \psi \rangle = \int_{E^*} \tilde{\psi}(y) \, d\nu_\Psi(y) \ge 0 . \quad \square$$

§ 6. Concluding remarks and examples

We conclude by giving some examples of positive functionals. We consider the velocity process of a Brownian particle in one dimensional space. If we set

$$E_0 = L^2_{real}(\mathbf{R}), \qquad D = 1 + u^2 - \left(\frac{d}{du}\right)^2 \quad \text{and} \quad (\xi, \eta)_p = \int_{\mathbf{R}} [D^p \xi(u)] D^p \eta(u)\, du\,,$$

it turns out that $E = \mathscr{S}_{real}(\mathbf{R})$ (the real Schwartz space), $E^* = \mathscr{S}'_{real}(\mathbf{R})$ (the space of real tempered distributions), $H = \mathscr{S}_{complex}(\mathbf{R})$ and so on.

(A) The ordinary case.

Let $\Psi \in (L^2)$. *The following* (a), (b), (c), *and* (d) *are equivalent.*

(a) $\Psi \geq 0$.

(b) $\Psi(x) \geq 0$ *μ-a.e.* $x \in E^*$.

(c) $(\mathscr{T}\Psi)(\xi)$ *is a positive-definite function of* $\xi \in E$.

(d) $\langle \Psi, \varphi\bar{\varphi} \rangle \geq 0$ *for all* $\varphi \in (L^2)$. *That is,* Ψ *is multiplicatively positive on* (L^2).

The equivalence of (a), (b), (c) and (d) is easily proved by Bochner-Minlos' theorem.

The measure ν_Ψ corresponding to Ψ satisfies $d\nu_\Psi(x) = \Psi(x)\, d\mu(x)$, i.e., ν_Ψ is absolutely continuous with respect to μ.

(B) The generalized case.

(B.1) The indicator function $I_{[0 \wedge t, 0 \vee t]}$ of the interval $[0 \wedge t, 0 \vee t]$ is approximated in $L^2(\mathbf{R})$ by a sequence of C^∞- and rapidly decreasing functions $\{\xi_n\}_{n=1}^\infty \subset E$. Therefore $\langle x, I_{[0 \wedge t, 0 \vee t]} \rangle = (L^2)\text{-}\lim_{n\to\infty} \langle x, \xi_n \rangle$ is well-defined as a function in (L^2) (see [7]). Let us define the "Brownian motion" $B(t)$:

$$B(t) = B(t, x) = \begin{cases} \langle x, I_{[0,t]} \rangle & \text{if } t \geq 0 \\ -\langle x, I_{[t,0]} \rangle & \text{if } t \leq 0 \end{cases}, \quad (x \in E^*)\,.$$

This definition of the Brownian motion $B(t)$ gives us the Gaussian random measure $W(\cdot)$ which satisfies $W([S, t]) = B(t) - B(s)$ for $s < t$ and it turns out that $\mathfrak{I}_n(f_n)$ is the n-ple Wiener-Itô integral $I_n(f_n)$ of the symmetric function, $f_n \in \mathscr{S}(\mathbf{R}^n)$, with respect to the random measure W.

Let us show that some of the multiplicative renormalizations stated in [20] or [25] are positive.

(i) $:\exp [\lambda \dot{B}(t)]:$ (λ and t are fixed real numbers).

This (generalized) functional is defined as a limit functional

$$\lim_{h\to 0}\left(\mathrm{E}\left\{\exp\left[\lambda\frac{B(t+h)-B(t)}{h}\right]\right\}\right)^{-1}\exp\left[\lambda\frac{B(t+h)-B(t)}{h}\right]\quad\text{in}\quad(\mathscr{S}')$$

and its $\mathscr{T}$-transform is $C(\xi)\exp[i\lambda\xi(t)]$. As is well-known, $C(\xi)\exp[i\lambda\xi(t)]$ is positive-definite in $\xi\in E$.

Accordingly, $\Psi = :\exp[\lambda\dot{B}(t)]:$ is a positive generalized Brownian functional. The measure ν_Ψ corresponding to Ψ is the one such as the distribution of $\langle x, \xi\rangle$ with respect to ν_Ψ is the normal distribution $N(\lambda\xi(t), \|\xi\|_0^2)$. ν_Ψ and μ are mutually singular.

(ii) $\Psi = :\exp\left[c\int_{\mathbf{R}}\dot{B}(u)^2\,du\right]:\quad (c<1/2)$.

Let $\triangle=\{\triangle_k\}$ be a partition of a finite interval T into subintervals $\triangle_k$ with the same length $|\triangle|$ and write $\Psi_\triangle=\exp[c\sum_k|\triangle_k B/|\triangle||^2|\triangle|]$, where $\triangle_k B$ is the difference $B(t_k,\cdot)-B(t_{k-1},\cdot)$. The $\mathscr{T}$-transform of $\Psi_\triangle/\mathrm{E}[\Psi_\triangle]\in(L^2)$ is $C(\xi)\exp[c(2c-1)^{-1}\sum_k|\triangle|(\xi, I_{\triangle_k})_0^2]$, which is positive-definite in $\xi\in E$ and converges also to a positive-definite function $C_T(\xi)$ of $\xi\in E$ as $|\triangle|\to 0$:

$$C_T(\xi)=C(\xi)\exp\left[\frac{c}{2c-1}\int_T\xi(u)^2\,du\right].$$

Moreover, $C_T(\xi)$ tends to $\exp[-2^{-1}(1-2c)^{-1}\|\xi\|_0^2]$ as T spreads out to $(-\infty,\infty)$. This functional of ξ is the characteristic functional of the measure of Gaussian white noise with variance $(1-2c)^{-1}$. We can prove that there exists a positive generalized Brownian functional $\Psi\in(\mathscr{S}')$, denoted by $:\exp[c\int_R\dot{B}(u)^2du]:$, the $\mathscr{T}$-transform of which is

$$\langle\Psi, e^{i\langle\cdot,\xi\rangle}\rangle=\exp\left[-\frac{1}{2}\frac{1}{1-2c}\|\xi\|_0^2\right].$$

To see how $\Psi = :\exp[c\int_R\dot{B}(u)^2du]:$ acts on $\Psi=\sum_{n=0}^\infty I_n(f_n)\in(\mathscr{S})$, let us expand $\langle\Psi,\exp[i\langle\cdot,\xi\rangle]\rangle$ in the following way. (We can obtain the Fock representation of Ψ by doing this.)

$$\begin{aligned}\langle\Psi, e^{i\langle\cdot,\xi\rangle}\rangle&=\exp\left[-\frac{1}{2}\frac{1}{1-2c}\|\xi\|_0^2\right]=C(\xi)\exp\left[\frac{c}{2c-1}\|\xi\|_0^2\right]\\&=C(\xi)\sum_{n=0}^\infty\frac{1}{n!}\left(\frac{c}{2c-1}\langle\delta(u-v),\xi\hat{\otimes}\xi(u,v)\rangle\right)^n\\&=\sum_{n=0}^\infty i^{2n}C(\xi)\frac{1}{n!}\left(\frac{c}{1-2c}\right)^n\langle\delta(*-\cdot)^{\hat{\otimes}n},\xi^{\hat{\otimes}2n}\rangle\end{aligned}$$

Thus,

$$\langle \Psi, \psi \rangle = \sum_{n=0}^{\infty} \frac{(2n)!}{n!} \left(\frac{c}{1-2c} \right)^n \iint_{R^n} f_{2n}(u_1, u_1, \cdots, u_n, u_n) du_1 \cdots du_n .$$

As expected, this is certainly the integral of $\tilde{\psi}(x)$ with respect to the Gaussian white noise with variance $(1-2c)^{-1}$.

(B.2) We give another example: δ_x for an arbitrarily fixed $x \in E^*$. As mentioned in Corollary 4.1, this functional belongs to $(\mathscr{S}')$. The $\mathscr{T}$-transform of δ_x is given by $\langle \delta_x, \exp[i\langle \cdot, \xi \rangle] \rangle = \exp[i\langle x, \xi \rangle] (\xi \in E)$. Obviously this is a positive-definite function of $\xi \in E$. Thus δ_x is a positive generalized functional. The measure ν corresponding to δ_x satisfies $\nu(\{x\}) = \nu(E^*) = 1$.

Further, if $x \in E^*$ is fixed as above, we have

$$\begin{aligned} e^{i\langle x, \xi \rangle} &= C(\xi) \exp[i\langle x, \xi \rangle] \exp[2^{-1}\langle \delta(u-v), \xi \hat{\otimes} \xi(u,v) \rangle] \\ &= C(\xi) \left(\sum_{m=0}^{\infty} \frac{i^m}{m!} \langle x \xi \rangle^m \right) \left(\sum_{k=0}^{\infty} \frac{i^k}{k!} \langle 2^{-1} \delta(u-v), \xi \hat{\otimes} \xi(u,v) \rangle^k \right) \\ &= \sum_{n=0}^{\infty} i^n C(\xi) \left(\sum_{m+2k=n} \frac{(-2)^{-k}}{m!k!} \langle x, \xi \rangle^m \langle \delta(u-v), \xi \hat{\otimes} \xi(u,v) \rangle^k \right) \\ &= \sum_{n=0}^{\infty} i^n C(\xi) \left\langle \sum_{k=0}^{[n/2]} \frac{(-2)^{-k}}{(n-2k)!k!} x^{\hat{\otimes}(n-2k)} \hat{\otimes} \{\delta(*-\cdot)\}^{\hat{\otimes}k}, \xi^{\hat{\otimes}n} \right\rangle , \end{aligned}$$

where

$$\begin{aligned} & x^{\hat{\otimes}(n-2k)} \hat{\otimes} \{\delta(*-\cdot)\}^{\hat{\otimes}k}(u_1, \cdots, u_{n-2k}, \cdots, u_n) \\ &= \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}(n)} x(u_{\sigma(1)}) \cdots x(u_{\sigma(n-2k)}) \delta(u_{\sigma(n-2k+1)} - u_{\sigma(n-2k+2)}) \cdots \delta(u_{\sigma(n-1)} - u_{\sigma(n)}) . \end{aligned}$$

From the above we can see how δ_x acts on $\psi \in (\mathscr{S})$. In fact if $\psi(\cdot) = \sum_{n=0}^{\infty} I_n(f_n)(\cdot)$ with $f_n \in \mathscr{S}^{\hat{\otimes}n}$,

$$\langle \delta_x, \psi \rangle = \sum_{n=0}^{\infty} n! \left\langle \sum_{k=0}^{[n/2]} \frac{(-2)^{-k}}{(n-2k)!k!} x^{\hat{\otimes}(n-2k)} \hat{\otimes} \{\delta(*-\cdot)\}^{\hat{\otimes}k}, f_n \right\rangle .$$

On the other hand the definition of δ_x gives $\langle \delta_x, \psi \rangle = \tilde{\psi}(x)$. Hence

$$\tilde{\psi}(x) = \sum_{n=0}^{\infty} \left\langle \sum_{n=0}^{[n/2]} \frac{n!(-2)^{-k}}{(n-2k)!k!} x^{\hat{\otimes}(n-2k)} \hat{\otimes} \{\delta(*-\cdot)\}^{\hat{\otimes}k}, f_n \right\rangle .$$

However we can easily see by direct computation that the n-th term of the right hand side of this equality is equal to $\mathfrak{I}_n(f_n)(x)$ in (4.6).

This fact suggests the validity of Proposition 4.2 from a somewhat different point of view.

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Department of Mathematics,
Faculty of General Education,
Kumamoto University