POSITIVE LINEAR MAPS OF OPERATOR ALGEBRAS

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1. Introduction and basic concepts

1.1. *Introduction.* This paper will be concerned with positive linear maps between C^*-algebras. Motivated by the theory of states and other special maps, two different approaches will be taken. If $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras the set of all positive linear maps of $\mathfrak{A}$ into $\mathfrak{B}$ which carry the identity operator in $\mathfrak{A}$ into a fixed positive operator in $\mathfrak{B}$, is a convex set. The main problem dealt with in this paper will be the study of the extreme points of this convex set. The other approach taken is that of decomposing the maps into the composition of nicely handled ones. A general results of this type is due to Stinespring [20]. Adding a strict positivity condition on the maps he characterized them by being of the form $V^* \varrho V$, where V is a bounded linear map of the underlying Hilbert space into another Hilbert space, and ϱ is a *-representation. Another result of general nature of importance to us is due to Kadison. He showed a Schwarz inequality for positive linear maps between C^*-algebras [11]. Positive linear maps are also studied in [3], [13], [14], and [15].

This paper is divided into eight chapters. In chapter 2 the maps are studied in their most general setting—partially ordered vector spaces. The first section contains the necessary formal definitions and the most general techniques. The last part contains results closely related to what Bonsall calls perfect ideals of partially ordered vector spaces [2]. From chapter 3 on the spaces are C^*-algebras. We first show how close extremal maps are to being multiplicative (Theorem 3.1), and then see that C^*-homomorphisms are extremal (Theorem 3.5), and when the maps generalizing vector states are extremal (Theorem 3.9).

In chapter 4 a geometrical condition stronger than extremality is imposed on the maps. It is shown that for identity preserving maps of an abelian C^*-algebra

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into a matrix algebra, extremality is equivalent to this geometrical condition (Theorem 4.10). It follows that, in this case, the extremal maps are the ones which are "approximately" *-homomorphisms (section 4.3).

In chapter 5 we classify all maps from a C^*-algebra $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{H})$—the bounded operators on the Hilbert space $\mathfrak{H}$—such that the composition of vector states of $\mathfrak{B}(\mathfrak{H})$ and the maps are pure states of $\mathfrak{A}$ (Theorem 5.6). As a consequence of this we find all maps of $\mathfrak{A}$ into a C^*-algebra $\mathfrak{B}$ such that the composition of pure states of $\mathfrak{B}$ and the maps are pure states of $\mathfrak{A}$ (Theorem 5.7). In particular it follows that every C^*-homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ is "locally" either a *-homomorphism or a *-anti-homomorphism (Corollary 5.9).

Chapter 6 is devoted to decomposition theory. Using Stinespring's result we show a general decomposition for positive linear maps (Theorem 6.2). As a consequence it is seen when order-homomorphisms are C^*-homomorphisms (Theorem 6.4). Finally, a Radon-Nikodym theorem is proved (Theorem 6.5). Another aspect of decomposition theory is studied in chapter 7. Using Kadison's Schwarz inequality it is shown that, "locally", every positive linear map is decomposable in a form similar to the decomposition in [20] (Theorem 7.4), and is globally "almost" decomposable (Theorem 7.6).

Finally, in chapter 8 we compute all the extremal identity preserving positive endomorphisms of the 2×2 matrices.

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1.2. *Notation and basic concepts.* A *partially ordered vector space* is a vector space over the reals, V, with a partial ordering given by a set of positive elements, V^+, the so-called "positive cone" of V. When $a-b$ is in V^+ we write $a \geqslant b$. Moreover, if a and b are in V^+ then so are $a+b$ and αb for α a positive real; if $-a$ is also in V^+ then $a=0$. V is a partially ordered vector space with an *order unit* if there exists an element e in V such that for every a in V there exists a positive real α with $-\alpha e \leqslant a \leqslant \alpha e$.

By a *C^*-algebra* we shall mean a complex Banach algebra with a unit and with an operation $A \rightarrow A^*$ satisfying

$$(\alpha A + B)^* = \bar{\alpha} A^* + B^*,\ (AB)^* = B^* A^*,\ (A^*)^* = A \text{ and } \|A^* A\| = \|A^*\|\,\|A\|,$$

whenever A and B are in the algebra and α is a complex number. A *positive* element in a C^*-algebra is one which is self-adjoint ($A = A^*$) and whose spectrum consists of non negative reals. Then the self-adjoint elements in a C^*-algebra form a partially ordered vector space, with the unit as an order unit. A linear map of one C^*-algebra (resp. partially ordered vector space) into another is said to be *positive* if it carries positive elements into positive elements. A *C^*-homomorphism* of a C^*-algebra into another is a positive linear map preserving squares of self-adjoint elements. A **-representation* of a C^*-algebra is a homomorphism ϕ, whose image lies in some $\mathfrak{B}(\mathfrak{H})$, satisfying $\phi(A^*)=\phi(A)^*$. The Gelfand—Neumark Theorem [4] asserts that a C^*-algebra has a faithful norm preserving *-representation as a C^*-algebra of operators on a Hilbert space. For a complete proof see [7].

If $\mathfrak{A}$ is a C^*-algebra acting on a Hilbert space $\mathfrak{H}$, $\mathfrak{A}^-$ denotes the weak closure of $\mathfrak{A}$. If $\mathfrak{A}$ contains the identity operator on $\mathfrak{H}$, $\mathfrak{A}^-$ is a von Neumann algebra. We refer the reader to [3] for questions concerning von Neumann algebras. We denote by M_n the $n \times n$ complex matrices. If $\mathfrak{S}$ is a family of operators acting on a Hilbert space $\mathfrak{H}$ and $\mathfrak{V}$ is a set of vectors in $\mathfrak{H}$ then $[\mathfrak{S}\mathfrak{V}]$ denotes the subspace of $\mathfrak{H}$ generated by vectors of the form Tx with T in $\mathfrak{S}$ and x in $\mathfrak{V}$. Since we identify each subspace with the orthogonal projection on it $[\mathfrak{S}\mathfrak{V}]$ also denotes the projection on this subspace. If $\mathfrak{S}$ is a C^*-algebra then $[\mathfrak{S}\mathfrak{V}]$ is a projection in the commutant $\mathfrak{S}'$ of $\mathfrak{S}$.

2. Maps of partially ordered vector spaces

2.1. *Definitions and basic results.* Let A and B be partially ordered vector spaces. Let a be an order unit for A and b be a positive element in B. We denote by $\mathfrak{D}((A, a), (B, b))$ the set of positive linear maps of A into B which carry a into b. If it is clear which order unit we consider, we also write $\mathfrak{D}(A, B, b)$ instead of $\mathfrak{D}((A, a), (B, b))$, and if b is an order unit we may also write $\mathfrak{D}(A, B)$ when no confusion is possible. It is clear that $\mathfrak{D}((A, a), (B, b))$ is a convex set.

We say a map ϕ in $\mathfrak{D}((A, a), (B, b))$ is *strongly positive* if $\phi^{-1}(x)$ contains a positive element for each positive element x in the image of ϕ. Following [13], if b is an order unit for B, we say ϕ is an *order-homomorphism* if ϕ is strongly positive and the null space of ϕ is linearly generated by positive elements. An injective order-homomorphism is an *order-isomorphism*. If N is the null space of ϕ and ϕ is in $\mathfrak{D}(A, B, b)$, b an order unit for B, then it is immediate that ϕ is strongly positive if and only if the canonical linear isomorphism $A/N \rightarrow B$ is an order-isomorphism. If R denotes the real numbers with 1 as order unit then a *state* of A is a map in $\mathfrak{D}(A, R)$. An extreme point of $\mathfrak{D}(A, R)$ is called a *pure state* of A. A map which is an extreme point of $\mathfrak{D}((A, a), (B, b))$ is said to be *extreme*. If τ and ϕ are positive linear maps of A into B we write $\tau \leqslant \phi$ if $\phi - \tau$ is a positive linear map.

LEMMA 2.1. *Let A and B be partially ordered vector spaces, a an order unit for A, and b a positive element in B. Then a map ϕ in $\mathfrak{D}((A, a), (B, b))$ is extreme if and only if whenever $\tau \in \mathfrak{D}((A, a), (B, \lambda b))$ and $\tau \leqslant \phi$ then $\tau = \lambda \phi$.*

Proof. Suppose $\tau \in \mathfrak{D}(A, B, \lambda b)$, $\tau \leqslant \phi$ and ϕ is extreme. If $\lambda = 0$ then $\tau = 0$ since a is an order unit. Similarly $\lambda = 1$ implies $\tau = \phi$. If $\lambda \neq 0, 1$, then

$$\phi = \lambda(\lambda^{-1}\tau) + (1-\lambda)\,((1-\lambda)^{-1}(\phi - \tau))$$

is the convex sum of two maps in $\mathfrak{D}(A, B, b)$. Since ϕ is extreme $\lambda^{-1}\tau = \phi$. The converse is trivial.

LEMMA 2.2. *Let A, B, and C be partially ordered vector spaces with order units a, b, and c respectively. Let d be a positive element in C, and let ϕ be a map in $\mathfrak{D}(A, B)$.*

(i) *If ϕ is a surjective order-isomorphism and ϱ is a map in $\mathfrak{D}(B, C, d)$ then $\varrho \circ \phi$ is extreme in $\mathfrak{D}(A, C, d)$ if and only if ϱ is extreme.*

(ii) *If ϱ is a surjective order-isomorphism in $\mathfrak{D}(B, C, c)$ then $\varrho \circ \phi$ is extreme in $\mathfrak{D}(A, C, c)$ if and only if ϕ is extreme in $\mathfrak{D}(A, B)$.*

The proof is an immediate consequence of Lemma 2.1.

LEMMA 2.3. *Let A, B, and C be partially ordered vector spaces. Let a be an order unit for A, b an order unit for B, and c a positive element in C. Let ϕ be a surjective order-homomorphism in $\mathfrak{D}(A, B)$. Let ϱ be a map in $\mathfrak{D}(B, C, c)$. Then $\varrho \circ \phi$ is extreme in $\mathfrak{D}(A, C, c)$ if and only if ϱ is extreme.*

Proof. Suppose ϱ is extreme, and let τ be a map in $\mathfrak{D}(A, C, \lambda c)$ such that $\tau \leqslant \varrho \circ \phi$. Let N be the null space of ϕ. Then the null space of τ contains N since N is generated by positive elements. $\phi = \psi \circ i$, where ψ is the canonical order-isomorphism $A/N \to B$ and i is the map $A \to A/N$. Thus $\tau = w \circ i$ with w a linear map $A/N \to C$ such that $w(a+N) = \lambda c$. If $x + N$ is positive in A/N then $\phi(x) \geqslant 0$. Hence there exists a positive element y in A such that $y + N = x + N$. Thus $0 \leqslant \tau(y) = w(x+N) \leqslant \varrho \circ \phi(y) = \varrho \circ \phi(x) = \varrho \circ \psi(x+N)$, so $0 \leqslant w \leqslant \varrho \circ \psi$. Since ϱ is extreme, so is $\varrho \circ \psi$ by Lemma 2.2. Hence $w = \lambda \varrho \circ \psi$. Thus $\tau = w \circ i = \lambda \varrho \circ \psi \circ i = \lambda \varrho \circ \phi$, and $\varrho \circ \phi$ is extreme. The converse is a straightforward application of Lemma 2.1.

LEMMA 2.4. *Let A and B be partially ordered vector spaces with order units a and b respectively. Let ϕ be a map in $\mathfrak{D}(A, B)$. Suppose there exists a separating family $\mathfrak{F}$ of pure states of B such that $f \circ \phi$ is a pure state of A for each f in $\mathfrak{F}$. Then ϕ is extreme.*

The proof is trivial.

DEFINITION 2.5. *If A and B are partially ordered vector spaces with order units then a map ϕ in $\mathfrak{D}(A, B)$ is of class 0 if $f \circ \phi$ is a pure state of A for each pure state f of B.*

It is clear from Lemma 2.4 that if the pure states of B separate points of B then a map of class 0 is extreme. We omit the trivial proof of the following lemma.

LEMMA 2.6. *If A and B are partially ordered vector spaces with order units a and b respectively and ϕ is a pure state of A then the map $x \to \phi(x) b$ is of class 0 in $\mathfrak{D}(A, B)$, and is denoted by ϕ. We say ϕ is a pure state in $\mathfrak{D}(A, B)$.*

2.2. *Perfect ideals.* We recall from [9] that an *order ideal* of a partially ordered vector space is a linear subspace I with the property that x is in I whenever $-y \leqslant x \leqslant y$ for some y in I. By [2] a *perfect ideal* of a partially ordered vector space is an order ideal I such that if x is in I and $\varepsilon > 0$ is given, there exists w_ε in I such that

$$-(\varepsilon a + w_\varepsilon) \leqslant x \leqslant \varepsilon a + w_\varepsilon,$$

where a is the order unit. Bonsall [2] has shown that a state of a partially ordered vector space is pure if and only if its null space is a maximal perfect ideal. We shall establish analogous results for surjective maps of class 0.

PROPOSITION 2.7. *Let A and B be partially ordered vector spaces with order units a and b respectively. Let ϕ be a strongly positive surjective map in $\mathfrak{D}(A, B)$ whose null space is a perfect ideal. Then ϕ is of class 0.*

Proof. If M is a perfect ideal of B then $\phi^{-1}(M)$ is a perfect ideal of A. In fact, let x be in $\phi^{-1}(M)$ and $\varepsilon > 0$. There exists y in M such that $\phi(x) = y$. Let $\delta = \varepsilon/3$. Then there exists $w(=w_\delta)$ in M such that

$$-(\delta b + w) \leqslant y \leqslant \delta b + w.$$

Let $z(=z_\delta)$ in $\phi^{-1}(M)$ be such that $\phi(z) = w$. Then $-\phi(\delta a + z) \leqslant \phi(x) \leqslant \phi(\delta a + z)$. Since ϕ is strongly positive there exist $f(=f_\delta)$ and $g(=g_\delta)$ in N—the null space of ϕ—such

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that $x \leq \delta a + z + f$, and $-x \leq \delta a + z + g$. N is perfect. Hence there exist $h(=h_\delta)$ and $k(=k_\delta)$ in N such that $\pm f \leq \delta a + h$ and $\pm g \leq \delta a + k$. Thus $\pm f$, $\pm g \leq 2\delta a + h + k$. Therefore $\pm x \leq 3\delta a + z + h + k = \varepsilon a + v_\varepsilon$, where $v_\varepsilon = z + h + k$ is in $\phi^{-1}(M) \supset N$. Now let f be a pure state of B. Then its null space M is a maximal perfect ideal. By the above the null space of the state $f \circ \phi$ is $\phi^{-1}(M)$, a maximal perfect ideal of A—maximal because $f \circ \phi$ is a state. Hence ϕ is of class 0.

Remark 2.8. The assumption that ϕ be strongly positive is essential. In Example 8.13 we shall find an example of a bijective map in $\mathfrak{D}(A, B)$ which is not of class 0.

LEMMA 2.9. *Let A and B be partially ordered vector spaces with order units a and b respectively. Let ϕ be a strongly positive surjective map in $\mathfrak{D}(A, B)$. Let I be an order ideal of A containing the null space N of ϕ. Then $\phi(I)$ is an order ideal of B. Moreover, if I is perfect (resp. maximal) then $\phi(I)$ is perfect (resp. maximal).*

Proof. Let $\phi(x)$ be an element in $\phi(I)$. Then x is in I. Suppose $-\phi(x) \leq \phi(y) \leq \phi(x)$. Then $\phi(x) \geq 0$. Hence there exists $w \geq 0$ in A such that $\phi(w) = \phi(x)$, so w is in I. Since $\phi(x-y) \geq 0$ there exists $z \geq 0$ in A such that $\phi(z) = \phi(x-y) = \phi(w-y)$. Similarly there exists $z' \geq 0$ in A such that $\phi(z') = \phi(x+y) = \phi(w+y)$, Hence there exist n and n' in $N \subset I$ such that $w = z + y + n \geq 0$ and $z' = w + y + n' \geq 0$. Thus $w \geq y + n$, and $y \geq -(w+n')$. Hence $-(w+n') \leq y \leq w - n$, where $w + n'$ and $w - n$ are in I. Thus $y \in I$, and $\phi(y) \in \phi(I)$. Since ϕ is surjective and b is not in $\phi(I)$, $\phi(I)$ is an order ideal.

It is straightforward to show that $\phi(I)$ is perfect if I is. If I is maximal let J be a maximal order ideal of B containing $\phi(I)$. J is the null space of a state f of B. Thus $\phi^{-1}(J) \supset \phi^{-1}(\phi(I)) \supset I$ is the null space of the state $f \circ \phi$. Thus $\phi^{-1}(J) = I$, $J = \phi(I)$, and $\phi(I)$ is maximal.

PROPOSITION 2.10. *Let A and B be partially ordered vector spaces with order units a and b respectively. Suppose ϕ in $\mathfrak{D}(A, B)$ is surjective. Then the two conditions below are related as follows:* (i) *implies* (ii)*; if ϕ is strongly positive then* (ii) *implies* (i).

(i) *There exists a separating family $\mathfrak{F}$ of pure states of B such that $f \circ \phi$ is a pure state of A for each f in $\mathfrak{F}$.*

(ii) *The null space of ϕ is the intersection of maximal perfect ideals.*

Proof. Since the null space of a pure state of A is a maximal perfect ideal it is trivial that (i) implies (ii). Suppose ϕ is strongly positive and that (ii) is satisfied. Let N be the null space of ϕ. $N = \bigcap_{M \in \mathfrak{N}} M$, M maximal perfect ideals of A. Then $\bigcap_{M \in \mathfrak{N}} \phi(M) = \{0\}$. Indeed, $M = \phi^{-1}(\phi(M))$ for each M in $\mathfrak{N}$. Thus

$$\phi^{-1}(\bigcap_{M \in \mathfrak{N}} \phi(M)) = \bigcap_{M \in \mathfrak{N}} \phi^{-1}(\phi(M)) = \bigcap_{M \in \mathfrak{N}} M = N.$$

Hence, if x is in $\bigcap_{M \in \mathfrak{N}} \phi(M)$ then $\phi^{-1}(x) \subset N$, so if $y \in \phi^{-1}(x)$ then $x = \phi(y) = 0$. By Lemma 2.9 $\phi(M)$ is a maximal perfect ideal of B for each M in $\mathfrak{N}$. Let $\mathfrak{F}$ be the family of pure states f_M of B with null spaces $\phi(M)$ respectively for each M in $\mathfrak{N}$. Then $\mathfrak{F}$ is separating, and $f_M \circ \phi$, having M as null space, is a pure state of A for each f_M in $\mathfrak{F}$.

We apply the last results to prove a general theorem about perfect ideals. We say a partially ordered vector space A with an order unit is *semi simple* if the states of A separate points, i.e. if and only if the intersection of the maximal order ideals of A is $\{0\}$.

LEMMA 2.11. *Let A be a partially ordered vector space with an order unit. Then A is semi simple if and only if the pure states of A separate points. Moreover, if I is an order ideal of A then I is the intersection of the maximal order ideals containing I if and only if A/I is semi simple.*

Proof. The sufficiency of the first statement is obvious. Suppose A is semi simple. If x is in A let

$$\|x\| = \sup_{f \in \mathfrak{S}} |f(x)|,$$

where $\mathfrak{S}$ is the state space of A, i.e. $\mathfrak{S} = \mathfrak{D}(A, R)$. Since A is semi simple $\| \quad \|$ is a norm. If A^* is the space of all bounded linear functionals on A in the defined norm, the w^*-topology on A^* is the weakest topology on A^* for which the elements in A act as continuous linear functionals on A^*. By Alaoglu's Theorem [1] $\mathfrak{S}$ is w^*-compact. Since $\mathfrak{S}$ is also convex it follows from the Krein–Milman Theorem [17] that $\mathfrak{S}$ is the closed convex hull of its extreme points Hence the pure states of A separate points.

Let I be an order ideal of A. If $I = \bigcap M$, where the M's are maximal order ideals, then, as was shown in the proof of Proposition 2.10, $\bigcap \bar{M} = \{0\}$, where $x \to \bar{x}$ denotes the canonical map $v: A \to \bar{A} = A/I$. This map is strongly positive, so by Lemma 2.9 each $\bar{M}$ is a maximal order ideal of $\bar{A}$, and $\bar{A}$ is semi simple. Conversely, suppose $\bar{A}$ is semi simple. The map v defines a $1-1$ correspondence between maximal order ideals of A containing I and maximal order ideals of $\bar{A}$. Since $\bar{A}$ is semi simple,

$$I = v^{-1}(0) = v^{-1}(\bigcap \{\bar{M} : \bar{M} \text{ is maximal order ideal of } \bar{A}\})$$
$$= \bigcap \{v^{-1}(\bar{M}) : M \text{ is maximal order ideal of } \bar{A}\}$$
$$= \bigcap \{M : M \text{ is maximal order ideal of } A \text{ containing } I\}.$$

The proof is complete.

THEOREM 2.12. *Let A be a partially ordered vector space with an order unit. Let I be a perfect ideal of A. If I is the intersection of the maximal order ideals of A containing I, then I is the intersectiom of the maximal perfect ideals containing it.*

Proof. By Lemma 2.11 the pure states of A/I separate points. By Proposition 2.7 the canonical map $A \to A/I$ is of class 0. Thus by Proposition 2.10 I is the intersection of maximal perfect ideals.

3. Extremal maps of C^*-algebras

If $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras we study the extreme points of the set $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}, B)$ of all positive linear maps of $\mathfrak{A}$ into $\mathfrak{B}$, which carry the identity operator in $\mathfrak{A}$ into the positive operator B in $\mathfrak{B}$. It is immediate that the results in chapter 2 are directly applicable. By the Gelfand-Neumark Theorem each C^*-algebra has a faithful *-representation as a C^*-algebra of operators acting on a Hilbert space. In view of Lemma 2.2, then, it is thus no restriction to state and prove theorems about extremal maps in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}, B)$ in the case when $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras of operators on Hilbert spaces.

In general we cannot, a priori, tell whether there are "many" extreme points in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}, B)$. However, if $\mathfrak{B}$ is a von Neumann algebra then the extreme points generate $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}, B)$. In fact, let t be the point—open topology on the space of linear transformations of $\mathfrak{A}$ into $\mathfrak{B}$, where $\mathfrak{B}$ is taken in the weak topology. By [14] $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}, B)$ is t-compact, and hence is the t-closed convex hull of its extreme points.

3.1. *Properties of extremal maps.* The multiplicative properties of extremal maps are characterized in

THEOREM 3.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras, $\mathfrak{A}$ acting on the Hilbert space $\mathfrak{H}$. Let A' be an operator in the commutant $\mathfrak{A}'$ of $\mathfrak{A}$, and let $(\mathfrak{A}, A')$ be the C^*-algebra generated by $\mathfrak{A}$, A', and A'^*. Suppose ϕ is extreme in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$ and that ϕ has an extension $\bar{\phi}$ to $\mathfrak{D}((\mathfrak{A}, A'), \mathfrak{B})$ with $\bar{\phi}(A')$ in the center of $\mathfrak{B}$. Then $\bar{\phi}(A'A) = \bar{\phi}(A')\phi(A)$ for all A in $\mathfrak{A}$.*

Proof. Let $\mathfrak{C}$ denote the center of $\mathfrak{B}$. We may assume A' is self-adjoint, for if $A' = S + iT$ with S and T self-adjoint, then S and T are in $(\mathfrak{A}, A')$, and $\bar{\phi}(S)$ and $\bar{\phi}(T)$ are in $\mathfrak{C}$ since $\bar{\phi}(A')$ is. If the theorem is established for self-adjoint operators then

$$\bar{\phi}(A' A) = \bar{\phi}((S+iT)A) = \bar{\phi}(SA) + i\bar{\phi}(TA) = \bar{\phi}(S)\phi(A) + i\bar{\phi}(T)\phi(A) = \bar{\phi}(A')\phi(A).$$

If A' is self-adjoint, then, multiplying A' by a scalar, we may assume $\|A'\| < \frac{1}{2}$. Then $\|\bar{\phi}(A')\| < \frac{1}{2}$. By spectral theory $\frac{1}{2}I - A'$ and $\frac{1}{2}I - \bar{\phi}(A')$ are positive invertible operators in $\mathfrak{A}'$ and $\mathfrak{C}$ respectively, and there exists $k > 0$ such that $kI \leqslant \frac{1}{2}I - \bar{\phi}(A')$. Define the map ψ of $\mathfrak{A}$ into $\mathfrak{B}$ by

$$\psi(A) = \bar{\phi}(A(\tfrac{1}{2}I - A'))(\tfrac{1}{2}I - \bar{\phi}(A'))^{-1}.$$

Then clearly $\psi \in \mathfrak{D}(\mathfrak{A}, \mathfrak{B})$. With $B \geqslant 0$ in $\mathfrak{A}$, then

$$k\psi(B) \leqslant (\tfrac{1}{2}I - \bar{\phi}(A'))\psi(B) = \bar{\phi}(B(\tfrac{1}{2}I - A')) \leqslant \phi(B).$$

Thus $k\psi \leqslant \phi$. Since ϕ is extreme, $\psi = \phi$. Thus

$$(\tfrac{1}{2}I - \bar{\phi}(A'))\phi(A) = \bar{\phi}(A(\tfrac{1}{2}I - A')) = \tfrac{1}{2}\phi(A) - \bar{\phi}(AA'),$$

and $\bar{\phi}(A')\phi(A) = \bar{\phi}(A'A)$ for all A in $\mathfrak{A}$. The proof is complete.

Employing techniques similar to those used to prove Theorem 3.1 we show the following improvement over Lemma 2.1.

PROPOSITION 3.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras, and let ϕ be a map in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$. Let τ be a positive linear map of $\mathfrak{A}$ into $\mathfrak{B}$ such that $\tau(I)$ is in the center of $\mathfrak{B}$ and $\tau \leqslant \phi$. If ϕ is extreme then $\tau = \tau(I)\phi$.*

Proof. Multiplying τ by a scalar we may assume $\|\tau(I)\| < 1$. Hence, by spectral theory, $(I - \tau(I))$ is a positive invertible operator in the center of $\mathfrak{B}$, and there exists $k > 0$ such that $k(I - \tau(I))^{-1} \leqslant I$. Thus, if A is a positive operator in $\mathfrak{A}$ then

$$0 \leqslant k(I - \tau(I))^{-1}(\phi(A) - \tau(A)) \leqslant (\phi(A) - \tau(A)) \leqslant \phi(A),$$

and

$$k(I - \tau(I))^{-1}(\phi - \tau) \in \mathfrak{D}(\mathfrak{A}, \mathfrak{B}, kI).$$

Since ϕ is extreme $(I - \tau(I))^{-1}(\phi - \tau) = \phi$, and $\tau = \tau(I)\phi$.

COROLLARY 3.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras. Let B be a positive operator in the center of $\mathfrak{B}$. If ϕ is extreme in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$ then the map $A \to B\phi(A)$ is extreme in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}, B)$.*

Proof. Denote the map by $B\phi$. If τ is a map in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}, \lambda B)$ and $\tau \leqslant B\phi$, then $\tau \leqslant \|B\| \phi$, so by Proposition 3.2 $\tau = \tau(I)\phi = \lambda B\phi$.

R. V. Kadison pointed out the following result to us.

PROPOSITION 3.4. *Let $\mathfrak{B}$ be an abelian von Neumann algebra and $\mathfrak{A}$ a C^*-algebra acting on a Hilbert space $\mathfrak{H}$. Let ϕ be extreme in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$. Then there exists an extreme extension $\bar{\phi}$ of ϕ to $\mathfrak{D}(\mathfrak{B}(\mathfrak{H}), \mathfrak{B})$.*

Proof. By [15, Lemma 3] each map in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$ has an extension to $\mathfrak{D}(\mathfrak{B}(\mathfrak{H}), \mathfrak{B})$. Let $\mathfrak{C}$ be the set of extensions of ϕ. Then $\mathfrak{C}$ is a closed convex subset of $\mathfrak{D}(\mathfrak{B}(\mathfrak{H}), \mathfrak{B})$. Since $\mathfrak{B}$ is a von Neumann algebra, $\mathfrak{D}(\mathfrak{B}(\mathfrak{H}), \mathfrak{B})$ is compact. Let $\bar{\phi}$ be an extreme point of $\mathfrak{C}$. It is straightforward to show that $\bar{\phi}$ is extreme in $\mathfrak{D}(\mathfrak{B}(\mathfrak{H}), \mathfrak{B})$.

3.2. *Special extremal maps.* An important class of maps in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$ are the C^*-homomorphisms—maps ϕ such that $\phi(A^2) = \phi(A)^2$ whenever A is a self-adjoint operator. The argument of the following theorem is taken from [9, Lemma 3.2].

THEOREM 3.5. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras and ϕ a C^*-homomorphism in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$. Then ϕ is extreme.*

Proof. We show that if ϕ is not extreme then ϕ is not a C^*-homomorphism. By [11, Theorem 1] $\tau(A^2) \geqslant \tau(A)^2$ for A self-adjoint, whenever $\tau \in \mathfrak{D}(\mathfrak{A}, \mathfrak{B})$. Suppose $\phi = \frac{1}{2}(\varrho + \psi)$ with ϱ and ψ in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$, and suppose $\varrho \neq \psi$. Let A be a self-adjoint operator in $\mathfrak{A}$ such that $\varrho(A) \neq \psi(A)$. Then

$$\phi(A)^2 = \tfrac{1}{4}(\varrho(A) + \psi(A))^2 = \tfrac{1}{2}(\varrho(A)^2 + \psi(A)^2) - \tfrac{1}{4}(\varrho(A) - \psi(A))^2$$
$$< \tfrac{1}{2}(\varrho(A)^2 + \psi(A)^2) \leqslant \tfrac{1}{2}(\varrho(A^2) + \psi(A^2)) = \phi(A^2).$$

The proof is complete.

Combining Theorem 3.1 and Theorem 3.5 we obtain the following result, announced in [8].

COROLLARY 3.6. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be abelian C^*-algebras. Then the extreme points in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$ are the *-homomorphisms in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$.*

Remark 3.7. If $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras and ϕ is a positive linear map of $\mathfrak{A}$ into $\mathfrak{B}$ then the set $\mathfrak{J}$ (resp. $\mathfrak{L}$) of A in $\mathfrak{A}$ for which $\phi(A^*A) = 0$ (resp. $\phi(AA^*) = 0$), is a left (resp. right) ideal in $\mathfrak{A}$—the *left* (resp. *right*) *kernel* of ϕ. In fact, $\mathfrak{J}$ (resp. $\mathfrak{L}$)

is the intersection of the left (resp. right) kernels of the positive linear functionals $f \circ \phi$ as f runs through a separating family of states of $\mathfrak{B}$.

If $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras acting on Hilbert spaces $\mathfrak{K}$ and $\mathfrak{H}$ respectively and B is a positive operator in $\mathfrak{B}$ then the maps in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}, B)$ which are analogous to vector states are the ones of the form $A \to V^*AV$, where V is a bounded linear map of $\mathfrak{H}$ into $\mathfrak{K}$.

THEOREM 3.8. *Let $\mathfrak{A}$ be a C^*-algebra acting on the Hilbert space $\mathfrak{K}$. Let $\mathfrak{H}$ be a Hilbert space and V a bounded linear map of $\mathfrak{H}$ into $\mathfrak{K}$. Suppose there exist a projection E' in $\mathfrak{A}'$ and an operator S in $\mathfrak{A}^-$ such that $VV^* = SE'$. Then the map $A \to V^*AV$ is extreme in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}(\mathfrak{H}), V^*V)$.*

Proof. Let $V^* \cdot V$ denote the map $A \to V^*AV$. Then $V^* \cdot V$ is the composition of the homomorphism $A \to AE'$ of $\mathfrak{A}$ onto $\mathfrak{A}E'$ and the map $AE' \to V^*AE'V = V^*AV$. By Lemma 2.3 $V^* \cdot V$ is extreme if and only if the map $AE' \to V^*AE'V$ is extreme in $\mathfrak{D}(\mathfrak{A}E', \mathfrak{B}(\mathfrak{H}), V^*V)$. We may thus assume $E' = I$ and $VV^* = S \in \mathfrak{A}^-$. Suppose the theorem is proved in the case when $\mathfrak{A}$ is a von Neumann algebra. We show it is then true with $\mathfrak{A}$ a C^*-algebra. Let $\psi \in \mathfrak{D}(\mathfrak{A}, \mathfrak{B}(\mathfrak{H}), \lambda V^*V)$ satisfy $\psi \leqslant V^* \cdot V$. If ω_x is a vector state on $\mathfrak{B}(\mathfrak{H})$ then $\omega_x \circ \psi \leqslant \omega_{Vx}$, so $\omega_x \circ \psi$ is weakly continuous [3, p. 50]. Note that the map $V^* \cdot V$ is weakly continuous. By [13. Remark 2.2.3.] ψ has a (unique) positive linear extension mapping ψ^- of $\mathfrak{A}^-$ into $\mathfrak{B}(\mathfrak{H})$, which is weakly continuous on the unit sphere in $\mathfrak{A}^-$, and $0 \leqslant \psi^- \leqslant V^* \cdot V$ on $\mathfrak{A}^-$. By assumption the theorem holds for $V^* \cdot V$ on $\mathfrak{A}^-$. Hence $\psi^- = \lambda V^* \cdot V$ on $\mathfrak{A}^-$. Thus $\psi = \psi^- \mid \mathfrak{A} = \lambda V^* \cdot V$ on $\mathfrak{A}$, and the theorem is proved for $\mathfrak{A}$ a C^*-algebra.

We assume $\mathfrak{A}$ is a von Neumann algebra acting on $\mathfrak{K}$. If S is a positive operator in $\mathfrak{A}$ such that 0 is an isolated point of $\{0\} \cup \sigma(S)$, where $\sigma(S)$ denotes the spectrum of S, then $S \cdot S$ is extreme in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}(\mathfrak{K}), S^2)$. In fact, let $\psi \in \mathfrak{D}(\mathfrak{A}, \mathfrak{B}(\mathfrak{K}), \lambda S^2)$ satisfy $\psi \leqslant S \cdot S$. Let P be the range projection of S. Then $P \in \mathfrak{A}$. With $0 \leqslant A \leqslant I$ in $\mathfrak{A}$

$$0 \leqslant \psi(A) \leqslant \psi(I) = \lambda S^2 \leqslant \lambda \|S^2\| P,$$

so that $\psi(A) = P\psi(A)P$. Thus $\psi = P\psi P$. Moreover,

$$0 \leqslant \psi(I-P) \leqslant S(I-P)S = 0.$$

Thus $I - P$ is in the left and right kernels of ψ (Remark 3.7). Thus $\psi(A) = \psi(PAP)$ for all A in $\mathfrak{A}$, and ψ restricts to a map ψ_p in $\mathfrak{D}(P\mathfrak{A}P, P\mathfrak{B}(\mathfrak{K})P, \lambda S^2)$. If we show $\psi_p = \lambda S \cdot S$ on $P\mathfrak{A}P$, then for A in $\mathfrak{A}$,

$$\psi(A) = \psi(PAP) = \psi_p(PAP) = \lambda SAS,$$

and $S \cdot S$ is extreme. We may thus assume $P = I$ and S invertible. With $A \geqslant 0$ in $\mathfrak{A}$, $\psi(A) \leqslant SAS$, so $S^{-1}\psi(A)S^{-1} \leqslant A$, and $S^{-1}\psi(I)S^{-1} = \lambda I$. The identity map of $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{K})$ is extreme in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}(\mathfrak{K}), I)$ by Theorem 3.5. Thus $S^{-1}\psi(A)S^{-1} = \lambda A$ for all A in $\mathfrak{A}$, and $\psi = \lambda S \cdot S$ as asserted.

Let S be any positive operator in $\mathfrak{A}$. Let ψ be a map as above. For n a positive integer let E_n be the spectral projection of S corresponding to the set

$$\{\lambda \in \sigma(S) : \lambda > 1/n\}^{-}.$$

As $n \to \infty$ (E_n) converges to P—the range projection of S—strongly. Note that 0 is an isolated point of $\{0\} \cup \sigma(E_n S)$. Since $\mathfrak{A}$ is a von Neumann algebra $E_n \in \mathfrak{A}$ and $E_n S \geqslant 0$. By the last paragraph $E_n S \cdot E_n S$ is extreme in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}(\mathfrak{K}), E_n S^2)$, and $E_n \psi E_n \leqslant E_n S \cdot E_n S$. Thus $E_n \psi E_n = \lambda E_n S \cdot E_n S$. If $\mathfrak{A}_1$ is the unit ball in $\mathfrak{A}$ the map $(A, B) \to AB$ is a strongly continuous map of $\mathfrak{A}_1 \times \mathfrak{A}$ into $\mathfrak{A}$ [3. p. 32]. Thus the map $(B, A) \to BAB$ is a strongly continuous map of $\mathfrak{A}_1 \times \mathfrak{A}_1$ into $\mathfrak{A}_1$. Since $E_n \to P$ strongly $E_n A E_n \to PAP$ strongly. Hence, with A in $\mathfrak{A}_1$,

$$\begin{aligned}\psi(A) = P\psi(A)P &= \text{strong limit } (E_n \psi(A) E_n) \\ &= \text{strong limit } (\lambda E_n SASE_n) = \lambda PSASP = \lambda SAS.\end{aligned}$$

Thus $\psi = \lambda S \cdot S$, and $S \cdot S$ is extreme in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}(\mathfrak{K}), S^2)$ when $S \geqslant 0$ in $\mathfrak{A}$.

In the general case V is a bounded linear map of $\mathfrak{H}$ into $\mathfrak{K}$ and $VV^* \in \mathfrak{A}$. Let $\psi \in \mathfrak{D}(\mathfrak{A}, \mathfrak{B}(\mathfrak{H}), \lambda V^* V)$ satisfy $\psi \leqslant V^* \cdot V$. Then $V\psi V^* \leqslant VV^* \cdot VV^*$, and

$$V\psi V^* \in \mathfrak{D}(\mathfrak{A}, \mathfrak{B}(\mathfrak{K}), \lambda (VV^*)^2).$$

By the last paragraph $V\psi V^* = \lambda VV^* \cdot VV^*$. Again, with P the range projection of V^*, $\psi = P\psi P$, and the set $V^*(\mathfrak{K})$ is dense in P. With x and y in $\mathfrak{K}$ and $A \in \mathfrak{A}$,

$$0 = (V(\psi(A) - \lambda V^* A V) V^* x, y) = ((\psi(A) - \lambda V^* A V) V^* x, V^* y),$$

so by continuity, $((\psi(A) - \lambda V^* A V) w, z) = 0$ for all w, z in $\mathfrak{H}$ and all A in $\mathfrak{A}$. Thus $\psi = \lambda V^* \cdot V$, and $V^* \cdot V$ is extreme. The proof is complete.

Note that if $\mathfrak{A}$ is a C^*-algebra acting on the Hilbert space $\mathfrak{K}$ and containing the identity operator on $\mathfrak{K}$, and if x is a unit vector in $\mathfrak{K}$, then ω_x is pure on $\mathfrak{A}$ if and only if $[x] = [\mathfrak{A} x][\mathfrak{A}' x]$. We generalize this as follows.

THEOREM 3.9. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be Hilbert spaces and V be an isometry of $\mathfrak{H}$ into $\mathfrak{K}$. Let $\mathfrak{A}$ be a C^*-algebra acting on $\mathfrak{K}$ and containing the identity operator on $\mathfrak{K}$. Suppose $V^* \mathfrak{A}' V \subset (V^* \mathfrak{A} V)''$. Then the following three conditions are equivalent.*

(i) (*resp.* (ii)) *The map* $A \to V^* A V$ *is extreme in* $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}(\mathfrak{H}))$ (*resp.* $\mathfrak{D}(\mathfrak{A}, (V^*\mathfrak{A}V)''))$ *and* $V^*\mathfrak{A}'V \subset (V^*\mathfrak{A}V)'$.

(iii) $VV^* = [\mathfrak{A}V(\mathfrak{H})][\mathfrak{A}'V(\mathfrak{H})]$.

Proof. Clearly (i) implies (ii). Assume (ii). The operator $P = VV^*$ is a projection in $\mathfrak{B}(\mathfrak{K})$. By assumption $V^*\mathfrak{A}'V$ is contained in the center of the von Neumann algebra $(V^*\mathfrak{A}V)''$. Let $E = [\mathfrak{A}V(\mathfrak{H})]$. Then $E \in \mathfrak{A}'$ and $P \leqslant E$. Assume for the moment that $E = I$. Then vectors of the form $y = AVx$, with A in $\mathfrak{A}$ and x in $\mathfrak{H}$, generate a dense linear manifold in $\mathfrak{K}$, and if $A' \in \mathfrak{A}'$ then

$$PA'y = PA'AVx = VV^*A'PAVx = PA'Py,$$

using Theorem 3.1. Thus $PA' = PA'P = A'P$ for all self-adjoint A' in $\mathfrak{A}'$. Thus $P \in \mathfrak{A}''$. Thus $P = [\mathfrak{A}'P]$. In the general case $V^* \cdot V$ is the composition of the maps $A \to AE \to V^*AEV (= V^*AV)$. By Lemma 2.3 the second map is extreme in $\mathfrak{D}(\mathfrak{A}E, (V^*\mathfrak{A}V)'')$. By the above and [3. Proposition 1. p. 18]

$$P = [(\mathfrak{A}E)'P] = [E\mathfrak{A}'EP] = E[\mathfrak{A}'P] = [\mathfrak{A}P][\mathfrak{A}'P].$$

Now suppose $P = [\mathfrak{A}P][\mathfrak{A}'P]$. Since $[\mathfrak{A}P] \in \mathfrak{A}'$ and $[\mathfrak{A}'P] \in \mathfrak{A}'' = \mathfrak{A}^-$, $V^* \cdot V$ is extreme in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}(\mathfrak{H}))$ by Theorem 3.8. Let $A \in \mathfrak{A}$ and $A' \in \mathfrak{A}'$. Then

$$\begin{aligned}(V^*A'V)(V^*AV) &= V^*A'PAV = V^*A'[\mathfrak{A}'P][\mathfrak{A}P]AV = V^*[\mathfrak{A}'P]A'A[\mathfrak{A}P]V \\ &= V^*AA'V = V^*[\mathfrak{A}P]AA'[\mathfrak{A}'P]V = V^*AV^*VA'V,\end{aligned}$$

so that $V^*\mathfrak{A}'V \subset (V^*\mathfrak{A}V)'$. (i) holds, and the proof is complete.

3.3. *Classes of extremal maps.* We distinguish the extreme points in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}(\mathfrak{H}))$ into classes, one class for each ordinal number less than or equal to the dimension of $\mathfrak{H}$.

DEFINITION 3.10. *Let* $\mathfrak{F}$ *be a family of projections in* $\mathfrak{B}(\mathfrak{H})$, *where* $\mathfrak{H}$ *is a Hilbert space. Then* $\mathfrak{F}$ *is a separating family of dimension* α *if*

(i) dim $Q = \alpha$ *for each* Q *in* $\mathfrak{F}$.

(ii) *for each projection* P *in* $\mathfrak{B}(\mathfrak{H})$ *such that* dim $P > \alpha$ *and each vector* x *in* P *there exists* Q *in* $\mathfrak{F}$ *such that* $x \in Q$ *and* $Q \leqslant P$.

(iii) *if* $\alpha = \dim \mathfrak{H}$ *then for each* $x \in \mathfrak{H}$ *there exists* $Q \in \mathfrak{F}$ *such that* $x \in Q$.

PROPOSITION 3.11. *Let* $\mathfrak{A}$ *be a* C^**-algebra and* $\mathfrak{H}$ *a Hilbert space. Let* ϕ *be a map in* $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}(\mathfrak{H}))$. *Let* $\mathfrak{F}$ *be a separating family of projections of dimension* α *in* $\mathfrak{B}(\mathfrak{H})$ *such that the map* $Q\phi Q$ *is extreme in* $\mathfrak{D}(\mathfrak{A}, Q\mathfrak{B}(\mathfrak{H})Q)$ *for each* Q *in* $\mathfrak{F}$. *Then* ϕ *is extreme. If* ϕ *is extreme in* $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}(\mathfrak{H}))$ *then there exists a minimal ordinal num-*

ber α such that there exists a separating family of dimension α in $\mathfrak{B}(\mathfrak{H})$ such that the map $Q\phi Q$ is extreme in $\mathfrak{D}(\mathfrak{A}, Q\mathfrak{B}(\mathfrak{H})Q)$ for each Q in it. We say ϕ is extreme of class α.

Proof. Let τ be a map in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}(\mathfrak{H}), \lambda I)$ such that $\tau \leqslant \phi$. Then for each Q in $\mathfrak{F}$, $Q\tau Q = \lambda Q\phi Q$, since $Q\phi Q$ is extreme. If S is a self-adjoint operator in $\mathfrak{A}$ then $T = (\tau - \lambda\phi)(S)$ is self-adjoint in $\mathfrak{B}(\mathfrak{H})$ and $QTQ = 0$ for each Q in $\mathfrak{F}$. If $x \in \mathfrak{H}$ then there exists Q in $\mathfrak{F}$ such that $x \in Q$. Thus $(Tx, x) = (TQx, Qx) = (QTQx, x) = 0$, and $T = 0$. Thus $\tau = \lambda\phi$, and ϕ is extreme. If P is a projection in $\mathfrak{B}(\mathfrak{H})$ of dimension greater than α then, similarly, $P\phi P$ is extreme in $\mathfrak{D}(\mathfrak{A}, P\mathfrak{B}(\mathfrak{H})P)$. Suppose ϕ is an extremal map in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}(\mathfrak{H}))$. The family consisting of the identity operator alone is a separating family of dimension equal to dim $\mathfrak{H}$, and the map $I\phi I = \phi$ is extreme in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}(\mathfrak{H}))$. By the above there exists a minimal ordinal number α and a separating family $\mathfrak{F}$ of projections of dimension α in $\mathfrak{B}(\mathfrak{H})$ such that the map $Q\phi Q$ is extreme in $\mathfrak{D}(\mathfrak{A}, Q\mathfrak{B}(\mathfrak{H})Q)$ for all Q in $\mathfrak{F}$.

4. Geometrical conditions

We impose a geometrical condition on the maps in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$ and show that this condition is closely related to, however, is not in general equivalent to, extremality (see Example 8.13).

4.1. *Definition and basic properties.*

DEFINITION 4.1. *Let $\mathfrak{B}$ be a C^*-algebra acting on a Hilbert space $\mathfrak{H}$. If A is an operator in $\mathfrak{B}$, $r(A)$ denotes the range projection of A and $n(A)$ the null space of A. If $\mathfrak{A}$ is a C^*-algebra and $\phi \in \mathfrak{D}(\mathfrak{A}, \mathfrak{B}, B)$, B a positive operator in $\mathfrak{B}$, we denote by $r(\phi)$ the map of $\mathfrak{A}$ into $\mathfrak{B}^-$ given by $A \to r(\phi(A))$. If τ is another map in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}, B)$ we say $r(\tau) \leqslant r(\phi)$ if $r(\tau(A)) \leqslant r(\phi(A))$ for each positive operator A in $\mathfrak{A}$. ϕ is said to have minimal range if whenever τ is in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}, B)$ and $r(\tau) \leqslant r(\phi)$ then $\tau = \phi$.*

It is immediate that a map ϕ of minimal range in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}, B)$ is extreme. In fact, if $\tau \in \mathfrak{D}(\mathfrak{A}, \mathfrak{B}, \lambda B)$, $\lambda \neq 0$, and $\tau \leqslant \phi$, then $\lambda^{-1}\tau \in \mathfrak{D}(\mathfrak{A}, \mathfrak{B}, B)$ and $r(\lambda^{-1}\tau) \leqslant r(\phi)$, so $\lambda^{-1}\tau = \phi$, and ϕ is extreme.

Remark 4.2. If $\mathfrak{A}$ and $\mathfrak{B}$ are as above, $\mathfrak{C}$ the convex hull of the maps in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$ having minimal range, then ϕ in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$ has minimal range if and only if ϕ is in $\mathfrak{C}$ and if $\tau \in \mathfrak{C}$ with $r(\tau) \leqslant r(\phi)$ then $r(\tau) = r(\phi)$. The necessity is obvious. Conversely, if $\{A_\alpha\}_{\alpha \in I}$ are positive operators in $\mathfrak{B}$ then

$$n\,(\sum_{\alpha\in I} A_\alpha) = \bigwedge_{\alpha\in I} n(A_\alpha),$$

so
$$r\,(\sum_{\alpha\in I} A_\alpha) = I - \bigwedge_{\alpha\in I} n(A_\alpha) = \bigvee_{\alpha\in I} (I - n(A_\alpha)) = \bigvee_{\alpha\in I} r(A_\alpha).$$

Let $\phi = \sum_{i=1}^{n} \lambda_i \phi_i$ be a convex sum of extreme maps in $\mathfrak{C}$ satisfying the conditions above. Then for A positive in $\mathfrak{A}$,

$$r(\phi(A)) = r\,(\sum_{i=1}^{n} \lambda_i \phi_i(A)) = \bigvee_{i=1}^{n} r(\phi_i(A)),$$

since $\lambda_i \neq 0$. Therefore $r(\phi_i) \leqslant r(\phi)$ for $i = 1, \ldots, n$, so by assumption $r(\phi_i) = r(\phi)$, and $\phi_i = \phi$. Thus ϕ has minimal range.

LEMMA 4.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras, $\mathfrak{B}$ acting on the Hilbert space $\mathfrak{H}$. Then every pure state in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$ has minimal range.*

Proof. Let ϕ be a pure state in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$. Let $\mathfrak{J}$ be its left kernel. By [12] the null space of ϕ is $\mathfrak{J} + \mathfrak{J}^*$. If $\tau \in \mathfrak{D}(\mathfrak{A}, \mathfrak{B})$ and $r(\tau) \leqslant r(\phi)$ then the left kernel of τ contains that of ϕ, and hence the null space of τ contains that of ϕ. Thus τ is a state and equals ϕ.

PROPOSITION 4.4. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{H}$ a Hilbert space. Let ϕ be a map in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}(\mathfrak{H}))$. Suppose $P\phi P$ has minimal range in $\mathfrak{D}(\mathfrak{A}, P\mathfrak{B}(\mathfrak{H})P)$ for each projection P in a separating family $\mathfrak{F}$ of dimension α. Then ϕ has minimal range. In particular, if ϕ is of class 1 then ϕ has minimal range.*

Proof. Let P be a projection in $\mathfrak{F}$. If A and B are positive operators in $\mathfrak{B}(\mathfrak{H})$ and $r(A) \leqslant r(B)$ then $n(A) \geqslant n(B)$, so $n(PAP) \geqslant n(PBP)$, and $r(PAP) \leqslant r(PBP)$. Thus, if $\tau \in \mathfrak{D}(\mathfrak{A}, \mathfrak{B}(\mathfrak{H}))$ and $r(\tau) \leqslant r(\phi)$ then $r(P\tau P) \leqslant r(P\phi P)$, and $P\tau P = P\phi P$ for each P in $\mathfrak{F}$. Thus $\tau = \phi$. Suppose ϕ is of class 1. If x is a unit vector in $\mathfrak{H}$ then $[x]\phi[x]$ $(= [x]\omega_x \circ \phi)$ has minimal range in $\mathfrak{D}(\mathfrak{A}, [x]\mathfrak{B}(\mathfrak{H})[x])$ by Lemma 4.3. Thus ϕ has minimal range.

THEOREM 4.5. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras, $\mathfrak{B}$ acting on the Hilbert space $\mathfrak{H}$. Let ϕ be a map in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$, and suppose there exists a subset $\mathfrak{S}$ of $\mathfrak{A}$ consisting of positive operators of norm less than or equal to 1 satisfying the following conditions:*

(i) *There exists $k > 0$ such that $\phi(S)^2 \geqslant k\,\phi(S)$ for all S in $\mathfrak{S}$.*

(ii) *If A is positive in $\mathfrak{A}$ and $\varepsilon > 0$ is given, then there exist $A_1, \ldots, A_n$ in $\mathfrak{S}$ and positive real numbers $a_1, \ldots, a_n$ such that $\|\sum_{i=1}^{n} a_i A_i - A\| < \varepsilon$.*

Then if $\tau \in \mathfrak{D}(\mathfrak{A}, \mathfrak{B})$ *and* $r(\tau) \leqslant r(\phi)$ *then* $k\tau \leqslant \phi$. *In particular, if* ϕ *is extreme then* ϕ *has minimal range.*

Proof. If A is positive in $\mathfrak{A}$ we have to show $k\tau(A) \leqslant \phi(A)$. Assume first that $A \in \mathfrak{S}$. Consider the von Neumann algebras acting on $\mathfrak{H}$ generated by $\tau(A)$ and I, and $\phi(A)$ and I, respectively. Let $\delta > 0$ be given and let $\sum_{s=1}^{n} a_s F_s$ and $\sum_{j=1}^{m} b_j G_j$ be approximations to $\tau(A)$ and $\phi(A)$, respectively, with

$$\sum_{s=1}^{n} F_s = r(\tau(A)), \quad \sum_{j=1}^{m} G_j = r(\phi(A)),$$

$a_s \neq 0$ in $\sigma(\tau(A))$, $b_j \neq 0$ in $\sigma(\phi(A))$, and such that

$$\left\| \sum_{s=1}^{n} a_s F_s - \tau(A) \right\| \leqslant \delta \text{ and } \left\| \sum_{j=1}^{m} b_j G_j - \phi(A) \right\| \leqslant \delta.$$

Since $\phi(A)^2 \geqslant k\phi(A)$ it follows by spectral theory that $b_j \geqslant k$. In fact, if $b \neq 0$ is in $\sigma(\phi(A))$ then $b^2 \geqslant kb$, so $b \geqslant k$, since $\phi(A)$ goes into the identity function in $C(\sigma(\phi(A)))$ by the canonical isomorphism. Moreover, $\|A\| \leqslant 1$, so $0 < a_s \leqslant 1$. Thus

$$\begin{aligned}
\phi(A) - k\tau(A) &= (\phi(A) - \sum_j b_j G_j) + (\sum_j b_j G_j - k \sum_s a_s F_s) + k(\sum_s a_s F_s - \tau(A)) \\
&\geqslant -\delta + (\sum_j kG_j - k \sum_s F_s) - k\delta \\
&= -(1+k)\delta + k(r(\phi(A)) - r(\tau(A))) \\
&\geqslant -(1+k)\delta.
\end{aligned}$$

Since δ is arbitrary, $\phi(A) - k\tau(A) \geqslant 0$. Now let A be any positive operator in $\mathfrak{A}$. Let $\varepsilon > 0$ be given. Then there exist $A_1, \ldots, A_n$ in $\mathfrak{S}$ and $c_i > 0$, $i = 1, \ldots, n$, such that $\|\sum_{i=1}^{n} c_i A_i - A\| < \varepsilon$, and $k\tau(A_i) \leqslant \phi(A_i)$ by the preceding. Thus

$$\begin{aligned}
\phi(A) - k\tau(A) &= \phi(A - \sum_i c_i A_i) + \sum_i c_i(\phi(A_i) - k\tau(A_i)) - k\tau(A - \sum_i c_i A_i) \\
&\geqslant -\varepsilon - k\varepsilon = -(1+k)\varepsilon.
\end{aligned}$$

Since ε is arbitrary, $\phi(A) \geqslant k\tau(A)$, and $\phi \geqslant k\tau$. The proof is complete.

COROLLARY 4.6. *Let* $\mathfrak{A}$ *be a von Neumann algebra and* $\mathfrak{B}$ *a* C^**-algebra acting on the Hilbert space* $\mathfrak{H}$. *If* $\phi \in \mathfrak{D}(\mathfrak{A}, \mathfrak{B})$ *and there exists* $k > 0$ *such that* $\phi(E)^2 \geqslant k\phi(E)$ *for each projection* E *in* $\mathfrak{A}$, *then if* τ *is in* $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$ *and* $r(\tau) \leqslant r(\phi)$, *then* $k\tau \leqslant \phi$. *In particular, if* ϕ *is a* C^**-homomorphism then* ϕ *has minimal range.*

Proof. Let $\mathfrak{S}$ be the subset of $\mathfrak{A}$ consisting of all projections in $\mathfrak{A}$. Then $\mathfrak{S}$ satisfies the conditions of Theorem 4.5. If ϕ is a C^*-homomorphism then ϕ is extreme by Theorem 3.5, and $\phi(E)^2 = \phi(E)$, so ϕ has minimal range.

Example 4.7. Let Z be the integers and μ the Lebesgue measure, $\mu(n)=1$ for all $n \in Z$. As in [15] let $\mathfrak{A}_d$ be the "discrete algebra", i.e. the maximal abelian von Neumann algebra consisting of operators T_g, with g an essentially bounded μ-measurable function on Z, where $T_g(h)=gh$ for each h in $L_2(Z,\mu)$. In the terminology of [15] let $\mathfrak{D}$ be the unique diagonal process relative to $\mathfrak{A}_d$ [15, Theorem 1]. If $\mathfrak{A}_d$ acts on $\mathfrak{H}$ then $\mathfrak{D}$ has minimal range in $\mathfrak{D}(\mathfrak{B}(\mathfrak{H}),\mathfrak{A}_d)$. In fact, let τ be a map in $\mathfrak{D}(\mathfrak{B}(\mathfrak{H}),\mathfrak{A}_d)$ such that $r(\tau)\leqslant r(\mathfrak{D})$. Then in particular $r(\tau|\mathfrak{A}_d)\leqslant r(\mathfrak{D}|\mathfrak{A}_d)$. Since $\mathfrak{D}|\mathfrak{A}_d$ is the identity map $\tau|\mathfrak{A}_d=\mathfrak{D}|\mathfrak{A}_d$ by Corollary 4.6. Since $\mathfrak{D}$ is the unique positive extension of the identity map of $\mathfrak{A}_d$ to $\mathfrak{B}(\mathfrak{H})$, $\tau=\mathfrak{D}$, and $\mathfrak{D}$ has minimal range. Using Proposition 3.4 we could similarly prove the weaker result that $\mathfrak{D}$ is extreme.

4.2. *Maps from abelian C^*-algebras.*

DEFINITION 4.8. *Let X be a compact Hausdorff space. Let ϕ be a positive linear map of $C(X)$ into a C^*-algebra. Let Y be a closed subset of X. We say ϕ lives on Y if $\phi(f)=0$ for each function f in $C(X)$ such that support $f\cap Y=\phi$.*

LEMMA 4.9. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{C}=C(X)$ the center of $\mathfrak{A}$. Let $\mathfrak{B}$ be a matrix algebra. If ϕ is extreme in $\mathfrak{D}(\mathfrak{A},\mathfrak{B})$ then $\phi|\mathfrak{C}$ lives on a finite subset of X.*

Proof. Let N be the null space of ϕ. By Theorem 3.1 $N\cap\mathfrak{C}$ is an ideal in $\mathfrak{C}$, which generates a closed two-sided ideal $\mathfrak{J}$ in $\mathfrak{A}$, and $\mathfrak{J}\subset N$. Let ϱ be the canonical homomorphism $\mathfrak{A}\to\mathfrak{A}/\mathfrak{J}$. Then $\phi=\phi'\circ\varrho$, where $\phi'|\varrho(\mathfrak{C})$ is an injective positive linear map of the abelian C^*-algebra $\varrho(\mathfrak{C})$ into $\mathfrak{B}$. Say $\varrho(\mathfrak{C})=C(Y)$. To prove the lemma it suffices to show that Y consists of a finite number of points. $\mathfrak{B}$ is a matrix algebra, hence a subset of some M_n. We show Card $Y\leqslant n^2$. In fact, let $r\leqslant$ Card Y and let $x_1,\dots,x_r$ be r distinct points in Y. Imbed $C^r\cong Cx_1\oplus\dots\oplus Cx_r$ into $C(Y)$ as follows: for each x_i let f_i be a positive function in $C(Y)$ such that

$$f_i(x_j)=\delta_{ij}\,(1\leqslant i,j\leqslant r).$$

Map $Cx_1\oplus\dots\oplus Cx_r$ into $C(Y)$ by

$$\tau:\sum_{i=1}^{r}a_i x_i\to\sum_{i=1}^{r}a_i f_i.$$

Then τ is linear and injective. Thus $\phi'\circ\tau$ is a linear imbedding of C^r into C^{n^2}. But this is impossible unless $r\leqslant n^2$. Thus Card $Y\leqslant n^2$. The proof is complete.

THEOREM 4.10. *Let $\mathfrak{A}$ be an abelian C^*-algebra and $\mathfrak{B}$ a matrix algebra. Then every extremal map in $\mathfrak{D}(\mathfrak{A},\mathfrak{B})$ has minimal range.*

Proof. Let ϕ be extreme in $\mathfrak{D}(\mathfrak{A},\mathfrak{B})$ and let N be the null space of ϕ. Then N is an ideal in $\mathfrak{A}$ (by Theorem 3.1). Let $\tau\in\mathfrak{D}(\mathfrak{A},\mathfrak{B})$ and $r(\tau)\leqslant r(\phi)$. Then N is contained in the null space of τ. Let ϱ be the canonical homomorphism $\mathfrak{A}\to\mathfrak{A}/N$. Then $\phi=\phi'\circ\varrho$ and $\tau=\tau'\circ\varrho$ with ϕ' extreme in $\mathfrak{D}(\mathfrak{A}/N,\mathfrak{B})$ by Lemma 2.3, and $r(\tau')\leqslant r(\phi')$. $\mathfrak{A}/N\cong C(X)$, with X a finite set by Lemma 4.9. Thus there exist orthogonal minimal projections $E_1,\dots,E_r$ which generate $\mathfrak{A}/N$. Since $\mathfrak{B}$ is a matrix algebra there exists $k>0$ such that $\phi'(E_i)^2\geqslant k\phi'(E_i)$, $(1\leqslant i\leqslant r)$. By Theorem 4.5 $\tau'=\phi'$, so $\tau=\phi$, and ϕ has minimal range.

COROLLARY 4.11. *Let $\mathfrak{A}$ be an abelian C^*-algebra and $\mathfrak{H}$ a Hilbert space. Let $\phi\in\mathfrak{D}(\mathfrak{A},\mathfrak{B}(\mathfrak{H}))$. Suppose ϕ is extreme of class n, with n an integer. Then ϕ has minimal range.*

This is immediate from Theorem 4.10 and Proposition 4.4.

Remark 4.12. We outline a proof of Theorem 4.10 which does not make use of Theorem 4.5. Let ϕ be extreme in $\mathfrak{D}(\mathfrak{A},\mathfrak{B})$. By [20] $\phi=V^*\varrho V$, where V is an isometry of $\mathfrak{H}$—the (finite dimensional) Hilbert space on which $\mathfrak{B}$ acts—into a Hilbert space $\mathfrak{K}$, and ϱ is a *-representation of $\mathfrak{A}$ on $\mathfrak{K}$. By Lemma 4.9 we may assume $\varrho(\mathfrak{A})\cong C(X)$ with X a finite set. As in the proof of Theorem 4.10 we have to show that the map $A\to V^*AV$ has minimal range in $\mathfrak{D}(\varrho(\mathfrak{A}),\mathfrak{B})$ The map $B\to VBV^*$ is an isomorphism of $\mathfrak{B}$ into $\mathfrak{B}(\mathfrak{K})$. We may thus assume ϕ is the map $A\to PAP$ of $\mathfrak{A}(=\varrho(\mathfrak{A}))$ into $\mathfrak{B}\subset\mathfrak{B}(\mathfrak{K})$, where $P=VV^*$ is a finite dimensional projection. Let $\tau\in\mathfrak{D}(\mathfrak{A},\mathfrak{B})$ be such that $r(\tau)\leqslant r(\phi)$. If A and B are positive operators in $\mathfrak{B}(\mathfrak{K})$ and $r(A)\leqslant r(PBP)$ then there exists a positive operator C in $\mathfrak{B}(\mathfrak{K})$ such that $r(C)\leqslant r(B)$ and $PCP=A$. Thus, with E a minimal projection in $\mathfrak{A}$ then $r(\tau(E))\leqslant r(PEP)$, so there exists $C\geqslant 0$ in $\mathfrak{A}'$, $CE=C$ and $PCEP=\tau(E)$. Thus there exists a positive operator A' in $\mathfrak{A}'$ such that $\tau(A)=PA'AP$, for all A in $\mathfrak{A}$. By Theorem 3.1 $\tau(A)=(PA'P)PAP=PAP=\phi(A)$, and ϕ has minimal range.

Remark 4.13. Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras, $\mathfrak{B}$ acting on the Hilbert space $\mathfrak{H}$. Let τ and ϕ be maps in $\mathfrak{D}(\mathfrak{A},\mathfrak{B})$. If x is a unit vector in $\mathfrak{H}$ denote by I_x (resp. J_x) the left kernel of the state $\omega_x\circ\tau$ (resp. $\omega_x\circ\phi$). Then $r(\tau)\leqslant r(\phi)$ if and only if $I_x\supset J_x$ for each unit vector x in $\mathfrak{H}$. In fact, with $A\geqslant 0$ in $\mathfrak{A}$ then $r(\tau(A))\leqslant r(\phi(A))$ if and only if $n(\tau(A))\geqslant n(\phi(A))$ if and only if $\phi(A)x=0$ implies $\tau(A)x=0$ for all x if and only if $\omega_x\circ\phi(A)=0$ implies $\omega_x\circ\tau(A)=0$ for all x if and only if $I_x\supset J_x$ for each unit vector $x\in\mathfrak{H}$. If f is a state of $\mathfrak{B}$ denote by I_f (resp. J_f) the left kernel of the state $f\circ\tau$ (resp. $f\circ\phi$) of $\mathfrak{A}$. It is a plausible conjecture that if $\mathfrak{A}$ is an abelian C^*-algebra then ϕ is extreme if and only if, whenever $\tau \in \mathfrak{D}(\mathfrak{A}, \mathfrak{B})$ and $I_f \supset J_f$ for all states f then $\tau = \phi$.

Example 4.14. In view of Theorem 3.1 and Lemma 4.9 it might be conjectured that if $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras, $\mathfrak{C}$ the center of $\mathfrak{A}$, and ϕ extreme in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$, then ϕ restricted to $\mathfrak{C}$ is extreme in $\mathfrak{D}(\mathfrak{C}, \mathfrak{B})$. This is false, as the following example shows. Let $\mathfrak{A} = M_2 \oplus M_3$ and $\mathfrak{B} = M_4$. Let ϕ' be the map of $\mathfrak{A}$ into $\mathfrak{B}$ defined as follows: if $A \in M_2$ and $B \in M_3$ let

$$\phi'(A) = \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{(A)}} & 0 & 0 \\ & & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \phi'(B) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & (B) & \\ 0 & & & \end{pmatrix}.$$

Let $S = \phi'(I)$, and let $\phi = S^{-\frac{1}{2}} \phi' S^{-\frac{1}{2}}$. The center $\mathfrak{C}$ of $\mathfrak{A}$ equals $CI_2 \oplus CI_3$, where I_i is the identity operator in $M_i (i = 2, 3)$. Clearly ϕ (resp. $\phi | \mathfrak{C}$) is extreme in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$ (resp. $\mathfrak{D}(\mathfrak{C}, \mathfrak{B})$) if and only if ϕ' (resp. $\phi' | \mathfrak{C}$) is extreme in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}, S)$ (resp. $\mathfrak{D}(\mathfrak{C}, \mathfrak{B}, S)$). By Theorem 4.10 $\phi | \mathfrak{C}$ is not extreme in $\mathfrak{D}(\mathfrak{C}, \mathfrak{B})$. We show ϕ' has minimal range. Suppose $\tau \in \mathfrak{D}(\mathfrak{A}, \mathfrak{B}, S)$ and $r(\tau) \leqslant r(\phi')$. If E is a 1-dimensional projection in $\mathfrak{A}$ then either $E \in M_2$ or $E \in M_3$, and $\phi'(E) = F$ is a 1-dimensional projection in $\mathfrak{B}$, say $E \in M_2$. $r(\tau(E)) \leqslant F$, so $\tau(E) = \lambda F$. Let $P = (a_{ij}) \in M_2$ be the projection with $a_{11} = 1$. Since $\tau(I) = \phi'(I) = S$, and $1 = S_{11} = \phi'(P)_{11}$, $\tau(P)_{11} = 1$. Thus, by linearity,

$$\tau\left(\begin{pmatrix} w & x \\ y & z \end{pmatrix} \oplus B\right)_{11} = w,$$

where $B \in M_3$. Now
$$E = \begin{pmatrix} |x|^2 & y \\ \bar{y} & 1 - |x|^2 \end{pmatrix}.$$

Thus $\tau(E)_{11} = |x|^2 = \lambda F_{11} = \lambda |x|^2$. Thus $\lambda = 1$ unless $x = 0$. Let E be a projection as above with $x \neq 0, 1$. Let $G = I_2 - E$. Then $\tau(E) = \phi'(E)$ and $\tau(G) = \phi'(G)$. Moreover, $\tau(P) = \phi'(P)$. Thus

$$\tau(I_2 - P) = \tau(E) + \tau(G) - \tau(P) = \phi'(E) + \phi'(G) - \phi'(P) = \phi'(I_2 - P).$$

Thus $\tau(E) = \phi'(E)$ for each 1-dimensional projection E in M_2. Similarly $\tau(E) = \phi'(E)$ for each 1-dimensional projection E in M_3. Thus $\tau = \phi'$, and ϕ' has minimal range.

4.3. *Geometry.* We show that the extremal maps in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$, $\mathfrak{A}$ an abelian C^*-algebra and $\mathfrak{B}$ a matrix algebra, are "approximately" homomorphisms. In view of

Lemma 4.9 we may assume $\mathfrak{A}$ is an abelian matrix algebra and each extreme map in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$ is injective.

Definition 4.15. *Let* $\{S_\alpha\}_{\alpha \in I}$ *be a finite set of self-adjoint operators acting on a Hilbert space. We say the set is linearly independent if* $\sum_{\alpha \in I} k_\alpha S_\alpha = 0$ *with* k_α *real numbers implies* $k_\alpha = 0$. *The set is said to be a minimal set if there exist real constants* k_α *such that* $\sum_{\alpha \in I} k_\alpha S_\alpha = I$, *and if* $J \subset I$ *and* h_β *are real numbers then* $\sum_{\beta \in J} h_\beta S_\beta = I$ *implies* $J = I$.

We omit the easy proof of the following lemma.

Lemma 4.16. *Let* $\{S_\alpha\}_{\alpha \in I}$ *be a finite set of self-adjoint operators. Then the following three conditions are equivalent.*

(i) $\{S_\alpha\}_{\alpha \in I}$ *is a minimal set.*

(ii) $\{S_\alpha\}_{\alpha \in I}$ *is linearly independent and there exist real numbers* $k_\alpha \neq 0$ *such that* $\sum_{\alpha \in I} k_\alpha S_\alpha = I$.

(iii) *There exist unique* $k_\alpha \neq 0$ *such that* $\sum_{\alpha \in I} k_\alpha S_\alpha = I$.

We denote $\{S_\alpha\}_{\alpha \in I}$ by $\{S_\alpha, k_\alpha\}_{\alpha \in I}$. We say $\{S_\alpha, k_\alpha\}_{\alpha \in I}$ is a *positive minimal set* if each $k_\alpha > 0$.

Lemma 4.17. *Let* $\mathfrak{A}$ *be an abelian matrix algebra and* $\mathfrak{B}$ *a matrix algebra. Let* $E_1, \ldots, E_n$ *be the minimal projections in* $\mathfrak{A}$. *Let* ϕ *be extreme and injective in* $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$. *Let*

$$\phi(E_i) = \sum_{1 \leqslant j \leqslant n_i} k_{ij} F_{ij},$$

where $k_{ij} > 0$, *and* F_{ij} *are the spectral projections for* $\phi(E_i)$, $(i = 1, \ldots, n)$. *Then* $\{F_{ij}, k_{ij}\}$ *is a positive minimal set.*

Proof. Suppose $\sum_{i,j} h_{ij} F_{ij} = 0$. Multiplying by a constant we may assume $k_{ij} - h_{ij} \geqslant 0$ for all pairs (i, j). Let $B_i = \sum_j (k_{ij} - h_{ij}) F_{ij}$. Then $B_i \geqslant 0$. If we define τ in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$ by $\tau(E_i) = B_i$, then $r(\tau) \leqslant r(\phi)$, so $\tau = \phi$ by Theorem 4.10. Thus $h_{ij} = 0$, and $\{F_{ij}, k_{ij}\}$ is a minimal set by Lemma 4.16.

If x is a unit vector in the Hilbert space $\mathfrak{H}$ and F is a projection in $\mathfrak{B}(\mathfrak{H})$ then the *angle* (x, F) between x and the subspace F is the angle between 0 and $\pi/2$ radians given by $\cos(x, F) = \|Fx\|$. If $F = [y]$ with y a unit vector in $\mathfrak{H}$ then

$$\cos^2(x, [y]) = ([y]\,x, x) = ([x]\,y, y) = \cos^2(y, [x]),$$

so the *angle* $\langle x, y \rangle$ between x and y is well defined.

Let $\{[x_\alpha]\}_{\alpha\in I}$ be a set of 1-dimensional projections in $\mathfrak{B}(\mathfrak{H})$, $\mathfrak{H}$ finite dimensional. Suppose $\{[x_\alpha], k_\alpha\}$ is a positive minimal set. Then

$$\dim \mathfrak{H} = \operatorname{tr}(I) = \sum_{\alpha\in I} k_\alpha$$

while

$$1 = \sum_{\alpha\in I} k_\alpha \cos^2 (x, [x_\alpha])$$

if x is a unit vector in $\mathfrak{H}$. In particular the angles $\langle x_\alpha, x_\beta\rangle$ are "almost" $\pi/2$ for "almost" every pair (α, β) in $I\times I$. This condition is also sufficient in order that $\{[x_\alpha], k_\alpha\}$, be a positive minimal set. Let D be the determinant $|(\cos^2\langle x_i, y_j\rangle)|$, where $I=\{1, 2, \ldots, n\}$, and $D_k\,(k=1, \ldots, n)$ the determinant $|(A_{ij})|$, where $A_{ij}=\cos^2\langle x_i, x_j\rangle$ if $j\neq k$, and $A_{ik}=1$. Then D and D_k have 1's on the main diagonal, so if the angles $\langle x_i, x_j\rangle$ are sufficiently close to $\pi/2$ the entries off the diagonal will diminish, except the kth column in D_k, and $D>0$ and $D_k>0$. Thus, if $\{[x_i]\}_{i=1,\ldots,n}$ is a set of 1-dimensional projections such that there exist k_i for which $\sum_{i=1}^n k_i [x_i] = I$, then $k_i = D_i/D$ if $D\neq 0$. If the angles $\langle x_i, x_j\rangle$ are so large that $D>0$ and $D_k>0$ $(k=1, \ldots, n)$ then $\{[x_i], k_i\}$ is a positive minimal set. We summarize the last results as follows.

PROPOSITION 4.18. *Let $\{[x_i]\}_{i=1,\ldots,n}$ be a set of 1-dimensional projections such that there exist real numbers k_i $(i=1, \ldots, n)$ for which*

$$\sum_{i=1}^n k_i [x_i] = I.$$

If $\{[x_i], k_i\}$ is a positive minimal set then the angles $\langle x_i, x_j\rangle$ are so large that

$$1 = \sum_{i=1}^n k_i \cos^2 \langle x_i, x_j\rangle \quad (j=1, \ldots, n).$$

Conversely, if the angles $\langle x_i, x_j\rangle$ are so large that $D>0$ and $D_i>0$ $(i=1, \ldots, n)$ then $\{[x_i], k_i\}$ is a positive minimal set.

5. Maps of classes 0 and 1

5.1. *Two theorems.* We characterize all maps of class 1 in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}(\mathfrak{H}))$ and of class 0 in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$, where $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras and $\mathfrak{H}$ is a Hilbert space, in a way analogous to Segal's characterization [18] of pure states in terms of vector states and irreducible representations (Theorems 5.6 and 5.7). Recall that a map ϕ in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}(\mathfrak{H}))$ is of class 1 if and only if $\omega_x \phi$ is a pure state of $\mathfrak{A}$ for each vector state ω_x of $\mathfrak{B}(\mathfrak{H})$, and ϕ is of class 0 in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$ if and only if $f\circ\phi$ is a pure state of $\mathfrak{A}$ for each pure state f of $\mathfrak{B}$. Following [3] we say that a positive linear map of one C^*-algebra into another is *faithful* if its null space contains no non zero positive operators. We are indebted to R. Kadison for remarks which simplified the proof of Theorem 5.6 considerably. We need some lemmas. The first with its proof is almost a direct copy of [3, Proposition 3, p. 61].

LEMMA 5.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be von Neumann algebras and ϕ a positive linear map of $\mathfrak{A}$ into $\mathfrak{B}$. If ϕ is ultra weakly continuous then there exists a minimal projection E in $\mathfrak{A}$—the support of ϕ—such that $\phi(A)=\phi(EAE)$ for all A in $\mathfrak{A}$, and the map $EAE \to \phi(EAE)$ is faithful.*

Proof. Let $\mathfrak{J}$ denote the left kernel of ϕ. Then $\mathfrak{J}$ is the intersection of the left kernels of the states $f \circ \phi$, where f runs through the ultra weakly continuous states of $\mathfrak{B}$ (Remark 3.7). By [3, Theorem 1, p. 54] $\mathfrak{J}$ is ultra weakly closed. By [3, Corollary 3, p. 45] there exists a unique projection F in $\mathfrak{A}$ such that $\mathfrak{J}=\{T \in \mathfrak{A}: TF=T\}$. Since F is self-adjoint, F is also in the right kernel of ϕ. Let $E=I-F$. Then $\phi(A)=\phi(EAE)$.

LEMMA 5.2. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be Hilbert spaces and ϕ in $\mathfrak{D}(\mathfrak{B}(\mathfrak{K}), \mathfrak{B}(\mathfrak{H}))$ be of class 1 and ultra weakly continuous. Let x be a unit vector in $\mathfrak{H}$. Then $\omega_x\phi=\omega_y$, where y is a unit vector in $\mathfrak{K}$, and $\phi([y])=[x]$ or $\phi([y])=I$.*

Proof. Since $\omega_x\phi$ is an ultra weakly continuous pure state of $\mathfrak{B}(\mathfrak{K})$ it follows from [3, Theorem 1, p. 54] that $\omega_x\phi$ is a vector state ω_y. To simplify notation let $Y=\phi([y])$ and $X=[x]$. Then $0 \leqslant Y \leqslant I$ and $\omega_x(Y)=1$. Thus $YX=X \leqslant Y$.

To prove Y equals X or I we first assume the dimension n of $\mathfrak{H}$ is finite and use induction. If $n=1$ the lemma is trivial.

Suppose $n=2$ and that $Y \neq X$. We may then assume $\mathfrak{B}(\mathfrak{H})=M_2$ and

$$Y=\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix},$$

where $p \neq 0$. Let w be a unit vector in $\mathfrak{K}$ orthogonal to y. Let $F=[y]+[w]$. Then $F\mathfrak{B}(\mathfrak{K})F \cong M_2$. Let e_{11}, e_{12}, e_{21}, and e_{22} be the matrix units in $F\mathfrak{B}(\mathfrak{K})F$ and assume $[y]=e_{11}$, $[w]=e_{22}$. If ω_u is a vector state of M_2 then $\omega_u\phi$ is a vector state of $\mathfrak{B}(\mathfrak{K})$, and for some scalar $k>0$ $k\omega_u\phi$ is a vector state or 0 on $F\mathfrak{B}(\mathfrak{K})F$. Thus

$$\omega_u\phi(e_{11})\,\omega_u\phi(e_{22})=|\omega_u\phi(e_{12})|^2. \tag{1}$$

Now $0 \leqslant \phi(e_{11}+e_{22}) \leqslant I$. Hence

$$\phi(e_{22}) = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} \quad \text{and} \quad \phi(e_{12}) = \begin{pmatrix} 0 & r \\ s & t \end{pmatrix}.$$

If $u = (u_1, u_2)$ is a vector in C^2 the following equations

hold:

$$\omega_u \phi(e_{11}) = |u_1|^2 + p\,|u_2|^2,$$
$$\omega_u \phi(e_{22}) = q\,|u_2|^2,$$
$$\omega_u \phi(e_{12}) = t\,|u_2|^2 + r\bar{u}_1 u_2 + s u_1 \bar{u}_2.$$

Thus, using (1),

$$\begin{aligned} |t\,|u_2|^2 + r\bar{u}_1 u_2 + s u_1 \bar{u}_2|^2 &= |t|^2 |u_2|^4 + (|r|^2 + |s|^2)\,|u_1|^2\,|u_2|^2 \\ &\quad + 2\,\Re\,((r\bar{t} + \bar{s}t)\,|u_2|^2\,\bar{u}_1 u_2) + 2\,\Re\,(r\bar{s}\,(\bar{u}_1 u_2)^2) \\ &= q\,|u_2|^2\,(|u_1|^2 + p\,|u_2|^2). \end{aligned}$$

Now, if f_1, f_2, f_3 are complex valued functions of the two complex variables u_1 and u_2 such that

$$f_1(|u_1|, |u_2|) = \Re\,(f_2(|u_1|, |u_2|)\,\bar{u}_1 u_2 + f_3(|u_1|, |u_2|)\,(\bar{u}_1 u_2)^2),$$

then it is easily verified that $f_1 = f_2 = f_3 = 0$. Applying this to the above it follows that

$$r\bar{t} + \bar{s}t = 0 = r\bar{s}, \; |t|^2 = pq, \; \text{and} \; |r|^2 + |s|^2 = q.$$

Thus $q = 0$, and $\phi([w]) = \phi(e_{22}) = 0$. Since this holds for every unit vector w orthogonal to y, and since ϕ is ultra weakly continuous, $Y = I$, as asserted.

Suppose $n \geqslant 3$, and assume the lemma is proved whenever dim $\mathfrak{H} \leqslant n-1$. Let E be a projection in $\mathfrak{B}(\mathfrak{H})$ containing x and dim $E = k < n$. Then $E\phi E$ is of class 1 and ultra weakly continuous in

$$\mathfrak{D}(\mathfrak{B}(\mathfrak{K}), E\mathfrak{B}(\mathfrak{H})E) \cong \mathfrak{D}(\mathfrak{B}(\mathfrak{K}), M_k),$$

and $\omega_y = \omega_x \phi = \omega_x \circ E\phi E$. By induction assumption EYE equals X or E. If $EYE = X$ then

$$0 = E(Y-X)E = ((Y-X)^{\frac{1}{2}}E)^*\,(Y-X)^{\frac{1}{2}}E,$$

so $(Y-X)E = 0$, and $YE = X = EY$, taking adjoints. Similarly, if $EYE = E$ then $E(I-Y)E = 0$, and $EY = YE = E$. Thus Y commutes with every projection containing x. Since $n \geqslant 3$ this is possible only if Y equals X or I.

If $\mathfrak{H}$ is not finite dimensional it follows from the above that Y commutes with every finite dimensional projection containing x. Hence Y equals X or I. The proof is complete.

LEMMA 5.3. *Let ϕ in $\mathfrak{D}(M_n, M_n)$ be of class* 1 *and faithful. Then ϕ is either a *-isomorphism or a *-anti-isomorphism.*

Proof. If $n=1$ the lemma is trivial. Assume $n \geq 2$. Then ϕ is not a state. If x and z are orthogonal unit vectors in C^n then $\omega_x\phi = \omega_y$ and $\omega_z\phi = \omega_w$, where y and w are orthogonal unit vectors in C^n. In fact, by Lemma 5.2 $\phi([y]) = [x]$ and $\phi([w]) = [z]$, and

$$0 \leq \omega_y([w]) = \omega_x\phi([w]) = \omega_x([z]) = 0.$$

Thus $[w]y=0$, and y and w are orthogonal. Let $e_{ij}\,(i,j=1,\dots,n)$ denote the matrix units in M_n. Then $e_{ii}\ (i=1,\dots,n)$ are orthogonal 1-dimensional projections. By the above there exist n orthogonal 1-dimensional projections $F_i\,(i=1,\dots,n)$ such that $\phi(F_i)=e_{ii}$. Replacing ϕ by $\phi(U^*\cdot U)$, where U is a unitary operator in M_n, if necessary, we may assume $F_i = e_{ii}$. Let $k \neq j$. Let $E = e_{kk}+e_{jj}$. Then $\phi(E)=E$. If $A \in M_n$ and $0 \leq EAE \leq E$ then $0 \leq \phi(EAE) \leq \phi(E) = E$, so

$$\phi(EAE) = E\phi(EAE)E = \phi(EAE)E = E\phi(EAE).$$

Since operators EAE with $A \geq 0$ generate $EM_nE \cong M_2$ linearly the equation above holds for all operators EAE in EM_nE. Thus $E\phi E$ is faithful and belongs to $D(EM_nE, EM_nE)$. If x is a unit vector in E then $\omega_x\phi = \omega_y$ with y a unit vector in E. Indeed, $\omega_y(E) = \omega_x\phi(E) = \omega_x(E) = 1$, so $y \in E$. Thus $E\phi E$ is of class 1 and faithful in $D(EM_nE, EM_nE)$. Moreover, $e_{kj}E = Ee_{kj} = Ee_{kj}E$, so $e_{kj} \in EM_nE$. At this point we have to refer the reader to a result in chapter 8. It follows from Lemma 8.9 (i) that $E\phi E$ is either an isomorphism or an anti-isomorphism such that $\phi(e_{kj}) = E\phi E(e_{kj}) = e^{i\theta}e_{kj}$ or $e^{i\theta}e_{jk}$ with $0 \leq \theta < 2\pi$. This holds for all k, j. It follows that ϕ is injective. Since each 1-dimensional projection in M_n is in $\phi(M_n)$ as $\omega_x\phi = \omega_y$ with $\phi([y]) = [x]$, ϕ is surjective and strongly positive. Thus ϕ is an order-isomorphism of M_n onto itself. Thus ϕ is a C^*-isomorphism [11, Corollary 5], hence is a *-isomorphism or a *-anti-isomorphism [10, Corollary 11]. The proof is complete.

If $\mathfrak{K}$ is a Hilbert space defined by the operations $\{a, x\} \to ax$, $\{x, y\} \to x+y$, and $\{x, y\} \to (x, y)$, where a is a complex number and x and y are vectors in $\mathfrak{K}$, we denote by c the conjugate map $\mathfrak{K} \to \mathfrak{K}^c$, where $\mathfrak{K}^c$ is defined by the operations $\{a, x\} \to \bar{a}x$, $\{x, y\} \to x+y$, and $\{x, y\} \to (y, x)$.

LEMMA 5.4. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be Hilbert spaces, and let ϕ in $\mathfrak{D}(\mathfrak{B}(\mathfrak{K}), \mathfrak{B}(\mathfrak{H}))$ be of class* 1 *and ultra weakly continuous. Then ϕ is either a vector state of $\mathfrak{B}(\mathfrak{K})$, or there* *exists a linear isometry V of $\mathfrak{H}$ into $\mathfrak{K}$ such that $\phi(A) = V^*AV$, or $\phi(A) = V^*c^*A^*cV$ for all A in $\mathfrak{B}(\mathfrak{K})$.*

Proof. Let P be the support of ϕ (Lemma 5.1). Let p denote the map $A \to PAP$ of $\mathfrak{B}(\mathfrak{K})$ onto $P\mathfrak{B}(\mathfrak{K})P$. If we can show that ϕ restricted to $P\mathfrak{B}(\mathfrak{K})P$ is of the form described above then $\phi = (\phi | P\mathfrak{B}(\mathfrak{K})P) \circ p$ is of the form described. We may thus assume ϕ is faithful and not a state. Let E be a finite dimensional projection in $\mathfrak{B}(\mathfrak{H})$. Then

$$E = \sum_{i=1}^{n} [x_i]$$

with x_i mutually orthogonal unit vectors in $\mathfrak{H}$. $\omega_{x_i}\phi = \omega_{y_i}$. By Lemma 5.2 $\phi([y_i]) = [x_i]$, and as shown in the proof of Lemma 5.3 the y_i are mutually orthogonal. Let

$$F = \sum_{i=1}^{n} [y_i].$$

The map $E\phi E$ is of class 1 in $\mathfrak{D}(\mathfrak{B}(\mathfrak{K}), E\mathfrak{B}(\mathfrak{H})E)$. Let G be its support (Lemma 5.1). Then in particular, $E\phi(G)E = E\phi(F)E = E\phi(GFG)E$. Now $G(I-F)G \geqslant 0$ and $E\phi(G(I-F)G)E = 0$. Since $E\phi(G \cdot G)E$ is faithful on $G\mathfrak{B}(\mathfrak{K})G$, $G(I-F)G = 0$, and $G = FG$. Thus $F \geqslant G$. Since $0 \leqslant \phi(G) \leqslant I$ and $E\phi(G)E = E = \phi(F)$, $\phi(G) \geqslant \phi(F)$. Thus $\phi(F-G) = 0$. Since ϕ is faithful $F = G$. Thus $E\phi E$ is faithful and of class 1 in

$$\mathfrak{D}(F\mathfrak{B}(\mathfrak{K})F, E\mathfrak{B}(\mathfrak{H})E) \; (\cong \mathfrak{D}(M_n, M_n)).$$

By Lemma 5.3 $E\phi E$ is either an isomorphism or an anti-isomorphism on $F\mathfrak{B}(\mathfrak{K})F$. If A is an operator in $\mathfrak{B}(\mathfrak{K})$ such that $0 \leqslant A \leqslant I$ then $\phi(FAF) \leqslant \phi(F) = E$, and $E\phi(FAF)E = \phi(FAF)$. By Lemma 5.1, for all A in $\mathfrak{B}(\mathfrak{K})$, $\phi(FAF) = E\phi(FAF)E = E\phi(A)E$.

Let $\{E_l\}_{l \in J}$ be a monotonically increasing net of finite dimensional projections in $\mathfrak{B}(\mathfrak{H})$ converging ultra weakly to I. Let F_l be the (finite dimensional) projection in $\mathfrak{B}(\mathfrak{K})$ such that $\phi(F_l) = E_l$. Then $\{F_l\}_{l \in J}$ is a monotonically increasing net. In fact, if $E_k \geqslant E_l$ then, by the preceding,

$$\phi(F_l(I-F_k)F_l) = E_l\phi(I-F_k)E_l = E_lE_k\phi(I-F_k)E_kE_l = 0,$$

so that $F_l(I-F_k)F_l = 0$. Thus $F_kF_l = F_l$, and $F_k \geqslant F_l$. Let $\mathfrak{B}(\mathfrak{K})_1$ denote the unit ball in $\mathfrak{B}(\mathfrak{K})$. Then the map $\mathfrak{B}(\mathfrak{K})_1 \times \mathfrak{B}(\mathfrak{K}) \to \mathfrak{B}(\mathfrak{K})$ by $(S, T) \to ST$ is ultra strongly continuous [3, p. 35] and similarly for $\mathfrak{H}$. Now $E_l \to I$ ultra weakly, hence ultra strongly [3, p. 37]. Hence in particular, $E_lSE_l \to S$ ultra strongly for each operator

S in $\mathfrak{B}(\mathfrak{H})$. Thus $F_l \to I$ ultra weakly. In fact, if there exists an operator $T > 0$ such that $I - F_l \geq T$ for all $l \in J$ then

$$0 = \phi(F_l T F_l) = E_l \phi(T) E_l \to \phi(T),$$

and $\phi(T) = 0$. Since ϕ is faithful $T = 0$, and $F_l \to I$ ultra weakly. If A and B are operators in $\mathfrak{B}(\mathfrak{K})$ it follows that $F_l A F_l \to A$ and $F_l B F_l \to B$ ultra strongly, so $F_l A F_l B F_l \to AB$ ultra strongly, hence ultra weakly. If dim $F_k \geq 2$, $E_k \phi E_k \mid F_k \mathfrak{B}(\mathfrak{K}) F_k$ is an isomorphism or an anti-isomorphism, say an isomorphism. Then $E_l \phi E_l \mid F_l \mathfrak{B}(\mathfrak{K}) F_l$ is an isomorphism for all $l \in J$, and

$$\begin{aligned}\phi(AB) &= \lim_{E_l \to I} \phi((F_l A F_l)(F_l B F_l)) \\ &= \lim_{E_l \to I} E_l \phi(F_l A F_l) E_l \phi(F_l B F_l) E_l \\ &= \lim_{E_l \to I} E_l \phi(A) E_l \phi(B) E_l \\ &= \phi(A)\phi(B).\end{aligned}$$

Thus ϕ is a homomorphism or an anti-homomorphism. Since ϕ is faithful ϕ is injective. Also $\phi(\mathfrak{B}(\mathfrak{K}))$ is ultra strongly dense in $\mathfrak{B}(\mathfrak{H})$. By [3, Corollary 2, p. 57] $\phi(\mathfrak{B}(\mathfrak{K})) = \mathfrak{B}(\mathfrak{H})$. If ϕ is an isomorphism it follows from [3, Proposition 3, p. 253] that ϕ is spatial, say $\phi(A) = U^* A U$ with U an isometry of $\mathfrak{H}$ onto $\mathfrak{K}$. If ϕ is an anti-isomorphism then by [3, p. 10] $\phi(A) = U^* c^* A^* c U$. The proof is complete.

Remark 5.5. If $\mathfrak{B}$ is an irreducible C^*-algebra acting on a Hilbert space $\mathfrak{H}$ then each vector state of $\mathfrak{B}(\mathfrak{H})$ is pure on $\mathfrak{B}$. Hence, if $\mathfrak{A}$ is a C^*-algebra and ϕ is in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$ then $\omega_x \phi$ is a pure state of $\mathfrak{A}$ for each vector state ω_x of $\mathfrak{B}$ if and only if ϕ is of class 1 in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}(\mathfrak{H}))$. We say ϕ is *of class 1 in* $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$. It is thus no restriction to consider maps of class 1 in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}(\mathfrak{H}))$ rather than maps of class 1 into irreducibly represented C^*-algebras.

THEOREM 5.6. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{H}$ a Hilbert space. Then a map ϕ in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B}(\mathfrak{H}))$ is of class* 1 *if and only if either ϕ is a pure state of $\mathfrak{A}$ or $\phi = V^* \varrho V$, where V is a linear isometry of $\mathfrak{H}$ into a Hilbert space $\mathfrak{K}$, and ϱ is an irreducible *-homomorphism or *-anti-homomorphism of $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{K})$.*

Proof. It is clear that if ϕ is of one of the forms described then ϕ is of class 1. Assume ϕ is of class 1. Let ω_x and ω_y be vector states of $\mathfrak{B}(\mathfrak{H})$. Then $\omega_x \phi$ and $\omega_y \phi$ are pure states of $\mathfrak{A}$. We show they are unitarily equivalent. In fact, let z

be a unit vector in $\mathfrak{H}$ orthogonal to x and y (if dim $\mathfrak{H}=2$ argue similarly). Define unit vectors w_i $(i=1,\ldots,5)$ as follows:

$$w_1=x,\ w_2=2^{-\frac{1}{2}}(x+z),\ w_3=z,\ w_4=2^{-\frac{1}{2}}(y+z),\ w_5=y.$$

Then
$$\|w_i-w_{i+1}\|^2=2-2^{\frac{1}{2}}<1 \quad (i=1,\ldots,4).$$

If we can show $\omega_{w_i}\phi$ is unitarily equivalent to $\omega_{w_{i+1}}\circ\phi$ $(i=1,\ldots,4)$ then $\omega_y\phi$ is unitarily equivalent to $\omega_x\phi$. We may thus assume $\|x-y\|<1$. Then, with A in $\mathfrak{A}$ and $\|A\|\leqslant 1$,

$$|(\omega_x-\omega_y)(A)|\leqslant|(A(x-y),x)|+|(Ay,x-y)|\leqslant 2\|A\|\,\|x-y\|<2.$$

Hence $\|\omega_x\phi-\omega_y\phi\|<2$, so by [7, Corollary 9] $\omega_x\phi$ and $\omega_y\phi$ are unitarily equivalent. Let ψ be the irreducible *-representation of $\mathfrak{A}$ of the Hilbert space $\mathfrak{K}$ induced by $\omega_x\phi$, [18]. Then $\omega_y\phi=\omega_w\psi$ for each vector state ω_y of $\mathfrak{B}(\mathfrak{H})$. Thus $\phi=\eta\circ\psi$ with η of class 1 in $\mathfrak{D}(\psi(\mathfrak{A}),\mathfrak{B}(\mathfrak{H}))$, and $\omega_y(\eta(\psi(A)))=\omega_y(\phi(A))=\omega_w(\psi(A))$ for each A in $\mathfrak{A}$. Thus $\omega_y\eta=\omega_w$. By [13, Remark 2.2.3] η has an extension $\bar{\eta}$ to $\mathfrak{D}(\psi(\mathfrak{A})^-, \mathfrak{B}(\mathfrak{H}))\ (=\mathfrak{D}(\mathfrak{B}(\mathfrak{K}),\mathfrak{B}(\mathfrak{H})))$, which is ultra weakly continuous. $\omega_y\circ\bar{\eta}$ is an ultra weakly continuous state on $\mathfrak{B}(\mathfrak{K})$, equal to ω_w when restricted to $\psi(\mathfrak{A})$. By continuity $\omega_y\bar{\eta}=\omega_w$, and $\bar{\eta}$ is of class 1 in $\mathfrak{D}(\mathfrak{B}(\mathfrak{K}),\mathfrak{B}(\mathfrak{H}))$. An application of Lemma 5.4 completes the proof.

THEOREM 5.7. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*algebras and ϕ in $\mathfrak{D}(\mathfrak{A},\mathfrak{B})$. Then ϕ is of class 0 if and only if for each irreducible *-representation ψ of $\mathfrak{B}$, $\psi\circ\phi$ is either a pure state of $\mathfrak{A}$ or $\psi\circ\phi=V^*\varrho V$ with V and ϱ as in Theorem 5.6.*

Proof. Each irreducible *-representation of $\mathfrak{B}$ is cyclic and hence unitarily equivalent to the *-representation induced by a state. Thus, by Remark 5.5 and Theorem 5.6 it suffices to show ϕ is of class 0 if and only if $\psi\circ\phi$ is of class 1 in $\mathfrak{D}(\mathfrak{A},\psi(\mathfrak{B}))$ for each irreducible *-representation ψ due to a state. If f is a pure state of $\mathfrak{B}$ then $f=\omega_z\phi_f$, where ϕ_f is an irreducible *-representation of $\mathfrak{B}$ on a Hilbert space $\mathfrak{H}_f$. Moreover, $\omega_w\phi_f$ is a pure state of $\mathfrak{B}$ for each unit vector w in $\mathfrak{H}_f$. Thus, ϕ is of class 0 in $\mathfrak{D}(\mathfrak{A},\mathfrak{B})$ if and only if $\omega_w\phi_f\circ\phi$ is a pure state of $\mathfrak{A}$ for each pure state f of $\mathfrak{B}$ and each unit vector w in $\mathfrak{H}_f$ if and only if $\phi_f\circ\phi$ is of class 1 in $\mathfrak{D}(\mathfrak{A},\mathfrak{B}(\mathfrak{H}_f))$ for each pure state f. The proof is complete.

5.2. *Applications.*

COROLLARY 5.8. *If $\mathfrak{A}$ is a C^*-algebra and ϕ is of class 1 in $\mathfrak{D}(\mathfrak{A},\mathfrak{B}(\mathfrak{H}))$ then either $\phi(\mathfrak{A})$ is the scalars in $\mathfrak{B}(\mathfrak{H})$ or $\phi(\mathfrak{A})$ is strongly dense in $\mathfrak{B}(\mathfrak{H})$.*

Proof. By Theorem 5.6 it suffices to show that if $\mathfrak{B}$ is an irreducible C^*-algebra acting on a Hilbert space $\mathfrak{K}$ and V is a linear isometry of $\mathfrak{H}$ into $\mathfrak{K}$ then $V^*\mathfrak{B}V$ is strongly dense in $\mathfrak{B}(\mathfrak{H})$. Let $\varepsilon > 0$ be given, and let $x_1, \ldots, x_n$ be n unit vectors in $\mathfrak{H}$. Let B be in $\mathfrak{B}(\mathfrak{H})$. We have to show there exists A in $V^*\mathfrak{B}V$ such that $\|(A-B)x_i\| \leqslant \varepsilon$. The operator $VBV^* \in \mathfrak{B}(\mathfrak{K})$. Therefore there exists C in $\mathfrak{B}$ such that

$$\|(C - VBV^*)Vx_i\| \leqslant \varepsilon, \; i = 1, \ldots, n.$$

If $A = V^*CV$ then, since $B = V^*VBV^*V$,

$$\begin{aligned}\|(A-B)x_i\| = \|(V^*CV - V^*VBV^*V)x_i\| &= \|V^*(C - VBV^*)Vx_i\| \\ &\leqslant \|(C - VBV^*)Vx_i\| \leqslant \varepsilon.\end{aligned}$$

COROLLARY 5.9. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras and ϕ be a surjective C^*-homomorphism in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$. Then $\psi \circ \phi$ is either a *-homomorphism or a *-anti-homomorphism for each irreducible *-representation ψ of $\mathfrak{B}$.*

Proof. By Lemma 2.3 ϕ is of class 0 in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$. Let ψ be an irreducible *-representation of $\mathfrak{B}$ on a Hilbert space $\mathfrak{H}$. By Theorem 5.7 $\psi \circ \phi$ is either a pure state of $\mathfrak{A}$ or is of the form $V^*\varrho V$, where ϱ is an irreducible *-homomorphism or *-anti-homomorphism of $\mathfrak{A}$ on a C^*-algebra acting on a Hilbert space $\mathfrak{K}$, and V is an isometry of $\mathfrak{H}$ into $\mathfrak{K}$. Now $\psi \circ \phi$ is a C^*-homomorphism. If $\psi \circ \phi$ is a state, it is thus a homomorphism. We may therefore assume $\psi \circ \phi = V^*\varrho V$. Let P be the projection VV^* in $\mathfrak{B}(\mathfrak{K})$. Then the map $A \to PAP$ is a C^*-homomorphism of $\varrho(\mathfrak{A})$, since the map $B \to VBV^*$ is an isomorphism of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$. With A self-adjoint in $\varrho(\mathfrak{A})$,

$$\begin{aligned}(AP - PAP)^*(AP - PAP) &= (PA^2P - PAPAP) - (PAPAP - PAPPAP) \\ &= PA^2P - (PAP)^2 = 0.\end{aligned}$$

Thus $AP = PAP = PA$ for each self-adjoint operator A in $\varrho(\mathfrak{A})$. Thus $P \in \varrho(\mathfrak{A})'$, Thus $P = I$, and the map $A \to V^*AV$ is an isomorphism of $\varrho(\mathfrak{A})$. Thus $\psi \circ \phi$ is either a homomorphism or an anti-homomorphism of $\mathfrak{A}$.

COROLLARY 5.10. *If $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras and ϕ is of class 0 in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$ then ϕ maps the center of $\mathfrak{A}$ into the center of $\mathfrak{B}$, and $\phi(AC) = \phi(A)\phi(C)$ for all A in $\mathfrak{A}$ and C in the center of $\mathfrak{A}$.*

Proof. Let ψ be an irreducible *-representation of $\mathfrak{B}$. Let $\mathfrak{C}$ be the center of $\mathfrak{A}$. Then $\psi \circ \phi$ is either a pure state of $\mathfrak{A}$ or of the form $V^*\varrho V$ with V and ϱ as in

Theorem 5.6. Since ϱ is irreducible $\varrho(\mathfrak{C})$ is the scalars in $\varrho(\mathfrak{A})$. Thus $\psi \circ \phi(\mathfrak{C})$ is the scalars in $\psi(\mathfrak{B})$. Thus $\psi \circ \phi$ restricted to $\mathfrak{C}$ is a state, hence a homomorphism. Since the irreducible *-representations of $\mathfrak{B}$ separate points $\phi | \mathfrak{C}$ is a homomorphism. If f is a pure state of $\mathfrak{B}$ then $f | \phi(\mathfrak{C})$ is a homomorphism. By [14, Corollary] $\phi(\mathfrak{C})$ is contained in the center of $\mathfrak{B}$. By [14, Lemma] (or from Theorem 3.1 and Lemma 2.4) $\phi(AC) = \phi(A)\phi(C)$ for each A in $\mathfrak{A}$ and C in $\mathfrak{C}$. The proof is complete.

We recall that a C^*-algebra $\mathfrak{A}$ is a *GCR algebra* if it has a composition series $\{I_\alpha\}$ (an increasing family $\{I_\alpha\}$ of two-sided ideals indexed by the set of ordinals less than or equal to some ordinal γ such that $I_0 = 0$ and $I_\gamma = \mathfrak{A}$, and if α is a limit ordinal then $\bigcup_{\beta<\alpha} I_\beta$ is dense in I_α) such that $I_{\alpha+1}/I_\alpha$ is *CCR*. A *CCR algebra* is a C^*-algebra each of whose irreducible *-representations consists of completely continuous operators. Kaplansky has proved [16, Theorem 7.4] that the homomorphic image of a *GCR* algebra is *GCR*. We show a similar result for class 0 maps.

COROLLARY 5.11. *Let $\mathfrak{A}$ be a GCR algebra and let $\mathfrak{B}$ be a separable C^*-algebra. Suppose there exists a map ϕ of class* 0 *in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$ such that whenever ψ is an irreducible *-representation of $\mathfrak{B}$ on a Hilbert space of dimension greater than* 1 *then $\psi \circ \phi$ is not a state. Then $\mathfrak{B}$ is GCR.*

Proof. Let ψ be an irreducible *-representation of $\mathfrak{B}$ on the Hilbert space $\mathfrak{H}$. By [6, Theorem 1], to show $\mathfrak{B}$ is *GCR* it suffices to show $\psi(\mathfrak{B}) \supset \mathfrak{C}(\mathfrak{H})$—the completely continuous operators in $\mathfrak{B}(\mathfrak{H})$. By [5, Theorem 2] it suffices to show that $\psi(\mathfrak{B})$ contains some non zero operator of finite rank. If dim $\mathfrak{H} = 1$ this is trivial. Otherwise $\psi \circ \phi = V^* \varrho V$, where V is a linear isometry of $\mathfrak{H}$ into a Hilbert space $\mathfrak{K}$, and ϱ is an irreducible *-homomorphism or *-anti-homomorphism of $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{K})$. By [6, Theorem 1] $\varrho(\mathfrak{A}) \supset \mathfrak{C}(\mathfrak{K})$. Thus

$$\psi(\mathfrak{B}) \supset \psi \circ \phi(\mathfrak{A}) = V^* \varrho(\mathfrak{A}) V \supset V^* \mathfrak{C}(\mathfrak{K}) V,$$

which contains operators of finite rank. Thus $\psi(\mathfrak{B}) \supset \mathfrak{C}(\mathfrak{H})$, and $\mathfrak{B}$ is *GCR*.

6. Decomposition of positive maps

6.1. *General results.* Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras and ϕ a positive linear map of $\mathfrak{A}$ into $\mathfrak{B}$. Let 1_n denote the identity transformation of M_n onto itself. Following [20] we say ϕ is *completely positive* if $\phi \otimes 1_n$ is a positive linear map of the algebraic tensor product $\mathfrak{A} \otimes M_n$ into $\mathfrak{B} \otimes M_n$ for each integer $n \geqslant 1$. If $\mathfrak{B}$ acts on a Hilbert space $\mathfrak{H}$ then ϕ is completely positive if and only if $\phi = V^* \varrho V$, where V is a bounded linear map of $\mathfrak{H}$ into a Hilbert space $\mathfrak{K}$ and ϱ is a *-representation of $\mathfrak{A}$ on $\mathfrak{K}$ [20, Theorem 1].

LEMMA 6.1. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{B}$ an abelian C^*-algebra. Then every bounded positive linear map of $\mathfrak{A}$ into $\mathfrak{B}$ is completely positive.*

Proof. Every pure state of $\mathfrak{B} \otimes M_n$ is of the form $f \otimes g$, where f is a state of $\mathfrak{B}$ and g a state of M_n. Indeed, let h be a pure state of $\mathfrak{B} \otimes M_n$. The center of $\mathfrak{B} \otimes M_n$ is $\mathfrak{B} \otimes C_n$, where C_n denotes the algebra of operators of the form λI with λ a complex number and I the identity in M_n. By Theorem 3.1, if A is in $\mathfrak{B}$ and B in M_n then

$$h(A \otimes B) = h((A \otimes I)(I \otimes B)) = h(A \otimes I)\, h(I \otimes B).$$

Denote the state $A \to h(A \otimes I)$ of $\mathfrak{B}$ by f and the state $B \to h(I \otimes B)$ of M_n by g. Then, if $\sum_{i=1}^m A_i \otimes B_i$ is any element in $\mathfrak{B} \otimes M_n$ then

$$h(\sum_i A_i \otimes B_i) = \sum_i h(A_i \otimes B_i) = \sum_i h(A_i \otimes I)\, h(I \otimes B_i) = \sum_i f(A_i)\, g(B_i)$$

$$= \sum_i f \otimes g(A_i \otimes B_i) = f \otimes g(\sum_i A_i \otimes B_i).$$

Thus $h = f \otimes g$ as asserted.

Let ϕ be a positive linear map of $\mathfrak{A}$ into $\mathfrak{B}$. We have to show that for each integer $n \geq 1$, $\phi \otimes 1_n$ is a positive linear map of $\mathfrak{A} \otimes M_n$ into $\mathfrak{B} \otimes M_n$. Let h be a pure state of $\mathfrak{B} \otimes M_n$. By the preceding paragraph $h = f \otimes g$, where f and g are states of $\mathfrak{B}$ and M_n respectively. Thus

$$h \circ (\phi \otimes 1_n) = (f \otimes g) \circ (\phi \otimes 1_n) = (f \circ \phi) \otimes g,$$

which is the tensor product of two positive linear functionals, and is hence positive [21]. It follows that $\phi \otimes 1_n$ is positive, and ϕ is completely positive. The proof is complete.

THEOREM 6.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras and ϕ a bounded positive linear map of $\mathfrak{A}$ into $\mathfrak{B}$. Then $\phi = i \circ (V^* \varrho V)$, where i^{-1} is an order-isomorphism of $\mathfrak{B}$ into an abelian C^*-algebra $\mathfrak{C}$ acting on a Hilbert space $\mathfrak{H}$, V a bounded linear map of $\mathfrak{H}$ into a Hilbert space $\mathfrak{K}$, and ϱ a *-representation of $\mathfrak{A}$ on $\mathfrak{K}$. Moreover, if $\phi(I) = I$ we may assume $\mathfrak{H} \subset \mathfrak{K}$ and V is a projection.*

Proof. Let X be the pure state space of $\mathfrak{B}$ (the w^*-closure of the set of pure states of $\mathfrak{B}$). Let μ be the (canonical) order-isomorphism of $\mathfrak{B}$ into $C(X)$ ([9]). Let

η be a faithful representation of $C(X)$ as operators on a Hilbert space $\mathfrak{H}$. Let $\mathfrak{C}=\eta(C(X))$. Let $j=\eta\circ\mu$. Then $j\circ\phi$ is a positive linear map of $\mathfrak{A}$ into $\mathfrak{C}$, so is completely positive by Lemma 6.1. By [20, Theorem 1], $j\circ\phi=V^*\varrho V$, where V and ϱ are as described. Let $i=j^{-1}$ on $j(\mathfrak{B})$. Then $\phi=i\circ(V^*\varrho V)$. If $\phi(I)=I$ then $j(\phi(I))=I$, and V may be chosen to be an isometry of $\mathfrak{H}$ into $\mathfrak{K}$. The map χ of $\mathfrak{C}$ into $\mathfrak{B}(\mathfrak{K})$ defined by $A\to VAV^*$ is an isomorphism, and $P=VV^*$ is a projection in $\mathfrak{B}(\mathfrak{K})$. Let $i=(\chi\circ j)^{-1}$. Then $\phi=i\circ(P\varrho P)$. The proof is complete.

6.2. *Order-homomorphisms.* Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{A}_*$ the set of self-adjoint operators in $\mathfrak{A}$. We say a linear self-adjoint subset $\mathfrak{J}$ of $\mathfrak{A}$ is an *order ideal* if $\mathfrak{J}\cap\mathfrak{A}_*$ is an order ideal in $\mathfrak{A}_*$. If $\mathfrak{B}$ is a C^*-algebra then a map ϕ in $\mathfrak{D}(\mathfrak{A},\mathfrak{B})$ is an *order-homomorphism* if $\phi\,|\,\mathfrak{A}_*$ is an order-homomorphism of $\mathfrak{A}_*$ into $\mathfrak{B}_*$.

LEMMA 6.3. *Let $\mathfrak{J}$ be a uniformly closed order ideal in the C^*-algebra $\mathfrak{A}$ such that $\mathfrak{J}$ is linearly generated by positive operators. Then $\mathfrak{J}$ is a two-sided ideal in $\mathfrak{A}$ if and only if there exists a C^*-algebra $\mathfrak{B}$ and a bounded positive linear map of $\mathfrak{A}$ into $\mathfrak{B}$ whose null space is $\mathfrak{J}$.*

Proof. If $\mathfrak{J}$ is a two-sided ideal then $\mathfrak{J}$ is the null space of the canonical homomorphism $\mathfrak{A}\to\mathfrak{A}/\mathfrak{J}$. By [19] $\mathfrak{B}=\mathfrak{A}/\mathfrak{J}$ is a C^*-algebra. Conversely, suppose there exists a C^*-algebra $\mathfrak{B}$ and a bounded positive linear map ϕ of $\mathfrak{A}$ into $\mathfrak{B}$ whose null space is $\mathfrak{J}$. By Theorem 6.2 $\phi=i(V^*\varrho V)$, where i^{-1} is an order-isomorphism of $\mathfrak{B}$ into a C^*-algebra $\mathfrak{C}$ acting on a Hilbert space $\mathfrak{H}$, ϱ a *-representation of $\mathfrak{A}$ on a Hilbert space $\mathfrak{K}$, and V a bounded linear map of $\mathfrak{H}$ into $\mathfrak{K}$. $\mathfrak{J}$ is the null space of the map $V^*\varrho V$. The null space of the map $\varrho(A)\to V^*\varrho(A)V$ is $\varrho(\mathfrak{J})$, which is an order ideal in $\varrho(\mathfrak{A})$ by Lemma 2.9, and is linearly generated by positive operators since $\mathfrak{J}$ is. To show that $\mathfrak{J}$ is an ideal it suffices to show $\varrho(\mathfrak{J})$ is an ideal in $\varrho(\mathfrak{A})$. We may thus assume ϕ is of the form $A\to V^*AV$ and $\mathfrak{J}$ is the null space of ϕ. If $A\geqslant 0$ in $\mathfrak{J}$ then $V^*AV=0$, so $AV=0=V^*A$, taking adjoints. Since $\mathfrak{J}$ is linearly generated by positive operators $0=AV=V^*A$ for each A in $\mathfrak{J}$, and $\mathfrak{J}$ is a two-sided ideal. The proof is complete.

THEOREM 6.4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras and ϕ an order-homomorphism in $\mathfrak{D}(\mathfrak{A},\mathfrak{B})$. Then the null space of ϕ is a two-sided ideal in $\mathfrak{A}$ and $\phi(\mathfrak{A})$ is uniformly closed. ϕ is a C^*-homomorphism if and only if for each self-adjoint operator A in $\mathfrak{A}$, $\phi(A)^2\in\phi(\mathfrak{A})$. If ϕ is surjective and ψ is an irreducible *-representation of $\mathfrak{B}$ then $\psi\circ\phi$ is either a homomorphism or an anti-homomorphism of $\mathfrak{A}$.*

Proof. By Lemma 6.3 the null space of ϕ is a two-sided ideal in $\mathfrak{A}$. Factoring it out we may assume ϕ is an order-isomorphism. If A is a self-adjoint operator in $\mathfrak{A}$ then, by spectral theory,

$$\|A\| = \max\{|x|, |y|\},$$

where $$x = \inf\{a \in R: aI \geqslant A\},\ y = \sup\{b \in R: bI \leqslant A\}.$$

Since ϕ is an order-isomorphism in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$, $aI \geqslant A$ if and only if $aI \geqslant \phi(A)$ and $bI \leqslant A$ if and only if $bI \leqslant \phi(A)$. Thus $\|\phi(A)\| = \|A\|$, and ϕ is an isometry on self-adjoint operators. If (A_j) is a Cauchy sequence of operators in a C^*-algebra and $A_j = B_j + iC_j$ with B_j and C_j self-adjoint operators, then (B_j) and (C_j) are Cauchy sequences. Indeed, $B_j = \frac{1}{2}(A_j + A_j^*)$ and $C_j = (2i)^{-1}(A_j - A_j^*)$. Thus

$$\|B_j - B_k\| = \tfrac{1}{2}\|(A_j - A_k) + (A_j - A_k)^*\| \leqslant \|A_j - A_k\|,$$

and similarly, $\|C_j - C_k\| \leqslant \|A_j - A_k\|$. Let $(\phi(A_j))$ be a Cauchy sequence in $\phi(\mathfrak{A})$. Then $\phi(A_j) = \phi(B_j) + i\phi(C_j)$, with B_j and C_j self-adjoint in $\mathfrak{A}$. Thus $(\phi(B_j))$ and $(\phi(C_j))$ are Cauchy sequences. Since ϕ is an isometry on self-adjoint operators (B_j) and (C_j) are Cauchy sequences in $\mathfrak{A}$, say $B_j \to B$ and $C_j \to C$. Let $A = B + iC$. Then $A \in \mathfrak{A}$, and

$$\phi(A) = \phi(B) + i\phi(C) = \lim_{j\to\infty}(\phi(B_j) + i\phi(C_j)) = \lim_{j\to\infty}\phi(A_j).$$

Thus $\phi(A_j) \to \phi(A)$ in $\phi(\mathfrak{A})$, and $\phi(\mathfrak{A})$ is uniformly closed. If ϕ is a C^*-isomorphism then, clearly, $\phi(A)^2 = \phi(A^2) \in \phi(\mathfrak{A})$ for each self-adjoint operator A in $\mathfrak{A}$. Conversely, suppose this condition is satisfied. We proceed as in the proof of [11, Theorem 2]. Let A be self-adjoint in $\mathfrak{A}$. Let B be the self-adjoint operator in $\mathfrak{A}$ such that $\phi(B) = \phi(A)^2 \leqslant \phi(A^2)$, by [11, Theorem 1], so $B \leqslant A^2$. However, for ϕ^{-1} we can assert, $\phi^{-1}(\phi(A)^2) = B \geqslant (\phi^{-1}(\phi(A)))^2 = A^2$, (note that ϕ^{-1} is defined on the C^*-algebra generated by $\phi(A)$ and I). Thus $B = A^2$, and $\phi(A^2) = \phi(A)^2$, so ϕ is a C^*-isomorphism. If ϕ is surjective and ψ is an irreducible *-representation of $\mathfrak{B}$ then $\psi \circ \phi$ is a homomorphism or an anti-homomorphism by the above and Corollary 5.9.

Not all order-isomorphisms of one C^*-algebra into another are C^*-isomorphisms. In fact, if $\mathfrak{A}$ is a C^*-algebra and X its pure state space, then the canonical order-isomorphism μ of $\mathfrak{A}$ into $C(X)$ is a C^*-isomorphism if and only if μ is abelian. However, μ is extreme in $\mathfrak{D}(\mathfrak{A}, C(X))$. Indeed, let Y be the set of pure states of $\mathfrak{A}$. Then Y is dense in X. Let $\tau \in \mathfrak{D}(\mathfrak{A}, C(X), \lambda I)$, $\tau \leqslant \mu$. For each point y in Y the map $A \to \mu(A)(y)$ is a pure state of $\mathfrak{A}$, and if $A \geqslant 0$ then $\tau(A)(y) \leqslant \mu(A)(y)$. Thus $\tau(A)(y) = \lambda\mu(A)(y)$ for each A in $\mathfrak{A}$ and y in Y. By continuity $\tau(A) = \lambda\mu(A)$ for each A in $\mathfrak{A}$, hence $\tau = \lambda\mu$, and μ is extreme. Note that μ is of class 0 if and only if $Y = X$.

Not all order ideals generated by positive operators in a C^*-algebra $\mathfrak{A}$ are two-sided ideals. For example, if P is a projection in $\mathfrak{A}$ then $P\mathfrak{A}P$ is an order ideal in $\mathfrak{A}$ generated by positive operators.

6.3. *A Radon-Nikodym theorem.*

THEOREM 6.5. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras and $\phi \in \mathfrak{D}(\mathfrak{A}, \mathfrak{B})$. Then there exists a decomposition $\phi = i \circ (P\varrho P)$ of ϕ, where i^{-1} is an order-isomorphism of $\mathfrak{B}$, ϱ a *-representation of $\mathfrak{A}$ on a Hilbert space $\mathfrak{K}$, P a projection in $\mathfrak{B}(\mathfrak{K})$, such that $[\varrho(\mathfrak{A})P] = I$ and such that if ψ is a positive linear map of $\mathfrak{A}$ into $\mathfrak{B}$ and $\psi \leqslant \phi$ then there exists an operator S' in $\varrho(\mathfrak{A})'$, $0 \leqslant S \leqslant I$, such that*

$$\psi(A) = i(PS'\varrho(A)P)$$

for all A in $\mathfrak{A}$.

Proof. Applying the universal representation to the C^*-algebra $\mathfrak{C}$ in Theorem 6.2. we may assume each pure state of $\mathfrak{C}$ is a vector state, $\phi = i \circ (P\varrho P)$, and $[\varrho(\mathfrak{A})P] = I$, (note that $[\varrho(\mathfrak{A})P] \in \varrho(\mathfrak{A})')$. Since $\psi \leqslant \phi$, $\psi = i \circ \psi' \circ \varrho$ with

$$\psi' : \varrho(\mathfrak{A}) \to i^{-1}(\mathfrak{B}) \subset \mathfrak{C},$$

and $0 \leqslant \psi' \leqslant P \cdot P$. Let X be the set of unit vectors in $\mathfrak{K}$ such that ω_x is a pure state of $\mathfrak{C}$. If $x \in X$ then $[x] \leqslant [P\varrho(\mathfrak{A})x] \leqslant [\mathfrak{C}x] = [x]$ since ω_x is pure (see e.g. Theorem 3.9). With ω_x and ω_y distinct pure states of $\mathfrak{C}$, $[\varrho(\mathfrak{A})x][\varrho(\mathfrak{A})y] = 0$. In fact, with A and B in $\varrho(\mathfrak{A})$,

$$(Ax, By) = (x, A^*By) = (Px, A^*By) = (x, PA^*By) = 0,$$

since $PA^*By \in [y]$ and x is orthogonal to y. From [3, Lemma 1, p. 50] it follows that there exists S'_x in $\varrho(\mathfrak{A})'$, $0 \leqslant S'_x \leqslant I$ such that $\omega_x \psi' = \omega_x(S'_x \cdot\ )$. Let

$$S_x = [\varrho(\mathfrak{A})x] S'_x [\varrho(\mathfrak{A})x].$$

Then $\omega_x \psi' = \omega_x(S_x \cdot\ )$. Let $(e_l)_{l \in J}$ be an orthonormal basis for P with $e_l \in X$ for each $l \in J$. Let $S' = \sum_{l \in J} S_{e_l}$. Then $S' \in \varrho(\mathfrak{A})'$ and $0 \leqslant S' \leqslant I$. Moreover, if $A \in \varrho(\mathfrak{A})$ then

$$\omega_{e_l}(S'A) = \omega_{e_l}(\sum_{l \in J} S_{e_l} A) = \omega_{e_l}(S_{e_l}A) = \omega_{e_l}\psi'(A).$$

Thus, if $x = \sum_{l \in J} \lambda_l e_l$ is a vector in P then

$$\begin{aligned}\omega_x(S'A) &= (S'A\sum_{l\in J}\lambda_l e_l,\ \sum_{l\in J}\lambda_l e_l)\\ &= \sum_{l\in J}|\lambda_l|^2(S'Ae_l,\ e_l)\\ &= \sum_{l\in J}|\lambda_l|^2(\omega_{e_l}(S'A)\\ &= (\psi'(A)\sum_{l\in J}\lambda_l e_l,\ \sum_{l\in J}\lambda_l e_l)\\ &= \omega_x(\psi'(A)).\end{aligned}$$

Thus $PS'AP=\psi'(A)$ for all A in $\varrho(\mathfrak{A})$, and $\psi(A)=i(PS'\varrho(A)P)$ for each A in $\mathfrak{A}$. The proof is complete.

We cannot expect a much stronger type of Radon-Nikodym Theorem for positive linear maps of C^*-algebras. For example, let $\mathfrak{D}_3$ be the diagonal 3×3 matrices and ϕ be an injective but not extreme map in $\mathfrak{D}(\mathfrak{D}_3, M_2)$, (it is easy to find such a ϕ). Then there exists $\tau\in\mathfrak{D}(\mathfrak{D}_3, M_2, \frac{1}{2}I)$ such that $0\leqslant\tau\leqslant\phi$ and $\tau\neq\frac{1}{2}\phi$. If there exists $S'\in\mathfrak{D}_3(=\mathfrak{D}_3')$ such that $\tau=\phi(S'\cdot\ )$ then $S'=\frac{1}{2}I$, since ϕ is injective, so the "natural" form for the Radon-Nikodym Theorem cannot hold.

COROLLARY 6.6. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras and $\phi=i\circ(P\varrho P)$ be as in Theorem 6.5. Let*

$$Z=\{S'\in\varrho(\mathfrak{A})': PS'AP\in i^{-1}(\mathfrak{B}) \text{ for all } A\in\varrho(\mathfrak{A})\}.$$

Then ϕ is extreme if and only if the map $P\cdot P$ is injective on Z.

Proof. Let τ be a map in $\mathfrak{D}(\mathfrak{A},\mathfrak{B},\lambda I)$ such that $\tau\leqslant\phi$. Then there exists S' in Z such that $\tau(A)=i(PS'\varrho(A)P)$ for each A in $\mathfrak{A}$. If $P\cdot P$ is injective on Z then $S'=\lambda I$, since $\lambda I\in Z$ and $PS'P=\lambda P=P\lambda IP$. Thus $\tau=\lambda\phi$, and ϕ is extreme. Conversely, suppose ϕ is extreme and $S'\in Z$, $PS'P=0$. By Lemma 2.3 the map $i\circ(P\cdot P)$ is extreme in $\mathfrak{D}(\varrho(\mathfrak{A}),\mathfrak{B})$. By Theorem 3.1 $PS'\varrho(A)P=0$ for all A in $\mathfrak{A}$. Now $[\varrho(\mathfrak{A})P]=I$. Thus vectors of the form $\varrho(A)x$, $x\in P$ and $A\in\mathfrak{A}$, generate a dense linear manifold in $\mathfrak{K}$, and $0=PS'=S'^*P=\varrho(A)S'^*P=S'^*\varrho(A)P$ for all A in $\mathfrak{A}$. Thus $S'=0$. The proof is complete.

7. Local decomposition of positive maps

The decomposition theory developed in chapter VI is in some respects unsatisfactory. For example, in the notation of Theorem 6.2 the map $V^*\varrho V$ need not be extreme in $\mathfrak{D}(\mathfrak{A},\mathfrak{C})$ if ϕ is extreme in $\mathfrak{D}(\mathfrak{A},\mathfrak{B})$. The question studied in this chapter is the following: if $\mathfrak{A}$ is a C^*-algebra, $\mathfrak{H}$ a Hilbert space, and ϕ a positive linear map of $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{H})$, can ϕ be written in the form $V^*\varrho V$, where V is a bounded linear map of $\mathfrak{H}$ into a Hilbert space $\mathfrak{K}$, and ϱ is a C^*-homomorphism of $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{K})$? We shall see that "locally" ϕ is of this form (Theorem 7.4), and globally ϕ is "almost" of this form (Theorem 7.6).

DEFINITION 7.1. *Let ϕ be a positive linear map of a C^*-algebra $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{H})$, $\mathfrak{H}$ being a Hilbert space. We say ϕ is decomposable if there exists a Hilbert space $\mathfrak{K}$, a bounded linear map V of $\mathfrak{H}$ into $\mathfrak{K}$, and a C^*-homomorphism ϱ of $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{K})$ such that $\phi = V^*\varrho V$. ϕ is locally decomposable if for each non zero vector x in $\mathfrak{H}$ there exists a Hilbert space $\mathfrak{K}_x$, a linear map V_x of $\mathfrak{K}_x$ into $\mathfrak{H}$, such that $\|V_x\| \leqslant M$ for all x, and a C^*-homomorphism ϱ_x of $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{K}_x)$ such that*

$$V_x \varrho_x(A) V_x^* x = \phi(A)x$$

for all A in $\mathfrak{A}$. ϕ is locally completely positive if for each $x \neq 0$ in $\mathfrak{H}$ there exists a decomposition

$$V_x \varrho_x(\cdot) V_x^* x = \phi(\cdot)x$$

as above, with the property that each ϱ_x is a *-homomorphism.

LEMMA 7.2. *Let $\mathfrak{A}$ be a C^*-algebra, $\mathfrak{H}$ a Hilbert space, and ϕ a positive linear map of $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{H})$ with $\phi(I) \leqslant I$. If x is a non zero vector in $\mathfrak{H}$ then there is a *-representation ψ of $\mathfrak{A}$ as a C^*-algebra on a Hilbert space $\mathfrak{K}$, a vector y in $\mathfrak{K}$ cyclic under $\psi(\mathfrak{A})$, and a bounded linear mapping V of the set*

$$\{\psi(A)y : A \text{ self-adjoint in } \mathfrak{A}\}^-$$

into $\mathfrak{H}$, such that $V\psi(A)V^*x = \phi(A)x$ for each self-adjoint A in $\mathfrak{A}$.

Proof. Let $f = \omega_x \phi$. Say $\|x\| = 1$. Then f is a state of $\mathfrak{A}$. Let ϕ_f be the *-representation induced by f of $\mathfrak{A}$ on $\mathfrak{H}_f$, and let z $(= x_f)$ be a cyclic vector for $\phi_f(\mathfrak{A})$ in $\mathfrak{H}_f$ such that $\omega_z \phi_f = f$. For A self-adjoint in $\mathfrak{A}$, define $V\phi_f(A)z = \phi(A)x$. Now the set $\{\phi_f(A)x$: A is self-adjoint in $\mathfrak{A}\}^-$ is a real linear subspace of $\mathfrak{H}_f$ whose complexification is dense in $\mathfrak{H}_f$. If $\phi_f(A)z = 0$ then

$$0 = (\phi_f(A^2)z, z) = f(A^2) = (\phi(A^2)x, x) \geqslant (\phi(A)^2 x, x) \geqslant 0,$$

by use of [11, Theorem 1]. Thus $\phi(A)x = 0$. If follows that V is well defined and linear. Note that $V\phi_f(I)z = Vz = \phi(I)x$ and that

$$(V^*x, \phi_f(A)z) = (x, V\phi_f(A)z) = (x, \phi(A)x) = f(A) = (z, \phi_f(A)z)$$

for each self-adjoint A in $\mathfrak{A}$. Thus $V^* x = z$, and $V\phi_f(A)V^* x = \phi(A)x$ for each self-adjoint A in $\mathfrak{A}$. Moreover,

$$\|\phi(A)x\|^2 = (\phi(A)^2 x, x) \leqslant (\phi(A^2)x, x) = f(A^2) = \|\phi_f(A)z\|^2,$$

so that $\|V\| \leqslant 1$. Let $\psi = \phi_f$, $\mathfrak{K} = \mathfrak{H}_f$, and $y = z$. The proof is complete.

LEMMA 7.3. *Let ϕ be a positive linear map of the C^*-algebra $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{H})$, $\mathfrak{H}$ being a Hilbert space, such that $\phi(I) \leqslant I$. Then ϕ satisfies the inequality*

$$\phi(A^*A + AA^*) \geqslant \phi(A^*)\phi(A) + \phi(A)\phi(A^*)$$

for all A in $\mathfrak{A}$.

Proof. The operators $A + A^*$ and $i(A - A^*)$ are self-adjoint. By [11, Theorem 1]

$$\phi((A + A^*)^2) + \phi((i(A - A^*))^2) \geqslant \phi(A + A^*)^2 + \phi(i(A - A^*))^2.$$

A straightforward computation now yields the desired result.

THEOREM 7.4. *Every bounded positive linear map of a C^*-algebra $\mathfrak{A}$ into the bounded operators on a Hilbert space $\mathfrak{H}$ is locally decomposable. ϕ is locally completely positive if and only if there exists a scalar $\alpha > 0$ such that the Cauchy–Schwarz inequality*

$$(\alpha\phi)(A^*A) \geqslant (\alpha\phi)(A^*)(\alpha\phi)(A)$$

is satisfied for all A in $\mathfrak{A}$.

Proof. Multiplying ϕ by a scalar we may assume $\phi(I) \leqslant I$. Let x be a non zero vector in $\mathfrak{H}$ and f and ϕ_f as in Lemma 7.2. Define ϕ_f' in terms of the right kernel as a *-anti-homomorphism (i.e. $[A, B] = f(AB^*)$, $\mathfrak{L}_f = \{A : [A, A] = 0\}$, $\phi_f'(C)(A + \mathfrak{L}_f) = AC + \mathfrak{L}_f$) of $\mathfrak{A}$ on the Hilbert space $\mathfrak{H}_f'$, and let $\psi_f = \phi_f \oplus \phi_f'$. Let $\mathfrak{K}_f$ be the Hilbert space $\mathfrak{H}_f \oplus \mathfrak{H}_f'$ with the inner product

$$(z \oplus z', y \oplus y') = \tfrac{1}{2}(z, y) + \tfrac{1}{2}(z', y'),$$

where $y, z \in \mathfrak{H}_f$ and $y' z', \in \mathfrak{H}_f'$. ψ_f is a C^*-homomorphism of $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{K}_f)$. With x_f and y_f the "wave functions" of f for ϕ_f and ϕ_f', respectively, let $z_f = x_f \oplus y_f$. Define a map V' of the linear submanifold $\psi_f(\mathfrak{A})z_f$ of $\mathfrak{K}_f$ into $\mathfrak{H}$ by $V'\psi_f(A)z_f = \phi(A)x$, for each A in $\mathfrak{A}$. Note that if $\psi_f(A)z_f = 0$ then $\phi_f(A)x_f = 0 = \phi_f'(A)y_f$. Thus

$$\phi_f(A^*)\phi_f(A)x_f = \phi_f(A^*A)x_f = 0 = \phi_f'(A^*)\phi_f'(A)y_f = \phi_f'(AA^*)y_f,$$

so that $f(AA^*) = f(A^*A) = 0$. Thus by Lemma 7.3

$$0 = ((\phi(A^*A) + \phi(AA^*))x, x)$$
$$\geqslant ((\phi(A^*)\phi(A) + \phi(A)\phi(A^*))x, x) \geqslant 0,$$

and $\phi(A)x = \phi(A^*)x = 0$. Thus V' is well defined and linear. Moreover,

$$\|V'\| = \sup\{\|\phi(A)x\|: \|\psi_f(A)z_f\| = 1\}$$
$$= \sup\{\|\phi(A)x\|: \|\phi_f(A)x_f \oplus \phi_f'(A)y_f\|^2 = 1\}$$
$$= \sup\{\|\phi(A)x\|: (\phi(A^*A + AA^*)x, x) = 2\}.$$

By Lemma 7.3, if $(\phi(A^*A + AA^*)x, x) = 2$ then $((\phi(A^*)\phi(A) + \phi(A)\phi(A^*))x, x) \leqslant 2$, so that $\|\phi(A)x\|^2 \leqslant 2$. Thus $\|V'\| \leqslant 2^{\frac{1}{2}}$. Extend V' by continuity to all of the subspace $E = [\psi_f(\mathfrak{A})z_f]$, and call the extension V' Define the linear map V of $\mathfrak{K}_f$ into $\mathfrak{H}$ by V restricted to E equals V' and V restricted to $I - E$ equals 0. Then $\|V\| \leqslant 2^{\frac{1}{2}}$. As in Lemma 7.2 it is straightforward to show that $V\psi_f(A)V^*x = \phi(A)x$. Letting $V_x = V$ and $\varrho_x = \psi_f$ we see that ϕ is locally decomposable.

Suppose there exists $\alpha > 0$ such that $\alpha\phi$ satisfies the Cauchy–Schwarz inequality $(\alpha\phi)(A^*A) \geqslant (\alpha\phi)(A^*)(\alpha\phi)(A)$ for all A in $\mathfrak{A}$. By Lemma 7.2 there exists a *-representation ψ of $\mathfrak{A}$ as a C^*-algebra on a Hilbert space $\mathfrak{K}$, and a vector y in $\mathfrak{K}$, cyclic under $\psi(\mathfrak{A})$, and a linear mapping V of the set $\{\psi(A)y: A \text{ is self-adjoint in } \mathfrak{A}\}^-$ into $\mathfrak{H}$ such that $\|V\| \leqslant 1$ and $V\psi(A)V^*x = \phi(A)x$ for each self-adjoint A in $\mathfrak{A}$ (we still assume $\phi(I) \leqslant I$). As in Lemma 7.2 $f = \omega_x\phi$, $y = x_f$ and $\psi = \phi_f$. If $\phi_f(B)x_f = 0$ then $\phi_f(B^*B)x_f = 0$, so that

$$0 = f(B^*B) = (\phi(B^*B)x, x) \geqslant \alpha(\phi(B^*)\phi(B)x, x) \geqslant 0,$$

so $\phi(B)x = 0$. Thus V has a linear extension to the linear manifold $\phi_f(\mathfrak{A})x_f$. Since $\|\phi(B)x\|^2 \leqslant \alpha^{-1}\|\phi_f(B)x_f\|^2$, $\|V\| \leqslant \alpha^{-\frac{1}{2}}$. Since x_f is cyclic V has a continuous linear extension to $\mathfrak{H}_f$, and $V\psi(A)V^*x = \phi(A)x$ for all A in $\mathfrak{A}$. Letting $V_x = V$ and $\varrho_x = \psi$ we conclude that ϕ is locally completely positive.

Conversely, let ϕ be a locally completely positive map of $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{H})$. Then for each vector $x \neq 0$ in $\mathfrak{H}$ there exists a Hilbert space $\mathfrak{K}_x$, a linear map V_x of $\mathfrak{K}_x$ into $\mathfrak{H}$ with $\|V_x\| \leqslant M$ for all $x \neq 0$, and a *-representation ϱ_x of $\mathfrak{A}$ on $\mathfrak{K}_x$ such that $V_x\varrho_x(A)V_x^*x = \phi(A)x$ for all A in $\mathfrak{A}$. Let $\alpha = M^{-2}$. Then for x in $\mathfrak{H}$ and A in $\mathfrak{A}$,

$$(\alpha\phi(A^*A)x, x) = \alpha(V_x\varrho_x(A^*A)V_x^*x, x) = \alpha^2M^2\|\varrho_x(A)V_x^*x\|^2$$
$$\geqslant \alpha^2\|V_x\|^2\|\varrho_x(A)V_x^*x\|^2 \geqslant \alpha^2\|V_x\varrho_x(A)V_x^*x\|^2$$
$$= \alpha^2\|\phi(A)x\|^2 = (\alpha\phi(A^*)\alpha\phi(A)x, x),$$

and $\alpha\phi$ satisfies the Cauchy–Schwarz inequality $\alpha\phi(A^*A) \geqslant \alpha\phi(A^*)\alpha\phi(A)$. The proof is complete.

COROLLARY 7.5. *Let ϕ be a bounded positive linear map of one C^*-algebra $\mathfrak{A}$ into the bounded operators on a Hilbert space. Suppose ϕ is a trace, i.e. $\phi(A^*A)=\phi(AA^*)$ for all A in $\mathfrak{A}$. Then ϕ is locally positive.*

Proof. By Lemma 7.3 ϕ satisfies the inequality $2\phi(A^*A)\geqslant\phi(A^*)\phi(A)+\phi(A)\phi(A^*)$ for all A in $\mathfrak{A}$, so that $\frac{1}{2}\phi(A^*A)\geqslant\frac{1}{2}\phi(A^*)\frac{1}{2}\phi(A)$. By Theorem 7.4 ϕ is locally completely positive.

THEOREM 7.6. *Let $\mathfrak{A}$ be a C^*-algebra and ϕ a bounded positive linear map of $\mathfrak{A}$ into the bounded operators on a Hilbert space $\mathfrak{H}$. Then there exists a Hilbert space $\mathfrak{K}$, a continuous linear map V of $\mathfrak{H}$ into $\mathfrak{K}$, a C^*-homomorphism ϱ of $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{K})$, and a linear (not necessarily continuous) map W of $\mathfrak{K}$ into $\mathfrak{H}$ such that*

$$\phi=W\varrho V.$$

Proof. Let $(e_l)_{l\in J}$ be an orthonormal basis for $\mathfrak{H}$. By Theorem 7.4, for each $l\in J$ there exists a Hilbert space $\mathfrak{K}_l=\mathfrak{K}_{e_l}$, a bounded linear map $V_l=V_{e_l}$ of $\mathfrak{K}_l$ into $\mathfrak{H}$, and a C^*-homomorphism $\varrho_l=\varrho_{e_l}$ such that $V_l\,\varrho_l(A)V_l^*e_l=\phi(A)e_l$ for each A in $\mathfrak{A}$. Let $\mathfrak{K}=\oplus_{l\in J}\mathfrak{K}_l$. If $x\in\mathfrak{H}$ then $x=\sum_{l\in J}a_le_l$. Define the map V of $\mathfrak{H}$ into $\mathfrak{K}$ by

$$Vx=\sum_{l\in J}a_lV^*e_l=\sum_{l\in J}a_lz_l,\ \ z_l=V^*e_l.$$

Then V is linear. V is continuous since

$$\|Vx\|^2=\sum_{l\in J}|a_l|^2\|z_l\|^2=\sum_{l\in J}|a_l|^2(\phi(I)e_l,e_l)\leqslant\|\phi(I)\|\sum_{l\in J}|a_l|^2=\|\phi\|\,\|x\|^2.$$

Define the map W of $\mathfrak{K}$ into $\mathfrak{H}$ by $W(\sum_{l\in J}x_l)=\sum_{l\in J}V_lx_l$, where $x_l\in\mathfrak{K}_l$. Then W is linear. Let $\varrho=\oplus_{l\in J}\varrho_l$. Then ϱ is a C^*-homomorphism of $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{K})$, and $\phi=W\varrho V$. The proof is complete.

When $\phi(A)x=V_x\varrho_x(A)V_x^*x$ as in Theorem 7.4 we say $V_x\varrho_xV_x^*$ is a *local decomposition* of ϕ at x.

Remark 7.7. Let ϕ be a bounded positive linear map of a C^*-algebra $\mathfrak{A}$ into a C^*-algebra $\mathfrak{B}$ acting on a Hilbert space $\mathfrak{H}$. Let x be a non zero vector in $\mathfrak{H}$. Suppose the local decomposition $V_x\varrho_xV_x^*$ of ϕ at x is such that $V_x\varrho_x(A)V_x^*$ commutes with $\mathfrak{B}'$ at x for each A in $\mathfrak{A}$ (i.e. if $B'\in\mathfrak{B}'$ then $V_x\varrho_x(A)V_x^*B'x=B'V_x\varrho_x(A)V_x^*x$). Then $\phi(A)y=V_x\varrho_x(A)V_x^*y$ for all y in $[\mathfrak{B}'x]$. In fact, by continuity we may assume $y=B'x$ with B' in $\mathfrak{B}'$. Then

$$V_x\varrho_x(A)V_x^*y=V_x\varrho_x(A)V_x^*B'x=B'V_x\varrho_x(A)V_x^*x=B'\phi(A)x=\phi(A)y.$$

In particular, if x is a separating vector for $\mathfrak{B}$ then $[\mathfrak{B}'x]=I$, and ϕ is decomposable, and ϕ is completely positive if and only if ϕ is locally completely positive. The following proposition is another result to this effect.

PROPOSITION 7.8. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{H}$ a Hilbert space. Let ϕ be a positive linear map of $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{H})$ with $\phi(I)$ invertible. Suppose ϕ is decomposable, $\phi=V^*\varrho V$, where V is a bounded linear map of $\mathfrak{H}$ onto a Hilbert space $\mathfrak{K}$, and ϱ is a C^*-homomorphism of $\mathfrak{A}$ onto an algebra of operators acting on $\mathfrak{K}$. If ϕ is locally completely positive then ϕ is completely positive.*

Proof. First assume ϕ is a C^*-homomorphism of $\mathfrak{A}$ onto a C^*-algebra $\mathfrak{B}$ acting on $\mathfrak{H}$. Then ϕ is a *-homomorphism. In fact, if not then by Corollary 5.9 there exists an irreducible *-representation ψ of $\mathfrak{B}$ such that $\psi\circ\phi$ is an irreducible anti-homomorphism and $\psi(\mathfrak{B})$ acts on a Hilbert space $\mathfrak{H}_\psi$ of dimension greater than 1. Since ϕ is locally completely positive there exists by Theorem 7.4 $\alpha>0$ such that $\phi(A^*A)\geqslant\alpha\phi(A^*)\phi(A)$ for all A in $\mathfrak{A}$. Composing with ψ it follows that for every operator B in the irreducible C^*-algebra $\psi(\mathfrak{B})$ there exists $\alpha>0$ such that $BB^*\geqslant\alpha B^*B$. Using [12, Theorem 1] it is easy to show dim $\mathfrak{H}_\psi=1$, contrary to assumption. Thus ϕ is a homomorphism. In the general case replace V by $\varrho(I)V$. Then V is still surjective. We may thus assume $\varrho(I)$ is the identity operator in $\mathfrak{B}(\mathfrak{K})$, and $V^*V=\phi(I)$. By the preceding it suffices to show ϱ is locally completely positive. By assumption there exists $\alpha>0$ such that $\phi(A^*A)\geqslant\alpha\phi(A^*)\phi(A)$. Since $\phi(I)$ is invertible there exists $\gamma>0$ such that $\|V^*Vx\|=\|\phi(I)x\|\geqslant\gamma\|x\|$ for all $x\in\mathfrak{H}$. Thus, since V is surjective, there exists $\delta>0$ such that $\|V^*z\|\geqslant\delta\|z\|$ for all $z\in\mathfrak{K}$. If ϱ is not locally completely positive then for any $\beta>0$ there exist x in $\mathfrak{K}$ and A in $\mathfrak{A}$ such that

$$(\varrho(A^*A)x,x)\leqslant\beta\|\varrho(A)x\|^2.$$

Choose β so small that $\beta/\delta^2<\alpha$. Then if $x=Vy$,

$$\begin{aligned}\alpha\|\phi(A)y\|^2\leqslant(\phi(A^*A)y,y)&=(\varrho(A^*A)x,x)\leqslant\beta\|\varrho(A)Vy\|^2\\&\leqslant\beta/\delta^2\|V^*\varrho(A)Vy\|^2=\beta/\delta^2\|\phi(A)y\|^2<\alpha\|\phi(A)y\|^2,\end{aligned}$$

a contradiction. Thus ϱ is a *-homomorphism. The proof is complete.

8. Maps of 2×2 matrices

We classify the extreme points in $\mathfrak{D}(M_2,M_2)$. In order to make the classification as neat as possible we make the following

DEFINITION 8.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras. Let ϕ and τ be maps in $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$. Then ϕ and τ are unitarily equivalent if there exist unitary operators U in $\mathfrak{A}$ and V in $\mathfrak{B}$ such that $\phi = V^* \tau(U^* \cdot U) V$.*

It is clear from Lemma 2.2 that ϕ is extreme if and only if τ is extreme.

THEOREM 8.2. *Let ϕ be a map in $\mathfrak{D}(M_2, M_2)$. Then ϕ is extreme if and only if ϕ is unitarily equivalent to a map of the form*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & \alpha b + \beta c \\ \bar{\alpha} c + \bar{\beta} b & \gamma a + \varepsilon b + \bar{\varepsilon} c + \delta d \end{pmatrix},$$

where $|\varepsilon|^2 = 2\gamma(\delta - |\alpha|^2 - |\beta|^2)$ in the case when $\gamma \neq 0$, and $|\alpha|$ or $|\beta|$ equals 1 when $\gamma = 0$.

The proof is divided into some lemmas. We first recall some facts about M_n. If $x = (x_1, \ldots, x_n)$ is a unit vector in C^n then $[x] = (x_i \bar{x}_j)$. If (a_{ij}) is a matrix in M_n then

$$\omega_x((a_{ij})) = \sum_{i,j} a_{ij} \bar{x}_j x_i.$$

We denote by e_{ij} the matrix units in M_n.

LEMMA 8.3. *Let f be a state on M_n. Then f is pure if and only if*

$$|f(e_{ij})|^2 = f(e_{ii}) f(e_{jj}), \ 1 \leqslant i, \ j \leqslant n.$$

Proof. If f is a linear functional on M_n and $[x] = (x_i \bar{x}_j)$ is a 1-dimensional projection, let E be the matrix $(f(e_{ij}))$ in M_n. Then

$$\omega_x(E) = \sum_{i,j} f(e_{ij}) \bar{x}_j x_i = f([x]).$$

Thus f is positive if and only if $E \geqslant 0$. If f is a state then $0 \leqslant E \leqslant I$. If f is a pure state then $f = \omega_x$ for some unit vector x, and $f(e_{ij}) = x_i \bar{x}_j$, so that $|f(e_{ij})|^2 = f(e_{ii}) f(e_{jj})$. Conversely, suppose this equation is satisfied. Then E is a projection. Indeed,

$$(E^2)_{ii} = \sum_k f(e_{ik}) f(e_{ki}) = \sum_k f(e_{ii}) f(e_{kk}) = f(e_{ii}) f(I) = E_{ii}.$$

Since $0 \leqslant E \leqslant I$, $E - E^2 \geqslant 0$ and has zeros on the diagonal. Thus $E = E^2$. With x a unit vector in the range of E,

$$1 = \omega_x(E) = f([x]).$$

Thus $f = \omega_x$, and f is pure.

In particular we have proved

COROLLARY 8.4. *Let f be a linear functional on M_2. Then f is positive if and only if*

$$f(e_{11}) \geqslant 0, \; f(e_{22}) \geqslant 0, \text{ and } |f(e_{12})|^2 \leqslant f(e_{11}) f(e_{22}).$$

LEMMA 8.5. *Let ϕ be an extreme map in $\mathfrak{D}(M_2, M_2)$. Then there exists a vector state ω_x of M_2 such that $\omega_x \phi$ is a pure state of M_2.*

Proof. Suppose not. Then for all unit vectors x in C^2

$$\omega_x \phi(e_{11}) \, \omega_x \phi(e_{22}) > |\omega_x \phi(e_{12})|^2,$$

by Lemma 8.3 and Corollary 8.4. Since the unit sphere in C^2 is compact there exists $\alpha > 0$ such that

$$\alpha \leqslant \omega_x \phi(e_{11}) \, \omega_x \phi(e_{22}) - |\omega_x \phi(e_{12})|^2$$

for all unit vectors x in C^2. Since $|\omega_x \phi(e_{12})|^2 \leqslant 1$

$$(1 \pm \alpha) |\omega_x \phi(e_{12})|^2 \leqslant \omega_x \phi(e_{11}) \, \omega_x \phi(e_{22}).$$

Define two maps ψ^+ and $\psi^-$of M_2 into M_2 as follows: $\psi^\pm$ is linear,

$$\psi^\pm(e_{ii}) = \phi(e_{ii}) \; (i = 1, 2), \;\; \psi^\pm(e_{12}) = (1 \pm \delta) \phi(e_{12}), \;\; \psi^\pm(e_{21}) = (1 \pm \delta) \phi(e_{21}),$$

where $\delta > 0$ is such that $(1+\delta)^2 \leqslant 1 + \alpha$. By Corollary 8.4 $\psi^\pm \in \mathfrak{D}(M_2, M_2)$ and $\phi = \frac{1}{2}\psi^+ + \frac{1}{2}\psi^-$. Since ϕ is extreme we have arrived at a contradiction. The assertion follows.

LEMMA 8.6. *If ϕ is extreme in $\mathfrak{D}(M_2, M_2)$ ϕ is unitarily equivalent to a map of the form*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & \alpha b + \beta c \\ \bar{\alpha} c + \bar{\beta} b & \gamma a + \varepsilon b + \bar{\varepsilon} c + \delta d \end{pmatrix}.$$

Proof. Write ϕ in the form

$$\phi(A) = \begin{pmatrix} \phi_{11}(A) & \phi_{12}(A) \\ \phi_{21}(A) & \phi_{22}(A) \end{pmatrix}.$$

Up to unitary equivalence we may by Lemma 8.5 assume ϕ_{11} is the pure state $(a_{ij}) \to a_{11}$. Then $\phi_{11}(e_{22}) = 0$. Thus $\phi_{12}(e_{22}) = 0 = \phi_{12}(e_{11})$. The lemma follows.

We fix a map ϕ in $\mathfrak{D}(M_2, M_2)$ of the form described in Lemma 8.6.

LEMMA 8.7. $|\alpha| + |\beta| \leqslant \delta^{\frac{1}{2}}$.

Proof. Since ϕ is positive

$$(|\alpha|^2+|\beta|^2)|b|^2+2\Re(\alpha\bar{\beta}b^2)\leqslant\gamma a^2+a2\Re\varepsilon b+\delta ad, \tag{2}$$

whenever $a\geqslant 0$, $d\geqslant 0$, $|b|^2\leqslant ad$. Replace b by $-b$ in (2) and add the two inequalities obtained. Then

$$(|\alpha|^2+|\beta|^2)|b|^2+2\Re(\alpha\bar{\beta}b^2)\leqslant\gamma a^2+\delta ad.$$

Choose θ such that $\Re(\alpha\bar{\beta}b^2e^{2i\theta})=|\alpha\bar{\beta}b^2|=|\alpha||\beta||b|^2$. Then

$$(|\alpha|+|\beta|)^2|b|^2\leqslant\gamma a^2+\delta ad.$$

This holds for $|b|^2=ad$ and all $a\geqslant 0$. Hence $(|\alpha|+|\beta|)^2\leqslant\delta$.

LEMMA 8.8. *If* $|\alpha|+|\beta|=\delta^{\frac{1}{2}}$ *then there exists a real number* r *such that* $\varepsilon^2=-\alpha\bar{\beta}r^2$.

Proof. If $\alpha\beta=0$, say $\beta=0$, then $|\alpha|=\delta^{\frac{1}{2}}$, and (2) takes the form $\delta|b|^2\leqslant\gamma a^2+2a\Re\varepsilon b+\delta ad$. In particular, if b is such that $\Re\varepsilon b=-|\varepsilon|\sqrt{ad}$ and $|b|^2=ad$, then $2|\varepsilon|\sqrt{ad}\leqslant\gamma a$. Since this holds for all d, $\varepsilon=0$. Assume $\alpha\beta\neq 0$. If $|b|^2=ad$ and $\Re(\alpha\bar{\beta}b^2)=|\alpha||\beta||b|^2$ then (2) takes the form $\gamma a+2\Re\varepsilon b\geqslant 0$. This holds for every $d\geqslant 0$. Thus $\Re\varepsilon b\geqslant 0$. Apply this to $-b$ then $0\leqslant\Re(\varepsilon(-b))=-\Re\varepsilon b$, so $\Re\varepsilon b=0$. Now $\Re(\alpha\bar{\beta}b^2)=|\alpha||\beta||b|^2$ and $b=|b|e^{i\theta}$ implies $e^{2i\theta}\alpha\bar{\beta}=|\alpha||\beta|$, and $e^{i\theta}=(|\alpha||\beta|)^{\frac{1}{2}}(\alpha\bar{\beta})^{-\frac{1}{2}}$. Thus $\Re(\varepsilon(|\alpha||\beta|/\alpha\bar{\beta})^{\frac{1}{2}})=0$, or $\varepsilon(\alpha\bar{\beta})^{-\frac{1}{2}}$ is purely imaginary. The lemma follows.

LEMMA 8.9. *Suppose* $|\varepsilon|=\sqrt{\gamma\delta}$. *Then* ϕ_{22} *is a vector state, say due to the unit vector* (x, w). ϕ *is extreme if and only if one of three cases occurs.*

(i) *if* $x=0$ *then* $|\alpha|$ *or* $|\beta|$ *equals* 1.

(ii) *if* $w=0$ *then* $\alpha=\beta=0$.

(iii) *if* $xw\neq 0$ *then* $\alpha=-\frac{1}{2}e^{i\theta}|w|(w\bar{x}/\bar{w}x)$, $\beta=\frac{1}{2}e^{i\theta}|w|$.

Proof. Case (i), $x=0$. Say $\phi=\frac{1}{2}((\phi+\phi')+(\phi-\phi'))$ with $\phi\pm\phi'$ in $\mathfrak{D}(M_2, M_2)$. Then $\alpha=\frac{1}{2}((\alpha+\alpha')+(\alpha-\alpha'))$ and similarly for β. Then a necessary and sufficient condition for $\phi\pm\phi'$ to be in $\mathfrak{D}(M_2, M_2)$ is that $|\alpha\pm\alpha'|+|\beta\pm\beta'|\leqslant 1$, as follows from inequality (2). If $\alpha\beta\neq 0$ choose k such that

$$0<k<1,\ (1\pm k)|\alpha|\leqslant 1,\ \text{and}\ 0\leqslant\left(1\mp k\frac{|\alpha|}{|\beta|}\right)|\beta|\leqslant 1.$$

Let $\alpha'=k\alpha$, $\beta'=-k|\alpha||\beta|^{-1}\beta$. Then

$$|\alpha\pm\alpha'|+|\beta\pm\beta'|=(1\pm k)|\alpha|+\left(1\mp k\frac{|\alpha|}{|\beta|}\right)|\beta|=|\alpha|+|\beta|\leqslant 1.$$

Thus if $|\alpha|+|\beta|<1$ or $\alpha\beta\neq 0$ then ϕ is not extreme. Thus, if ϕ is extreme then $|\alpha|$ or $|\beta|$ equals 1. Conversely, if $|\alpha|$ or $|\beta|$ equals 1 then ϕ is even extreme of class 0.

Case (ii), $w=0$. Then ϕ is a pure state, and clearly $\alpha=\beta=0$.

Case (iii), $xw\neq 0$. Let F be the projection

$$\begin{pmatrix} |w|^2 & -\dfrac{|w|^2 x}{w} \\ -\dfrac{|x|^2 w}{x} & |x|^2 \end{pmatrix}.$$

Then $\phi_{22}(F)=0$, so $\phi_{12}(F)=0$. Hence $\alpha=-\beta(w\bar{x}/\bar{w}x)$. Since ϕ_{11} and ϕ_{22} are pure states it follows that ϕ is extreme if and only if $|\beta|$ is maximal, i.e. by Lemma 8.7, if and only if $|\alpha|+|\beta|=|w|$, i.e. if and only if $|\beta|(|(w\bar{x}/\bar{w}x)|+1)=|w|$, or $\beta=\frac{1}{2}e^{i\theta}|w|$. The proof is complete.

LEMMA 8.10. *$|\varepsilon|^2\leqslant 2\gamma(\delta-|\alpha|^2-|\beta|^2)$. If $|\alpha|+|\beta|=\delta^{\frac{1}{2}}$ then $|r|\leqslant 2\gamma^{\frac{1}{2}}$, where r is the number found in Lemma 8.8.*

Proof. If $\gamma=0$ then clearly $\varepsilon=0$. Assume $\gamma\neq 0$. In inequality (2) replace b by ib and obtain

$$(|\alpha|^2+|\beta|^2)|b|^2-2\Re(\alpha\bar{\beta}b^2)\leqslant\gamma a^2+2a\Re\varepsilon ib+\delta ad.$$

Let b be such that $\Re\varepsilon b(1+i)=-2^{\frac{1}{2}}|\varepsilon||b|$, and suppose $|b|^2=ad=1$. Adding the above inequality and inequality (2) we obtain for $a\geqslant 0$,

$$|\alpha|^2+|\beta|^2\leqslant\gamma a^2-2^{\frac{1}{2}}a|\varepsilon|+\delta.$$

The function $f(x)=\gamma x^2-2^{\frac{1}{2}}|\varepsilon|x+\delta$ has its minimum for $x=2^{-\frac{1}{2}}\gamma^{-1}|\varepsilon|$. Thus $|\alpha|^2+|\beta|^2\leqslant-|\varepsilon|^2(2\gamma)^{-1}+\delta$, and the first assertion of the lemma follows. If $|\alpha|+|\beta|=\delta^{\frac{1}{2}}$ then $|\varepsilon|^2\leqslant 2\gamma 2|\alpha||\beta|$. By Lemma 8.8 $|\varepsilon|^2=|\alpha||\beta|r^2$. Thus $r^2\leqslant 4\gamma$.

LEMMA 8.11. *If $|\varepsilon|^2=2\gamma(\delta-|\alpha|^2-|\beta|^2)$ and $\gamma\neq 0$ then $|\alpha|+|\beta|=\delta^{\frac{1}{2}}$.*

Proof. There exists θ between 0 and 2π such that $\varepsilon=e^{i\theta}(2\gamma(\delta-|\alpha|^2-|\beta|^2))^{\frac{1}{2}}$. Let $b=e^{i(\pi-\theta)}$. Let $ad=1$. Then inequality (2) becomes

$$|\alpha|^2+|\beta|^2+2\Re(\alpha\bar{\beta}e^{-2i\theta})\leqslant\gamma a^2-2a(2\gamma(\delta-|\alpha|^2-|\beta|^2))^{\frac{1}{2}}+\delta.$$

The function

$$f(x)=\gamma x^2-2(2\gamma(\delta-|\alpha|^2-|\beta|^2))^{\frac{1}{2}}x+\delta$$

has its minimum for $x=(2\gamma^{-1}(\delta-|\alpha|^2-|\beta|^2))^{\frac{1}{2}}$. Then $f(x)=2(|\alpha|^2+|\beta|^2)-\delta$, and by the inequality above and Lemma 8.7

$$\delta\leqslant|\alpha|^2+|\beta|^2-2\Re(\alpha\bar{\beta}e^{-2i\theta})\leqslant(|\alpha|+|\beta|)^2\leqslant\delta.$$

The proof is complete.

LEMMA 8.12. *If ϕ is extreme and $\gamma \neq 0$ then* $|\varepsilon|^2 = 2\gamma(\delta - |\alpha|^2 - |\beta|^2)$.

Proof. Suppose first $|\alpha| + |\beta| < \delta^{\frac{1}{2}}$. Then, if $|\varepsilon| = \sqrt{\delta\gamma}$ then ϕ is not extreme by Lemma 8.9. If $|\varepsilon| < \sqrt{\delta\gamma}$ then by Lemma 8.11 $|\varepsilon|^2 < 2\gamma(\delta - |\alpha|^2 - |\beta|^2)$, so there is room for perturbations on each one of α, β, and ε. Thus ϕ is not extreme in that case.

If $|\alpha| + |\beta| = \delta^{\frac{1}{2}}$, but $|\varepsilon|^2 < 2\gamma(\delta - |\alpha|^2 - |\beta|^2)$ then $|\varepsilon| < \sqrt{\delta\gamma}$, because if $|\varepsilon| = \sqrt{\delta\gamma}$ and $|\alpha| + |\beta| = \delta^{\frac{1}{2}}$ then by Lemma 8.10 $\delta\gamma \leqslant 2\gamma(\delta - |\alpha|^2 - |\beta|^2)$, and $0 \leqslant (|\alpha| + |\beta|)^2 - 2|\alpha|^2 - 2|\beta|^2 = -(|\alpha| - |\beta|)^2 \leqslant 0$, so that $2\gamma(\delta - |\alpha|^2 - |\beta|^2) = 2\gamma(\delta - \frac{1}{2}\delta) = \gamma\delta = |\varepsilon|^2$. By Lemma 8.8 $\varepsilon^2 = -\alpha\bar{\beta}r^2$, where by Lemma 8.10 $|r| \leqslant 2\gamma^{\frac{1}{2}}$. If $|r| < 2\gamma^{\frac{1}{2}}$ then there is room for perturbations on ε, and ϕ is not extreme. If $r = 2\gamma^{\frac{1}{2}}$ then $4\gamma|\alpha||\beta| = |\varepsilon|^2 < 2\gamma(\delta - |\alpha|^2 - |\beta|^2)$, contrary to the assumption that $|\alpha| + |\beta| = \delta^{\frac{1}{2}}$. Thus $|\varepsilon|^2 = 2\gamma(\delta - |\alpha|^2 - |\beta|^2)$.

Proof of Theorem 8.2. It remains to show that if $|\varepsilon|^2 = 2\gamma(\delta - |\alpha|^2 - |\beta|^2)$ and $\gamma \neq 0$ then ϕ is extreme. Suppose $\phi = \frac{1}{2}(\phi_1 + \phi_2)$ with $\phi_1 = \phi + \phi'$, $\phi_2 = \phi - \phi'$, and ϕ_1, $\phi_2 \in \mathfrak{D}(M_2, M_2)$, and

$$\phi'\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 0 & \alpha' b + \beta' c \\ \bar{\alpha}' c + \bar{\beta}' b & \gamma' a + \varepsilon' b + \bar{\varepsilon}' c + \delta' d \end{pmatrix}.$$

We have to show $\alpha' = \beta' = \gamma' = \delta' = \varepsilon' = 0$. Notice that $\gamma' = -\delta'$. We may assume $\gamma' \geqslant 0$. By Lemma 8.10 and the parallellogram law

$$\begin{aligned} 2|\varepsilon|^2 + 2|\varepsilon'|^2 &= |\varepsilon + \varepsilon'|^2 + |\varepsilon - \varepsilon'|^2 \\ &\leqslant 2(\gamma + \gamma')(\delta + \delta' - |\alpha + \alpha'|^2 - |\beta + \beta'|^2) \\ &\quad + 2(\gamma - \gamma')(\delta - \delta' - |\alpha - \alpha'|^2 - |\beta - \beta'|^2) \\ &= 4\gamma(\delta - |\alpha|^2 - |\alpha'|^2 - |\beta|^2 - |\beta'|^2) \\ &\quad + 2\gamma'(2\delta' + (|\alpha - \alpha'|^2 - |\alpha + \alpha'|^2) + (|\beta - \beta'|^2 - |\beta + \beta'|^2)). \end{aligned}$$

Now $|\alpha - \alpha'|^2 - |\alpha + \alpha'|^2 = -4\Re\alpha\bar{\alpha}'$. Thus by Lemma 8.11

$$|\varepsilon'|^2 \leqslant -2\gamma(|\alpha'|^2 + |\beta'|^2) - 2\gamma'^2 + 4\gamma'(|\alpha||\alpha'| + |\beta||\beta'|). \tag{3}$$

If $\gamma' = 0$ then $\alpha' = \beta' = \varepsilon' = 0$, and ϕ is extreme. If $\alpha' = \beta' = 0$ then $\gamma' = \varepsilon' = 0$, and again ϕ is extreme. We shall show $\alpha' = \beta' = 0$. Apply inequality (2) to ϕ_1 and ϕ_2, add the two inequalities obtained, and use the parallellogram law. Then

$$(|\alpha|^2 + |\alpha'|^2 + |\beta|^2 + |\beta'|^2)|b|^2 + 2\Re((\alpha\bar{\beta} + \alpha'\bar{\beta}')b^2) \leqslant \gamma a^2 + 2a\Re(\varepsilon b) + \delta ad.$$

Therefore

$$(|\alpha'|^2 + |\beta'|^2)|b|^2 + 2\Re(\alpha'\bar{\beta}'b^2) \leqslant (\gamma a^2 + 2a\Re\varepsilon b + \delta ad) - ((|\alpha|^2 + |\beta|^2)|b|^2 + 2\Re(\alpha\bar{\beta}b^2)). \tag{4}$$

By our assumption on ε there exists $b \neq 0$ such that the right side of inequality (4) is zero. Thus

$$0 \geqslant (|\alpha'|^2 + |\beta'|^2)|b|^2 + 2\,\Re(\alpha'\bar{\beta}' b^2) \geqslant (|\alpha'| - |\beta'|)^2 |b|^2 \geqslant 0.$$

Thus $|\alpha'| = |\beta'|$. Then $\alpha'\bar{\beta}' = |\alpha'|^2 e^{i\theta}$, $\alpha\bar{\beta} = e^{i\varrho}|\alpha||\beta|$. Let $b = e^{i\varphi}$ and $ad = |b|^2 = 1$. Then $\Re(\alpha'\bar{\beta}' b^2) = |\alpha'|^2 \cos(\theta + 2\varphi)$, $\Re(\alpha\bar{\beta} b^2) = |\alpha||\beta| \cos(\varrho + 2\varphi)$. By Lemma 8.8 and Lemma 8.11 $\varepsilon^2 = -\alpha\bar{\beta} r^2$ and $\varepsilon = i e^{\frac{1}{2} i\varrho}\, 2\sqrt{\gamma|\alpha||\beta|}$. Thus $\Re\,\varepsilon b = -2\sqrt{\gamma|\alpha||\beta|}\, \sin(\tfrac{1}{2}\varrho + \varphi)$. Thus inequality (4) reads

$$2|\alpha'|^2(1 + \cos(\theta + 2\varphi)) \leqslant \gamma a^2 - 4a\sqrt{\gamma|\alpha||\beta|}\, \sin(\tfrac{1}{2}\varrho + \varphi) + 2|\alpha||\beta|(1 - \cos(\varrho + 2\varphi)).$$

Now $1 - \cos 2u = 2\sin^2 u$. Thus

$$0 \leqslant 2|\alpha'|^2(1 + \cos(\theta + 2\varphi)) \leqslant \gamma a^2 - 4a\sqrt{\gamma|\alpha||\beta|}\, \sin(\tfrac{1}{2}\varrho + \varphi) + 4|\alpha||\beta| \sin^2(\tfrac{1}{2}\varrho + \varphi). \quad (5)$$

For each φ such that $\sin(\tfrac{1}{2}\varrho + \varphi) \geqslant 0$ let

$$a_\varphi = 2\sqrt{\gamma^{-1}|\alpha||\beta|}\, \sin(\tfrac{1}{2}\varrho + \varphi).$$

Then the right side of inequality (5) is equal to zero. Letting φ vary it follows that $\alpha' = 0 = \beta'$. Thus ϕ is extreme. The proof is complete.

Example 8.13. Let ϕ be the map in $\mathfrak{D}(M_2, M_2)$ determined by

$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & \delta^{\frac{1}{2}} b \\ \delta^{\frac{1}{2}} c & \gamma a + \delta d \end{pmatrix}$$

with $0 < \delta < 1$. By Theorem 8.2 ϕ is extreme. ϕ is also bijective and not of class 0. Hence the assumption that ϕ be strongly postitive is necessary in Proposition 2.7 and Proposition 2.10. Moreover, ϕ does not have minimal range. In fact, if $\omega_x \phi$ is a vector state ω_y then ω_x is the state $(a_{ij}) \to a_{11}$. Indeed, let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then the following equations hold:

$$|y_1|^2 = |x_1|^2 + \gamma|x_2|^2,$$
$$|y_2|^2 = \delta|x_2|^2,$$
$$\bar{y}_1 y_2 = \delta^{\frac{1}{2}} \bar{x}_1 x_2.$$

Thus $\delta|x_1|^2|x_2|^2 = |y_1|^2|y_2|^2 = \delta|x_2|^2(|x_1|^2 + \gamma|x_2|^2)$, and $\delta\gamma|x_2|^4 = 0$, so that $x_2 = 0$. Thus ω_x is the state we asserted. Let i be the identity mapping of M_2 onto itself. Then

$r(i) \leqslant r(\phi)$. By Remark 4.13 it suffices to show the left kernel of ω_x contains that of $\omega_x \phi$ for each vector state ω_x of M_2. But $\omega_x \phi$ is either faithful or is the state $(a_{ij}) \to a_{11}$, since a non vector state on M_2 is faithful. Thus $r(i) \leqslant r(\phi)$. Since $i \neq \phi$, ϕ does not have minimal range.

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