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POSITIVE MASS THEOREM AND THE BOUNDARY BEHAVIORS OF COMPACT MANIFOLDS WITH NONNEGATIVE SCALAR CURVATURE

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Abstract

In this paper, we study the boundary behaviors of compact manifolds with nonnegative scalar curvature and nonempty boundary. Using a general version of Positive Mass Theorem of Schoen-Yau and Witten, we prove the following theorem: For any compact manifold with boundary and nonnegative scalar curvature, if it is spin and its boundary can be isometrically embedded into Euclidean space as a strictly convex hypersurface, then the integral of mean curvature of the boundary of the manifold cannot be greater than the integral of mean curvature of the embedded image as a hypersurface in Euclidean space. Moreover, equality holds if and only if the manifold is isometric with a domain in the Euclidean space. Conversely, under the assumption that the theorem is true, then one can prove the ADM mass of an asymptotically flat manifold is nonnegative, which is part of the Positive Mass Theorem.

0. Introduction

The structure of a manifold with positive or nonnegative scalar curvature has been studied extensively. There are many beautiful results for compact manifolds without boundary, see [16, 21, 22, 9, 10, 11]. For example, in [16], Lichnerowicz found that some compact manifolds admit no Riemannian metrics with positive scalar curvature. In [21, 22] Schoen and Yau proved that every torus T^n with $n \leq 7$ admits no metric with positive scalar curvature, and admits no non-flat metric with

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nonnegative scalar curvature. This was also proved later by Gromov and Lawson [11] for $n > 7$.

For complete noncompact manifolds, the most famous result is the Positive Mass Theorem (PMT), first proved by Schoen and Yau [23, 24] and later by Witten [28] using spinors, see also [20, 1]. One of their results is as follows: Suppose (M, g) is an asymptotically flat manifold such that g behaves like Euclidean at infinity near each end, and suppose its scalar curvature is nonnegative, then (M, g) is actually the Euclidean space if the ADM mass of one of the ends is zero.

It is natural to ask what we can say about manifolds with boundary and with nonnegative scalar curvature. In a recent work of Yau [30], it was proved that if Ω is a noncompact complete three manifold with boundary and with scalar curvature not less than $-3/2c^2$. Suppose one of the components of $\partial\Omega$ has nonpositive Euler number and mean curvature is not less than c and suppose $\mathrm{Area}(\partial B) \geq c \cdot \mathrm{Vol}(B)$ for any ball B in Ω. Then Ω is a isometric to the warped product of the flat torus with a half line. This is a result on the effect of mean curvature of the boundary that can influence the internal geometry of a manifold.

In this work, we will study boundary behaviors of compact manifolds with nonnegative scalar curvature. It turns out that the question is related to the Positive Mass Theorem. The results in this work might also be related to the study of the quasi-local mass defined in [2]. In fact, it was pointed out by the referee that Theorem 1 below can be interpreted as positivity of quasilocal mass defined by Brown and York in [4, 5]. Hawking and Horowitz [12] also gave a similar derivation of quasilocal mass as in [4, 5].

Moreover, a new definition of quasilocal mass for a compact spacelike hypersurface in a time orientable spacetime has been introduced by Liu and Yau [17] and has been proved to be positive using Theorem 1.

Consider a compact oriented three manifold Ω^3 with smooth boundary $\partial\Omega$. Suppose each component Σ of the boundary has positive Gaussian curvature, then Σ can be isometrically embedded in $\mathbb{R}^3$. Moreover, the embedding is unique up to an isometry of $\mathbb{R}^3$, see [19, 13], for example. We will prove:

Theorem 1. *Let (Ω^3, g) be a compact manifold of dimension three with smooth boundary and with nonnegative scalar curvature. Suppose $\partial\Omega$ has finitely many components Σ_i so that each component has positive Gaussian curvature and positive mean curvature H with respect to the* *unit outward normal. Then for each boundary component* Σ_i*,*

$$\int_{\Sigma_i} H d\sigma \le \int_{\Sigma_i} H_0^{(i)} d\sigma \tag{0.1}$$

where $H_0^{(i)}$ *is the mean curvature of* Σ_i *with respect to the outward normal when it is isometrically embedded in* $\mathbb{R}^3$*,* $d\sigma$ *is the volume form on* Σ_i *induced from* g*. Moreover, if equality holds in* (0.1) *for some* Σ_i*, then* $\partial\Omega$ *has only one component and* Ω *is a domain in* $\mathbb{R}^3$*.*

A similar result is still true in higher dimensions if we assume that each component Σ_i can be realized as a strictly convex hypersurface in the Euclidean space and if in addition Ω is spin. See Theorem 4.1 for more details. Some results similar to Theorem 1 have also been obtained by Miao [18].

The idea of the proof of Theorem 1 is as follows. We use the methods introduced by Bartnik [3] to glue the manifold Ω to another one so that the resulting manifold N is asymptotically flat. This can be accomplished as in [3] (see also [27]) by solving a parabolic partial differential equation of some foliation, so that the mean curvatures on the boundary of Ω and $N \setminus \Omega$ match along $\partial\Omega$. Note that the manifold N is only Lipschitz. Next, we will prove that the positive mass theorem is still true for such a manifold, see Theorem 3.1. This theorem is believed to be true, but the authors are unable to find an explicit reference in the literature and it seems the proof involves some technical points. We will give a detailed proof of the result. We should remark that a positive mass theorem of nonsmooth three-dimensional manifolds was also obtained in [18]. After obtaining N, it can be shown that there is a monotonicity on the difference of the integrals of the mean curvatures of the boundary as a submanifold in Ω and as a submanifold in the Euclidean space. Then one can conclude the theorem is true.

It is interesting to see that in some sense Theorem 1 is equivalent to the positive mass theorem. In fact, we can prove that:

Theorem 2. *Suppose* (0.1) *is true for any compact Riemannian three manifold* Ω *with boundary satisfying the assumptions in Theorem* 1*. Let* (N, g) *be an asymptotically flat manifold* (*in a certain sense*) *with nonnegative scalar curvature which is in* $L^1(N)$*. Then the* ADM *mass* m_E *is nonnegative for each end* E *of* N*.*

The paper is organized as follows. In §1, the equation of foliation is derived. In §2, we will solve the equation of foliation and obtain necessary estimates for later applications. In §3, we will prove a of positive mass theorem for a class of manifolds with Lipschitz metrics. Theorem 1 and its higher dimension analog will be proved in §4. Theorem 2 will be proved in §5.

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1. The equation of foliation with prescribed scalar curvature

In this section, we will derive the equation of foliation with prescribed scalar curvature. The equation has been basically obtained in [3], see also [27]. All manifolds in this work are assumed to be orientable.

Let Σ be a smooth compact manifold without boundary with dimension $n-1$ and let $N = [a, \infty) \times \Sigma$ equipped with a Riemannian metric of the form

$$ds_0^2 = d\rho^2 + g_\rho \tag{1.1}$$

for a point $(\rho, x) \in N$. Here g_ρ is the induced metric on Σ_ρ which is the level surface ρ=constant. Note that for fixed $x \in \Sigma$, (ρ, x), $a \le \rho < \infty$ is a geodesic. Given a function $\mathcal{R}$ on N, we want to find the equation for $u > 0$ such that

$$ds^2 = u^2 d\rho^2 + g_\rho \tag{1.2}$$

has scalar curvature $\mathcal{R}$. Let ω_i, $1 \le i \le n-1$ be a local orthonormal coframe on Σ_0. Parallel translate ω_i on the direction $\frac{\partial}{\partial\rho}$. Let $\omega_n = d\rho$. Let e_i, $1 \le i \le n$ be the dual frame of ω_i, and let ω_{ij} be the connection forms. Then the structure equations of ds_0^2 are

$$d\omega_i = \sum_{j=1}^n \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

and

$$d\omega_{ij} - \sum_{k=1}^n \omega_{ik} \wedge \omega_{kj} = -\frac{1}{2} \sum_{k,l=1}^n R^0_{ijkl} \omega_k \wedge \omega_l.$$

where R^0_{ijkl} is the curvature tensor with respect to ds_0^2. The second fundamental form h^0_{ij}, $1 \le i, j \le n-1$ of Σ_ρ with respect to the normal $e_n = \frac{\partial}{\partial\rho}$ is given by

$$\omega_{ni} = \sum_{j=1}^{n-1} h^0_{ij}\omega_j. \tag{1.3}$$

Let $\eta_i = \omega_i$, $1 \le i \le n-1$ and let $\eta_n = u\omega_n$. Then η_i is an orthonormal coframe with respect to ds^2. Let η_{ij} be the connection forms of η_i. Direct computations show that

$$\eta_{ij} = \omega_{ij}, \quad 1 \le i, j \le n-1, \tag{1.4}$$

and

$$\eta_{ni} = -(\log u)_i \eta_n + u^{-1}\omega_{ni}, \quad 1 \le i \le n-1 \tag{1.5}$$

where $(\log u)_i$ is the derivative of $\log u$ in the e_i direction. In particular, the second fundamental form h_{ij} of Σ_ρ with respect to ds^2 is given by

$$h_{ij} = u^{-1}h^0_{ij}. \tag{1.6}$$

We want to compare the curvature tensor R_{ijkl} of ds^2 with R^0_{ijkl}. For any $1 \le i, j \le n-1$, apply the Gauss equations to Σ_ρ, noting that the metric on Σ_ρ induced by ds_0^2 and ds^2 are the same, we have:

$$\begin{aligned} R_{ijij} &= R^\rho_{ijij} + h^2_{ij} - h_{ii}h_{jj} \\ &= R^\rho_{ijij} + u^{-2}\left((h^0_{ij})^2 - h^0_{ii}h^0_{jj}\right) \\ &= R^\rho_{ijij} + u^{-2}\left(R^0_{ijij} - R^\rho_{ijij}\right) \\ &= \left(1 - u^{-2}\right)R^\rho_{ijij} + u^{-2}R^0_{ijij} \end{aligned} \tag{1.7}$$

where R^ρ_{ijij} is the intrinsic curvature tensor of Σ_ρ. To compare R_{nini}

with R^0_{nini}, we have

(1.8)

$$
\begin{aligned}
&-\frac{1}{2}\sum_{k,l=1}^{n} R_{nikl}\eta_k\wedge\eta_l\\
&= d\eta_{ni}-\sum_{k=1}^{n-1}\eta_{nk}\wedge\eta_{ki}\\
&= -\sum_{j=1}^{n-1}(\log u)_{ij}\omega_j\wedge\eta_n-(\log u)_i d\eta_n-u^{-2}\sum_{j=1}^{n}u_j\omega_j\wedge\omega_{ni}\\
&\quad+u^{-1}d\omega_{ni}-\sum_{k=1}^{n-1}\eta_{nk}\wedge\eta_{ki}\\
&= -\sum_{j=1}^{n-1}(\log u)_{ij}\eta_j\wedge\eta_n-\sum_{j=1}^{n-1}(\log u)_i\left(-(\log u)_j\eta_n+u^{-1}\omega_{nj}\right)\wedge\eta_j\\
&\quad-u^{-2}\sum_{j=1}^{n}u_j\omega_j\wedge\omega_{ni}+\left(u^{-1}\sum_{k=1}^{n-1}\omega_{nk}\wedge\omega_{ki}-\sum_{k=1}^{n-1}\eta_{nk}\wedge\eta_{ki}\right)\\
&\quad-u^{-1}\cdot\frac{1}{2}\sum_{k,l=1}^{n}R^0_{nikl}\omega_k\wedge\omega_l\\
&= \text{I+II+III+IV+V}.
\end{aligned}
$$

Here $(\log u)_{ij}=e_j\left(e_i(\log u)\right)$. Since $\omega_{nj}(e_n)=0$ for all j, ω_{nj} is a linear combination of $\omega_1,\dots,\omega_{n-1}$. The coefficient of $\eta_n\wedge\eta_i$ in II is $[(\log u)_i]^2$. By (1.3), the coefficient of $\eta_n\wedge\eta_i$ in III is $-u^{-3}\frac{\partial u}{\partial\rho}h^0_{ii}$. Moreover, by (1.5)

$$u^{-1}\sum_{k=1}^{n-1}\omega_{nk}\wedge\omega_{ki}-\sum_{k=1}^{n-1}\eta_{nk}\wedge\eta_{ki}=\sum_{k=1}^{n-1}(\log u)_k\eta_n\wedge\eta_{ki}.$$

The coefficient of $\eta_n\wedge\eta_i$ in IV is:

$$\sum_{k=1}^{n-1}(\log u)_k\eta_{ki}(e_i).$$

Hence compare the coefficients of $\eta_n\wedge\eta_i$ in (1.8), we have

$$-R_{nini}=(\log u)_{ii}+[(\log u)_i]^2-u^{-3}u_\rho h^0_{ii}+\sum_{k=1}^{n-1}(\log u)_k\eta_{ki}(e_i)-u^{-2}R^0_{nini}.$$

Since

$$\eta_{ki}(e_i) = \omega_{ki}(e_i) = -\langle \nabla_{e_i} e_i, e_k \rangle,$$

$$\sum_{i=1}^{n-1} (\log u)_{ii} + \sum_{k=1}^{n-1} (\log u)_k \eta_{ki}(e_i) = \Delta_\rho \log u,$$

where Δ_ρ is the Laplacian on Σ_ρ with respect to the induced metric from ds_0^2. Hence

$$\sum_{i=1}^{n-1} R_{nini} = -u^{-1}\Delta_\rho u + u^{-3}\frac{\partial u}{\partial \rho} H^0 + u^{-2}\sum_{i=1}^{n-1} R^0_{nini}. \tag{1.9}$$

where H^0 is the mean curvature of Σ_ρ with respect to the metric ds_0^2. Combining (1.7) and (1.9), the scalar curvature $\mathcal{R}$ of ds^2 is given by

$$\begin{aligned} \mathcal{R} &= (1-u^{-2})\mathcal{R}^\rho + u^{-2}\sum_{i,j}^{n-1} R^0_{ijij} + 2\sum_{i=1}^{n-1} R_{nini} \\ &= (1-u^{-2})\mathcal{R}^\rho + u^{-2}\mathcal{R}^0 - 2u^{-1}\Delta_\rho u + 2u^{-3}\frac{\partial u}{\partial \rho} H^0 \end{aligned}$$

where $\mathcal{R}^0$ is the scalar curvature of N with respect to ds_0^2 and $\mathcal{R}^\rho$ is the scalar curvature of Σ_ρ with the induced metric. Hence $u^2 d\rho^2 + g_\rho$ has the scalar curvature $\mathcal{R}$, if and only if u satisfies

$$H^0 \frac{\partial u}{\partial \rho} = u^2 \Delta_\rho u + \frac{1}{2}(u-u^3)\mathcal{R}^\rho - \frac{1}{2}u\mathcal{R}^0 + \frac{u^3}{2}\mathcal{R}. \tag{1.10}$$

Example 1. Let $N = \mathbb{R}^3 \backslash B(1)$ with the standard Euclidean metric. Then $N = [1,\infty) \times \Sigma$ where Σ is diffeomorphic to $\mathbb{S}^2$. The metric on N is given by $d\rho^2 + g_\rho$, where (Σ_ρ, g_ρ) is the standard sphere with radius ρ. Suppose we want to find u with scalar curvature $\mathcal{R} = 0$. Then u satisfies:

$$2\rho^{-1}\frac{\partial u}{\partial \rho} = u^2 \rho^{-2}\Delta_{\mathbb{S}^2} u + (u-u^3)\rho^{-2}$$

where $\mathbb{S}^2$ is the standard unit sphere. Hence we have

$$2\rho \frac{\partial u}{\partial \rho} = u^2 \Delta_{\mathbb{S}^2} u + (u-u^3).$$

This is a special form of the equation derived in [3].

Example 2. Let Σ_0 be a smooth compact strictly convex hypersurface in R^n. Let r be the distance function from Σ_0. Then the metric on the exterior N of Σ_0 is given by $dr^2 + g_r$, where g_r is the induced metric on Σ_r, which is the hypersurface with distance r from Σ_0. The function u with prescribed scalar curvature $\mathcal{R} = 0$ is given by

$$2H_0 \frac{\partial u}{\partial r} = 2u^2 \Delta_r u + (u - u^3)\mathcal{R}^r$$

where H_0 is the mean curvature of Σ_r, $\mathcal{R}^r$ is the scalar curvature of Σ_r with the induced metric from $\mathbb{R}^n$ and Δ_r is the Laplacian on Σ_r.

Example 3. Let $N = \mathbb{H}^3 \setminus B(1)$ with the standard hyperbolic metric. Then $N = [1, \infty) \times \Sigma$ where Σ is diffeomorphic to $\mathbb{S}^2$. Then the metric on N is given by $d\rho^2 + (\sinh\rho)^2 g_0$, where g_0 is the standard metric on the standard unit sphere in $\mathbb{R}^3$. Suppose we want to find u with scalar curvature $\mathcal{R} = -6$. Then by a direct computation, we know u satisfies:

$$\sinh(2\rho)\frac{\partial u}{\partial \rho} = u^2 \Delta_{\mathbb{S}^2} u + (u - u^3)(1 + 3\sinh^2 \rho).$$

2. Solution to the equation of foliation

In this section, we will solve the equation in Example 2 in §1. Namely:

Let Σ_0 be a compact strictly convex hypersurface in $\mathbb{R}^n$, $\mathbf{X}$ be the position vector of a point on Σ_0, and let $\mathbf{N}$ be the unit outward normal of Σ_0 at $\mathbf{X}$. Let Σ_r be the convex hypersurface described by $\mathbf{Y} = \mathbf{X} + r\mathbf{N}$, with $r \geq 0$. The Euclidean space outside Σ_0 can be represented by

$$(\Sigma_0 \times (0, \infty), dr^2 + g_r)$$

where g_r is the induced metric on Σ_r. Consider the following initial value problem

$$\begin{cases} 2H_0 \frac{\partial u}{\partial r} & = 2u^2 \Delta_r u + (u - u^3)\mathcal{R}^r \quad \text{on } \Sigma_0 \times [0, \infty) \\ u(x, 0) & = \quad u_0(x) \end{cases} \tag{2.1}$$

where $u_0(x) > 0$ is a smooth function on Σ_0, H_0 and $\mathcal{R}^r$ are the mean curvature and scalar curvature of Σ_r respectively, and Δ_r is the Laplacian operator on Σ_r.

We will solve (2.1) and show that the metric $ds^2 = u^2dr^2 + g_r$ is asymptotically flat outside Σ_0. We will also compute the mass of ds^2. We basically follow the argument in [3], see also [27]. However, some estimates are obtained with different methods.

Lemma 2.1. *Let $(x_1, \ldots, x_{n-1})$ be local coordinates on an open set in Σ_0. For any integer $k \geq 0$ and any multi-index α there is a constant C such that for $r \geq 1$*

$$\left| \left(r\frac{\partial}{\partial r} \right)^k \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} \right) \left(H_0(x,r) - \frac{n-1}{r} \right) \right| \leq \frac{C}{r^2}$$

and

$$\left| \left(r\frac{\partial}{\partial r} \right)^k \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} \right) \left(\mathcal{R}^r(x,r) - \frac{(n-1)(n-2)}{r^2} \right) \right| \leq \frac{C}{r^3}.$$

Proof. Let x be a point on Σ_0 and choose local coordinates $(x_1, \ldots, x_{n-1})$ near x such that $\frac{\partial}{\partial x_i}$ is orthonormal at x and such that $\frac{\partial \mathbf{N}}{\partial x_i} = k_i \frac{\partial \mathbf{X}}{\partial x_i}$. Namely, $k_i > 0$ are the principal curvatures of Σ_0 at $\mathbf{X}$. Direct computations show that at the point $\mathbf{Y} = \mathbf{X} + r\mathbf{N}$,

$$H_0 - \frac{n-1}{r} = -\frac{1}{r}\sum_{i=1}^{n-1} \frac{1}{1+rk_i} = -\frac{1}{r}\frac{\sum_{i=0}^{n-2} b_i r^i}{\sum_{i=0}^{n-1} a_i r^i}, \tag{2.2}$$

and

$$\begin{aligned} \mathcal{R}^r - \frac{(n-1)(n-2)}{r^2} &= -\frac{1}{r^2} \sum_{1\leq i,j\leq n-1,\ i\neq j} \frac{1+rk_i+rk_j}{(1+rk_i)(1+rk_j)} \\ &= -\frac{1}{r^2}\frac{\sum_{i=0}^{n-2} d_i r^i}{\sum_{i=0}^{n-1} c_i r^i}, \end{aligned} \tag{2.3}$$

where $a_i,\ b_i,\ c_i,\ d_i$ are smooth functions on Σ_0, such that $a_{n-1} > 0$ and $c_{n-1} > 0$.

Now if $(x_1, \ldots, x_{n-1})$ are any local coordinates near a point x_0, and if

$$f(x,r) = \frac{\sum_{i=0}^{p} \beta_i r^i}{\sum_{i=0}^{q} \gamma_i r^i}$$

where β_i and γ_i are smooth functions on Σ_0 with $\gamma_q > 0$, then for each j,

$$\frac{\partial f}{\partial x_j} = \frac{\sum_{i=0}^{p+q} \widetilde{\beta}_i r^i}{\sum_{i=0}^{2q} \widetilde{\gamma}_i r^i}$$

and
$$r\frac{\partial f}{\partial r}=\frac{\sum_{i=0}^{p+q}\hat{\beta}_i r^i}{\sum_{i=0}^{2q}\hat{\gamma}_i r^i}$$
where $\widetilde{\beta}_i$, $\widetilde{\gamma}_i$, $\hat{\beta}_i$ and $\hat{\gamma}_i$ are smooth functions on Σ_0 with $\widetilde{\gamma}_{2q}>0$ and $\hat{\gamma}_{2q}>0$.

Combining these observations with (2.2) and (2.3), the results follow. q.e.d.

Next, we will obtain preliminary estimates for the upper and lower bounds for the solution u of (2.1).

Lemma 2.2. *If u is defined for all r, then there is a constant C independent of r such that*
$$|u(x,r)-1|\le Cr^{2-n}$$
for $r\ge 1$. In fact, if u is defined on $[0,R)$, then for $0\le r<R$, we have
$$\begin{aligned}&\left[1+C_2\exp\left(-\int_0^r\xi(s)ds\right)\right]^{-\frac{1}{2}}\\ &\le u(x,r)\le\left[1-C_1\exp\left(-\int_0^r\varphi(s)ds\right)\right]^{-\frac{1}{2}}\end{aligned}$$
where
$$\varphi(r)=\min_{x\in\Sigma_0}\frac{\mathcal{R}^r(x,r)}{H_0(x,r)}>0,\qquad \psi(r)=\max_{x\in\Sigma_0}\frac{\mathcal{R}^r(x,r)}{H_0(x,r)}>0,$$
$$C_1=1-\left(\max_{\Sigma_0}u_0+1\right)^{-2},\qquad C_2=\left(\min_{\Sigma_0}u_0\right)^{-2}-1,$$
and $\xi(r)=\varphi(r)$ if $\min_{\Sigma_0}u_0\le 1$, $\xi(r)=\psi(r)$ if $\min_{\Sigma_0}u_0>1$.

Proof. Let
$$f(r)=\left[1-C_1\exp\left(-\int_0^r\varphi(s)ds\right)\right]^{-\frac{1}{2}}.$$
Then $f(0)>u_0(x)$ for all $x\in\Sigma_0$. For any $\lambda>1$, we have
$$\begin{aligned}\frac{d}{dr}(\lambda f)&=\frac{1}{2}\left(\lambda f-\lambda f^3\right)\varphi\\ &>\frac{1}{2}(\lambda f-\lambda^3 f^3)\frac{\mathcal{R}^r}{H_0}\end{aligned}$$

where we have used the fact that $0 < C_1 < 1$ so that $f > 1$, the fact that $\lambda > 1$ and the definition of φ. An application of the maximum principle then shows that $u \le \lambda f$. Since $\lambda > 1$ is arbitrary, we have $u \le f$. Notice that $\varphi(r) = (n-2)/r + O(r^{-2})$, it is easy to see $u - 1 \le C' r^{2-n}$ for some C', if u is defined for all r.

To obtain the lower bound for u. Suppose $\min_{\Sigma_0} u_0 \le 1$. Let

$$h(r) = \left[1 + C_2 \exp\left(-\int_0^r \varphi(s)ds\right)\right]^{-\frac{1}{2}}.$$

It is easy to see that h is well-defined, $h(0) = \min_{\Sigma_0} u_0$ and $h < 1$. Then

$$\begin{aligned}\frac{dh}{dr} &= \frac{1}{2}(h - h^3)\varphi \\ &\le \frac{1}{2}(h - h^3)\frac{\mathcal{R}^r}{H_0}\end{aligned}$$

where we have used the fact that $h - h^3 > 0$ and the definition of φ. As before, we can conclude that $h \le u$.

Suppose $\min_{\Sigma_0} u_0 > 1$. Let

$$g(r) = \left[1 + C_2 \exp\left(-\int_0^r \psi(s)ds\right)\right]^{-\frac{1}{2}}.$$

Then g is well-defined because $-1 < C_2 < 0$. Moreover, $g(0) = \min_{\Sigma_0} u_0$ and $g > 1$.

$$\begin{aligned}\frac{dg}{dr} &= \frac{1}{2}(g - g^3)\psi \\ &\le \frac{1}{2}(g - g^3)\frac{\mathcal{R}^r}{H_0}\end{aligned}$$

where we have used the fact that $g - g^3 < 0$. We can obtain the required lower bound for u as before.

If u is defined for all r, we also have $u - 1 \ge -C'' r^{2-n}$ for some constant $C'' > 0$ if r is large enough. q.e.d.

Because of Lemma 2.2, we have:

Lemma 2.3. (2.1) *has a unique solution u for all r which satisfies the estimates in Lemma* 2.2.

We need some estimates for the metric g_r. More precisely, we need the fact that $r^{-2} g_r$ is asymptotically equal to the standard metric on

$\mathbb{S}^{n-1}$. Since Σ_0 is convex, the Gauss map $\mathbf{N} : \Sigma_0 \to \mathbb{S}^{n-1}$ is a diffeomorphism. Fix local coordinates $(x_1, \ldots, x_{n-1})$ on Σ_0 so that Σ_0 is given by $\mathbf{X}(x_1, \ldots, x_{n-1})$. Then the metric ${}_rg = {}_rg_{ij}dx_idx_j$ on Σ_r is given by

$${}_rg_{ij} = {}_og_{ij} + r\left[\langle \mathbf{N}_i, \mathbf{X}_j\rangle + \langle \mathbf{N}_j, \mathbf{X}_i\rangle\right] + r^2 b_{ij}$$

where ${}_og_{ij}$ is the metric on Σ_0 and b_{ij} is the standard metric on $\mathbb{S}^{n-1}$ in coordinates $(x_1, \ldots, x_{n-1})$ via the Gauss map $\mathbf{N}$. Let ${}_r\widetilde{g}_{ij} = r^{-2}\ {}_rg_{ij}$. Then we have the following estimate. The proof is similar to that of Lemma 2.2.

Lemma 2.4. *With the above notations, for any $k \geq 0$ and any multi-index α there is a constant C such that for $r \geq 1$,*

$$\left|\left(r\frac{\partial}{\partial r}\right)^k \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha}\right)\left({}_r\widetilde{g}_{ij} - b_{ij}\right)(x, r)\right| \leq \frac{C}{r}.$$

For $r \geq 1$, let $r = e^t$, and so $\frac{\partial}{\partial t} = r\frac{\partial}{\partial r}$. Equation (2.1) becomes

$$\frac{\partial u}{\partial t} = (rH_0)^{-1}u^2\widetilde{\Delta}_r u + \frac{1}{2}(u - u^3)r\mathcal{R}^r H_0^{-1}. \tag{2.4}$$

where $\widetilde{\Delta}_r$ is the Laplacian on Σ_r with respect to the metric ${}_r\widetilde{g}_{ij}$.

Lemma 2.5. *Let u be the solution of* (2.1), *then in local coordinates $(x_1, \ldots, x_{n-1})$ on Σ_0, for any k and α, there is a constant C such that*

$$\left|\left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha}\right)(u(x, r) - 1)\right| \leq Cr^{2-n}.$$

Proof. In local coordinates

$$\widetilde{\Delta}_r = \frac{1}{\sqrt{{}_r\widetilde{g}}}\frac{\partial}{\partial x_i}\left(\sqrt{{}_r\widetilde{g}}\ {}_r\widetilde{g}^{ij}\frac{\partial}{\partial x_j}\right),$$

where ${}_r\widetilde{g} = \det({}_r\widetilde{g}_{ij})$. Hence

$$\begin{aligned}
&(rH_0)^{-1}u^2\widetilde{\Delta}_r u \\
&= \frac{(rH_0)^{-1}u^2}{\sqrt{{}_r\widetilde{g}}}\frac{\partial}{\partial x_i}\left(\sqrt{{}_r\widetilde{g}}\ {}_r\widetilde{g}^{ij}\frac{\partial u}{\partial x_j}\right) \\
&= \frac{\partial}{\partial x_i}\left[(rH_0)^{-1}u^2\ {}_r\widetilde{g}^{ij}\frac{\partial u}{\partial x_j}\right] - \frac{\partial}{\partial x_i}\left[\frac{(rH_0)^{-1}u^2}{\sqrt{{}_r\widetilde{g}}}\right]\left[\sqrt{{}_r\widetilde{g}}\ {}_r\widetilde{g}^{ij}\frac{\partial u}{\partial x_j}\right] \\
&= \frac{\partial}{\partial x_i}\left[a_i(x, t, \partial u)\right] - a(x, t, \partial u)
\end{aligned}$$

where

$$a_i(x,t,\vec{p}) = (rH_0)^{-1} \ {}_r\widetilde{g}^{ij} p_j$$

and

$$a(x,t,\vec{p}) = \frac{\partial}{\partial x_i}\left[\frac{(rH_0)^{-1}}{\sqrt{{}_r\widetilde{g}}}\right]\sqrt{{}_r\widetilde{g}}\ {}_r\widetilde{g}^{ij}u^2 p_j + 2\,(rH_0)^{-1}\,u\ {}_r\widetilde{g}^{ij}p_i p_j.$$

Here in a, u is considered to be a given function. Hence, by Lemmas 2.1, 2.2 and 2.4, for $t=\log r$ large enough,

$$a_i p_i \geq C|p|^2$$

$$|a_i| \leq C'|p|$$

and

$$|a| \leq C''\left(1+|p|^2\right)$$

for some positive constants C, C', C'' independent of t. By [15, Th. V.1.1], for any $t_0 \geq 1$, there are constants $\beta > 0$ and $C_1 > 0$ independent of t_0, such that

$$\frac{|u(x,t)-u(x',t)|}{|x-x'|^{\beta}} + \frac{|u(x,t)-u(x,t')|}{|t-t'|^{\frac{\beta}{2}}} \leq C_1 \tag{2.5}$$

for all $x \neq x' \in \Sigma_0$ and $t \neq t'$ in $[t_0, t_0+1]$. Now consider the function $v = u-1$, we have

$$\frac{\partial v}{\partial t} - (rH_0)^{-1}\,u^2\ {}_r\widetilde{g}^{ij}\frac{\partial^2 v}{\partial x_i \partial x_j} + \frac{(rH_0)^{-1}\,u^2}{\sqrt{{}_r\widetilde{g}}}\frac{\partial}{\partial x_i}\left(\sqrt{{}_r\widetilde{g}}\ {}_r\widetilde{g}^{ij}\right)\frac{\partial u}{\partial x_j} - \frac{1}{2}\left(u^2+u\right) r\mathcal{R}^r H_0^{-1} v = 0.$$

By Lemmas 2.1, 2.2, 2.4 and (2.5), using the interior Schauder estimates [15, Th. IV.10.1] or Friedmann [15, Th.1, Chap. 4], the lemma follows.

q.e.d.

As in [3], let

$$m = \frac{1}{2}r^{n-2}\left(1-u^{-2}\right). \tag{2.6}$$

By Lemma 2.5, it is easy to see that:

Corollary 2.1. *With the notations in Lemma 2.5, in local coordinates* $(x_1, \ldots, x_{n-1})$ *on* Σ_0*, for any* k *and* α*, there is a constant* C *such that*

$$\left| \left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha}\right) m \right| \le C.$$

Direct computations show that m satisfies

(2.7)
$$\frac{\partial m}{\partial r} = u^2 H_0^{-1} \Delta_r m + 3u^4 r^{2-n} H_0^{-1} |\nabla_r m|^2 + \left(\frac{n-2}{r} - \mathcal{R}^r H_0^{-1}\right) m$$

where ∇_r is the gradient with respect to the metric ${}_r g_{ij}$. Hence
(2.7′)
$$\frac{\partial m}{\partial t} = u^2 \left(r H_0\right)^{-1} \widetilde{\Delta}_r m + 3u^4 r^{-1-n} H_0^{-1} |\widetilde{\nabla}_r m|^2 + \left(n - 2 - \mathcal{R}^r r H_0^{-1}\right) m$$

where $\widetilde{\nabla}_r$ is the gradient with respect to the metric ${}_r\widetilde{g}_{ij}$ and $t = \log r$ as before.

Let Δ and ∇ be the Laplacian and the gradient with respect to pull back metric on Σ_0 of the standard metric on $\mathbb{S}^{n-1}$ through the Gauss map. Let $(x_1, \ldots, x_{n-1})$ be local coordinates on Σ_0 as in the setting of Lemma 2.3.

Lemma 2.6. *With the above notations,*

$$\frac{\partial m}{\partial t} = \frac{1}{n-1} \Delta m + f(x, t)$$

where $f(t, x)$ *is a function such that in a local coordinates, for any* k *and* α*, there is a constant* C *such that*

$$\left| \left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha}\right) f(x,t) \right| \le C e^{-t}. \tag{2.8}$$

Proof. Here and below $f(x, t)$ will denote a function satisfying (2.8), but it may vary from line to line. By Lemmas 2.1, 2.2, 2.5, it is easy to see that

$$u^2 (r H_0)^{-1} = \frac{1}{n-1} + f. \tag{2.9}$$

By Lemma 2.4 and Corollary 2.1, we have

$$
\begin{aligned}
(2.10)\qquad \widetilde{\Delta}_r m &= {}_r\widetilde{g}^{ij}\frac{\partial^2 m}{\partial x_i \partial x_j} + \frac{1}{\sqrt{{}_r\widetilde{g}}}\frac{\partial}{\partial x_i}\left(\sqrt{{}_r\widetilde{g}}\,{}_r\widetilde{g}^{ij}\right)\frac{\partial m}{\partial x_j} \\
&= \Delta m + \left({}_r\widetilde{g}^{ij} - b^{ij}\right)\,{}_r\widetilde{g}^{ij}\frac{\partial^2 m}{\partial x_i \partial x_j} \\
&\quad + \left[\frac{1}{\sqrt{{}_r\widetilde{g}}}\frac{\partial}{\partial x_i}\left(\sqrt{{}_r\widetilde{g}}\,{}_r\widetilde{g}^{ij}\right) - \frac{1}{\sqrt{b}}\frac{\partial}{\partial x_i}\left(\sqrt{b}b^{ij}\right)\right]\frac{\partial m}{\partial x_j} \\
&= \Delta m + f,
\end{aligned}
$$

where $b = \det(b_{ij})$. Combining (2.9) and (2.10), we have

$$
(2.11)\qquad u^2\,(rH_0)^{-1}\,\widetilde{\Delta}_r m = \frac{1}{n-1}\Delta m + f.
$$

Similarly, one can prove that

$$
3u^4 r^{-1-n} H_0^{-1}|\widetilde{\nabla}_r m|^2 + \left(n - 2 - \mathcal{R}^r r H_0^{-1}\right) m = f.
$$

By (2.7′), the lemma follows. q.e.d.

Lemma 2.7. *In local coordinates on Σ_0, there is a constant m_0*

$$
|m - m_0| + |\nabla m|(x,t) + \left|\frac{\partial m}{\partial t}\right|(x,t) \le Ce^{-t}
$$

for some constant C for all x, t.

Proof. Let $a(t) = ⨍_{\mathbb{S}^{n-1}} m(x,t)$. Here and below, the volume form of $\mathbb{S}^{n-1}$ is understood to be the standard one if there is no specification. Let $\widetilde{m}(x,t) = m(x,t) - a(t)$. Then

$$
(2.12)\qquad \frac{da}{dt} = ⨍_{\mathbb{S}^{n-1}} f(x,t).
$$

where f is the function in Lemma 2.6. In particular, we have

$$
\left|\frac{da}{dt}\right| \le Ce^{-t}.
$$

Hence

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}^{n-1}}\widetilde{m}^2 &= 2\int_{\mathbb{S}^{n-1}}\widetilde{m}\frac{\partial\widetilde{m}}{\partial t}\\ &= \frac{2}{n-1}\int_{\mathbb{S}^{n-1}}\Delta\widetilde{m} + 2\int_{\mathbb{S}^{n-1}}\widetilde{m}\left(f-\frac{da}{dt}\right)\\ &\le -\frac{2}{n-1}\int_{\mathbb{S}^{n-1}}|\nabla\widetilde{m}|^2 + C_1e^{-t}\left(\int_{\mathbb{S}^{n-1}}|\widetilde{m}|^2\right)^{\frac12}\\ &\le -2\int_{\mathbb{S}^{n-1}}|\widetilde{m}|^2 + C_1e^{-t}\left(\int_{\mathbb{S}^{n-1}}|\widetilde{m}|^2\right)^{\frac12}\end{aligned}$$

for some constant C_1 independent of t where we have used (2.12), Lemma 2.6, the fact that $\int_{\mathbb{S}^{n-1}}\widetilde{m}=0$ and the first eigenvalue of $\mathbb{S}^{n-1}$ is $n-1$. From this it is easy to see that

$$\left(\int_{\mathbb{S}^{n-1}}\widetilde{m}^2\right)^{\frac12} \le C_2e^{-t}(t+1) \tag{2.13}$$

for some constant C_2. On the other hand,

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\frac{1}{n-1}\Delta\right)\widetilde{m}^2 &= 2\widetilde{m}\left(\frac{\partial\widetilde{m}}{\partial t}-\frac{1}{n-1}\Delta\widetilde{m}\right)-\frac{2}{n-1}|\nabla\widetilde{m}|^2\\ &\le 2\widetilde{m}\left(f-\frac{da}{dt}\right)\\ &\le C_3e^{-t}\end{aligned} \tag{2.14}$$

for some constant C_3 independent of t, where we have used Corollary 2.1, Lemma 2.6 and (2.12). Hence we have

$$\left(\frac{\partial}{\partial t}-\frac{1}{n-1}\Delta\right)\left(\widetilde{m}^2+C_3e^{-t}\right)\le 0.$$

Using the mean value equality and (2.13), we have

$$\widetilde{m}^2(x,t)\le C_4e^{-t}(t+1)$$

for some C_4 independent of t and x. Put this back to (2.14) and iterate, we conclude that for any $0<\alpha<1$, there is a constant C_5 independent of x and t such that

$$|\widetilde{m}|(x,t)\le C_5e^{-\alpha t}. \tag{2.15}$$

Since $\widetilde{m}$ satisfies:

$$\frac{\partial \widetilde{m}}{\partial t} = \frac{1}{n-1}\Delta \widetilde{m} + f - \fint_{\mathbb{S}^{n-1}} f \tag{2.16}$$

where f is the function in Lemma 2.6, by the interior Schauder estimates [7, Chap 4, Th. 1], we conclude that for some $\beta > 0$

$$|\widetilde{m}|_{2+\beta,\mathbb{S}^{n-1}\times[t,t+1]} \le C_6 e^{-\alpha t} \tag{2.17}$$

for some constant C_6 independent of t. By the definition of f in Lemma 2.6, we have

$$\frac{\partial m}{\partial t} = \frac{1}{n-1}\Delta m + f$$

where $|f(x,t)| \le Ce^{(-1-\alpha)t}$. Hence (2.13) can be improved as

$$\left(\int_{\mathbb{S}^{n-1}} \widetilde{m}^2\right)^{\frac{1}{2}} \le Ce^{-t}$$

and (2.14) can be improved as

$$\left(\frac{\partial}{\partial t} - \frac{1}{n-1}\Delta\right)\widetilde{m}^2 \le Ce^{-2t}.$$

Hence, we have

$$|\widetilde{m}|(x,t) \le C_7 e^{-t} \tag{2.18}$$

for some constant C_7 independent of x and t. Using (2.16), (2.18), Lemma 2.6 and the interior Schauder estimate, (2.17) can be improved as

$$|\widetilde{m}|_{2+\beta,\mathbb{S}^{n-1}\times[t,t+1]} \le C_8 e^{-t}.$$

Use the definition of $\widetilde{m}$, we conclude that

$$|\nabla m|(x,t) + \left|\frac{\partial m}{\partial t}\right| \le C_9 e^{-t}.$$

From the fact that $|\frac{da}{dt}| \le Ce^{-t}$, we conclude that there is a constant m_0 such that $|a(t) - m_0| \le Ce^{-t}$. Combining these with (2.18), the lemma follows. q.e.d.

Lemma 2.8. *Let $z_1, \dots, z_n$ be the standard coordinates on $\mathbb{R}^n$ and let $\rho(z) = \left(\sum_{i=1}^n z_i^2\right)^{\frac{1}{2}}$. Then*

$$u(z) = 1 + \frac{m_0}{\rho^{n-2}} + v$$

where m_0 is the constant in Lemma 2.7, and v satisfies:

$$|v| = O(\rho^{1-n}),$$

and

$$|\nabla_0 v|(z) = O\left(\rho^{-n}(z)\right),$$

where $\nabla_0 v$ is the Euclidean gradient of v.

Proof. By the definitions of m and m_0 and the fact that $|r - \rho|$ is bounded, it is easy to see that $|v| = O\left(r^{1-n}\right)$. Let $\widetilde{v} = u - 1 + \frac{m_0}{r^{2-n}}$. By Lemma 2.7 and the definition of m and $\widetilde{v}$, in local coordinates of Σ_0, we have

$$\left|\frac{\partial \widetilde{v}}{\partial x_i}\right| = \left|\frac{\partial u}{\partial x_i}\right| \le C_1 r^{2-n} \left|\frac{\partial m}{\partial x_i}\right| \le C_1 r^{1-n}. \tag{2.19}$$

Also

$$\begin{aligned} r^{n-2} u \frac{\partial \widetilde{v}}{\partial r} &= r^{n-2} u \frac{\partial}{\partial r}\left(u - 1 - \frac{m_0}{r^{n-2}}\right) \\ &= u\left[\frac{\partial m}{\partial r} - \frac{n-2}{2} r^{n-3}\left(1 - u^{-2}\right) + (n-2) r^{-1} m_0\right] \\ &= u\left[\frac{\partial m}{\partial r} - (n-2) r^{-1}(m - m_0)\right]. \end{aligned} \tag{2.20}$$

By (2.19), (2.20), Lemma 2.7, the fact that $r \sim \rho$ and the fact that $r = e^t$, we have

$$|\nabla_0 \widetilde{v}| = O(r^{-n}). \tag{2.21}$$

If we use the notations in Lemma 2.1, we see that

$$\nabla_0 r = \mathbf{N}.$$

So

$$\frac{\partial r}{\partial z_i} = \mathbf{N}_i = \frac{z_i - x_i}{r} \tag{2.22}$$

where $\mathbf{N}_i$ is the i-th component of $\mathbf{N}$, and $\mathbf{X} = (x_1, \ldots, x_n)$ is the position vector on Σ_0. Since

$$v - \widetilde{v} = \frac{m_0}{\rho^{n-2}} - \frac{m_0}{r^{n-2}}.$$

Combining (2.21), (2.22) and the fact that $|r-\rho|$ is bounded, the lemma is proved. q.e.d.

By Lemma 2.4 and Lemma 2.5, we get $|u-1| = O(r^{2-n})$, $|\nabla_0 u| = O(r^{1-n})$, and $|\nabla_0^2 u| = O(r^{-n})$ by a direct computation, here, ∇_0 and ∇_0^2 are the gradient and Hessian operator of the Euclidean metric respectively. If we write

$$u^2 dr^2 + g_r = \sum_{i,j} g_{ij} dz_i dz_j.$$

Then direct computations show (see the computations in (2.24), (2.27) below, for example):

$$|g_{ij} - \delta_{ij}| + \rho|\nabla_0 g_{ij}| + \rho^2|\nabla_0^2 g_{ij}| \leq C\rho^{2-n}. \tag{2.23}$$

By the result in [1], the ADM mass of the metric $ds^2 = u^2 dr^2 + g_r$ is well-defined, because the scalar curvature of ds^2 is zero outside a compact set.

Lemma 2.9. *The* ADM *mass of the metric $u^2 dr^2 + g_r$ is equal to $c(n)m_0$, where $c(n)$ is a positive constant depending on n.*

Proof. Let z be the standard metric on $\mathbb{R}^n$, and consider the metric

$$g = u^2 dr^2 + g_r = dr^2 + g_r + (u^2-1)dr^2.$$

If we write $g = \sum_{i,j} g_{ij} dz_i dz_j$, then

$$g_{ij} = \delta_{ij} + b_{ij}$$

where $\sum_{i,j} b_{ij} dz_i dz_j = (u^2-1)dr^2$. Hence

$$b_{ij} = (u^2-1)\frac{\partial r}{\partial z_i}\frac{\partial r}{\partial z_j}. \tag{2.24}$$

The ADM mass of g is given by

$$\lim_{\rho\to\infty} \int_{\mathbb{S}^{n-1}} \left(\frac{\partial g_{ij}}{\partial z_i} - \frac{\partial g_{ii}}{\partial z_j}\right) \rho^{n-2} z_j dV_0 \tag{2.25}$$

where dV_0 is the standard metric on $\mathbb{S}^{n-1}$. By (2.22)

$$\frac{\partial^2 r}{\partial z_i \partial z_j} = \frac{\delta_{ij}}{r} - \frac{z_i z_j}{r^3} + O(r^{-2}) = \frac{\delta_{ij}}{\rho} - \frac{z_i z_j}{\rho^3} + O(\rho^{-2}) \tag{2.26}$$

$$\frac{\partial g_{ij}}{\partial z_i} = \frac{\partial b_{ij}}{\partial z_i} = 2u \frac{\partial u}{\partial z_i} \frac{\partial r}{\partial z_i} \frac{\partial r}{\partial z_j} + (u^2 - 1)\left(\frac{\partial^2 r}{\partial z_i^2} \frac{\partial r}{\partial z_j} + \frac{\partial r}{\partial z_i} \frac{\partial^2 r}{\partial z_i \partial z_j}\right). \tag{2.27}$$

Now

$$\begin{aligned} 2u \frac{\partial u}{\partial z_i} \frac{\partial r}{\partial z_i} \frac{\partial r}{\partial z_j} &= 2\left(1 + m_0 \rho^{2-n} + O\left(\rho^{1-n}\right)\right)\left(-(n-2) m_0 \rho^{-n} z_i + O\left(\rho^{-n}\right)\right) \\ &\quad \cdot \left(\rho^{-2} z_i z_j + O\left(\rho^{-2}\right)\right) \\ &= 2(n-2) m_0 \rho^{-n} z_j + O\left(\rho^{-n}\right). \end{aligned} \tag{2.28}$$

Here repeated indices mean summation.

$$(u^2 - 1)\left(\frac{\partial^2 r}{\partial z_i^2} \frac{\partial r}{\partial z_j} + \frac{\partial r}{\partial z_i} \frac{\partial^2 r}{\partial z_i \partial z_j}\right) = 2 m_0 (n-1) \rho^{-n} z_j + O\left(\rho^{-n}\right) \tag{2.29}$$

$$\frac{\partial g_{ii}}{\partial z_j} = \frac{\partial h_{ii}}{\partial z_j} = 2u \frac{\partial u}{\partial z_j} \frac{\partial r}{\partial z_i} \frac{\partial r}{\partial z_i} + 2(u^2 - 1)\left(\frac{\partial r}{\partial z_i} \frac{\partial^2 r}{\partial z_i \partial z_j}\right) \tag{2.30}$$

$$\begin{aligned} 2u \frac{\partial u}{\partial z_j} \frac{\partial r}{\partial z_i} \frac{\partial r}{\partial z_i} &= 2\left(1 - m_0 \rho^{2-n} + O\left(\rho^{1-n}\right)\right)\left((n-2) m_0 \rho^{-n} z_i + O\left(\rho^{-n}\right)\right) \\ &\quad \cdot \left(1 + O\left(\rho^{-2}\right)\right) \\ &= 2(n-2) m_0 \rho^{-n} z_j + O\left(\rho^{-n}\right) \end{aligned} \tag{2.31}$$

$$2(u^2 - 1)\left(\frac{\partial r}{\partial z_i} \frac{\partial^2 r}{\partial z_i \partial z_j}\right) = O\left(\rho^{-n}\right). \tag{2.32}$$

Combining (2.25), (2.27)-(2.32), the lemma follows. q.e.d.

Lemma 2.10.

$$\lim_{r\to\infty}\int_{\Sigma_r}(H_0-H)\,d\sigma_r=\lim_{r\to\infty}\int_{\Sigma_r}H_0(1-u^{-1})d\sigma_r=(n-1)\omega_{n-1}m_0$$

where ω_{n-1} is the volume of the standard sphere $\mathbb{S}^{n-1}$ and H_0 and H are the mean curvature of Σ_r with respect to the Euclidean metric and the metric $u^2dr^2+g_r$ respectively.

Proof. The result follows from (1.6), Lemmas 2.1, 2.8, 2.9 and the definition of Σ_r. q.e.d.

We can summarize the results in Lemmas 2.3, 2.8, 2.9 and 2.10 as follows:

Theorem 2.1. *The initial value problem* (2.1) *has a unique solution u on $\Sigma_0\times[0,\infty)$ such that:*

(a)

$$u(z)=1+\frac{m_0}{\rho^{n-2}}+v$$

where m_0 is a constant and v satisfies $|v|=O\left(\rho^{1-n}\right)$ and $|\nabla_0 v|=O\left(\rho^{-n}\right)$.

(b) *The metric $ds^2=u^2dr^2+g_r$ is asymptotically flat in the sense of* (2.23) *with scalar curvature $\mathcal{R}\equiv 0$ outside Σ_0.*

(c) *The* ADM *mass m_{ADM} of ds^2 is given by*

$$\begin{aligned}c(n)m_{\mathrm{ADM}}=(n-1)\omega_{n-1}m_0&=\lim_{r\to\infty}\int_{\Sigma_r}H_0(1-u^{-1})d\sigma_r\\&=\lim_{r\to\infty}\int_{\Sigma_r}(H_0-H)\,d\sigma_r,\end{aligned}$$

for some positive constant $c(n)$, where H_0 and H are the mean curvatures of σ_r with respect to the Euclidean metric and ds^2 respectively.

If we let $u_0\equiv k$ for $k\geq 1$, it is easy to see from Lemma 2.2, that the solution $u^{(k)}$ of (2.1) are uniformly bounded on $[a,\infty)$ for all $a>0$. Hence as in [3], we can solve (2.1) with initial value $u_0^{-1}=0$. In fact, by Lemma 2.2, u_0 satisfies:

$$\left[1-\exp\left(-\int_0^r \psi(s)ds\right)\right]^{-\frac{1}{2}} \le u_0(x,r) \le \left[1-\exp\left(-\int_0^r \varphi(s)ds\right)\right]^{-\frac{1}{2}}.$$

This means that Σ_0 is a minimal surface with respect to the asymptotically flat metric $u^2dr^2+g_r$. As in [3], we have the following:

Lemma 2.11. *Let*

$$M(r)=\int_{\Sigma_r} H_0(1-u^{-2})d\sigma_r,$$

then $M(r)$ *is nondecreasing.*

Proof. By the Gauss equations, it is easy to see that

$$\frac{\partial H_0}{\partial r}=-\sum_{i,j=1}^{n-1}\left(h_{ij}^0\right)^2$$

where h_{ij}^0 is the second fundamental form of Σ_r in $\mathbb{R}^n$. By direct computation, we see:

$$\begin{aligned}
&\frac{d}{dr}\int_{\Sigma_r} H_0(1-u^{-2})d\sigma_r\\
&=\int_{\Sigma_r}\left({H_0}^2(1-u^{-2})+2H_0u^{-3}\frac{\partial u}{\partial r}+\frac{\partial H_0}{\partial r}(1-u^{-2})\right)d\sigma_r\\
&=2\int_{\Sigma_r} u^{-1}\triangle_{\Sigma_r} u d\sigma_r\\
&=2\int_{\Sigma_r} u^{-2}|\nabla u|^2 d\sigma_r\\
&\ge 0
\end{aligned}$$

where we have used the fact that u satisfies (2.1) and that

$$H_0^2-\sum_{i,j=1}^{n-1}\left(h_{ij}^0\right)^2=\mathcal{R}^r.$$

Hence, $M(r)$ is nondecreasing. q.e.d.

Thus, as in [3, Corollary 1.1] we have:

Proposition 2.1. *Let u be the solution of* (2.1) *with initial value $u_0^{-1} = 0$. Let m_{ADM} be the* ADM *mass of the metric $u^2dr^2 + g_r$. Then*

$$m_{\mathrm{ADM}} \geq C(n) \int_{\Sigma_0} H_0 d\sigma_0$$

for some positive constant $C(n)$ depending only on n.

3. Positive mass theorem on manifolds with Lipschitz metric

In order to prove the main result we need to verify that the positive mass theorem is still true for some manifolds whose metrics may be only Lipschitz. In this section, we always assume that N^n is an orientable complete noncompact smooth manifold with dimension n, such that there is bounded domain $\Omega \subset N$ with smooth boundary $\partial\Omega$. We also assume that N is spin (which is always true if $n = 3$) and there is a continuous Riemannian metric g on N such that:

(i) g is smooth on $N \setminus \Omega$ and $\overline{\Omega}$, and is Lipschitz near $\partial\Omega$.

(ii) The mean curvatures at $\partial\Omega$ with respect to the outward normal and with respect to the metrics $g|_{N\setminus\Omega}$ and $g|_{\overline{\Omega}}$ are the same.

(iii) N has finitely many ends, each of which is asymptotically Euclidean in the following sense: There is a compact set K containing Ω such that $N \setminus K = \cup_{i=1}^{\ell} E_i$. Each E_i is diffeomorphic to $\mathbb{R}^n \setminus B_{R_i}(0)$ and in the standard coordinates in $\mathbb{R}^n$, the metric g satisfies

$$g_{ij} = \delta_{ij} + b_{ij},$$

with

$$\|b_{ij}\| + r\|\partial b_{ij}\| + r^2\|\partial\partial b_{ij}\| = O(r^{2-n}) \tag{3.1}$$

where r and ∂ denotes Euclidean distance and the standard gradient operator on $\mathbb{R}^n$, respectively.

(iv) The scalar curvature of $N \setminus \partial\Omega$ is nonnegative and is in $L^1(N)$.

We should remark that because of (i), the outward unit normal on $\partial\Omega$ is well-defined. Moreover, (iii) and (iv) imply that the ADM mass of each end of N is also well-defined by the proof in [1]. Explicitly, the ADM mass at each end E is given by

$$C(n)m_E = \lim_{r\to\infty} \int_{S(r)} (g_{ij,j} - g_{jj,i})\, dS_i$$

where $C(n)$ is a positive constant, $S(r)$ is the Euclidean sphere and dS_i is the normal surface area of $S(r)$.

We have the following:

Theorem 3.1. *Let (N, g) as above. Then $m_E \geq 0$ for any end E of N. Moreover, if the* ADM *mass of one of ends of N is zero, then N has only one end and N is flat.*

We will use the argument of Witten [28, 20, 1]. Let us first fix some notations. In the following, a local orthonormal frame e_i, $1 \leq 1 \leq n$ means that $e_i = a_{ij}\frac{\partial}{\partial x_j}$ with Lipschitz functions a_{ij} which are smooth on $N \setminus \Omega$ and $\overline{\Omega}$, where $(x_1, \ldots, x_n)$ are smooth local coordinates. By the assumptions on g, we can always find such a local frame near each point.

Let e_i be a local orthonormal frames and ω_i be the dual 1-forms. Let ω_{ij} be the connection forms of g and let $\{\sigma_I\}$ be the orthonormal base of fibers of the spinor bundle $\mathcal{S}$ with respect to $\{e_i\}$, ∇ be the covariant derivative on $\mathcal{S}$, then we have:

$$\nabla \sigma_I = -\frac{1}{4}\sum_{i,j} \omega_{ij} \otimes e_i \cdot e_j \cdot \sigma_I,$$

where "·" refers to Clifford multiplication. By the above notations, the Dirac operator can be expressed in the following way:

$$\mathcal{D} = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}.$$

A spinor Ψ is said to be in $W^{1,2}_{\mathrm{loc}}(U)$ in some open set U if near each point $x \in U$ there is a local orthonormal frame e_i, such that if σ_I is a base for $\mathcal{S}$ as above and

$$\Psi = \sum_I \Psi^I \sigma_I$$

then Ψ^I is in $W^{1,2}$ near x. That is to say, Ψ^I has weak derivatives so that Ψ^I together with its weak derivatives are in L^2_{loc}. Note this is well-defined because the transition functions from one orthonormal base to another one are Lipschitz. Note that it is also meaningful to say that Ψ is locally Hölder or locally Lipschitz. For an open set U, the norm $W^{1,2}$ norm $|||\Psi|||$ of a spinor Ψ is defined as

$$|||\Psi|||^2 = \int_U ||\nabla\Psi||^2 + ||\Psi||^2.$$

Near a point $x \in \partial\Omega$, choose an orthonormal frame e_i such that $e_n = \frac{\partial}{\partial\rho}$ where ρ is the signed distance function from $\partial\Omega$. $\rho > 0$ outside Ω and $\rho < 0$ in Ω. Moreover e_i, $1 \le i \le n-1$, are chosen so that they are obtained by parallel translation along the integral curves of $\frac{\partial}{\partial\rho}$ which are geodesics normal to $\partial\Omega$. We call this kind of frame to be an *adapted* orthonormal frame. Let ω_i be the dual of e_i and let ω_{ij} be the connection forms. It is easy to see that we have the following:

Lemma 3.1. *With the above notations, $\omega_{ij}(e_k)$ are Lipschitz for $1 \le i,j \le n-1$ and for all k. Moreover, $\omega_{ij}(e_n) = 0$ for all i, j.*

Under the *adapted* orthonormal frame, we have:

$$\mathcal{R} = -2\frac{\partial H}{\partial \rho} - \sum_{i,j} {h_{ij}}^2 - H^2 + \mathcal{R}^\rho,$$

where h_{ij} is components of the second fundamental form, H is the mean curvature and $\mathcal{R}^\rho$ is the scalar curvature of the hypersurface with distance ρ from $\partial\Omega$. Since H matches along $\partial\Omega$, it is Lipschitz. By this formula we see that $\mathcal{R}$ is well-defined in the distribution sense. This is important in the proof of the following Lichnerowicz formula.

Lemma 3.2. *Let U be a open set of N. For any spinor $\eta \in W_0^{1,2}(U)$, $\Psi \in W_{\mathrm{loc}}^{1,2}(U)$, we have:*

$$\int_U \langle \mathcal{D}\Psi, \mathcal{D}\eta\rangle = \int_U \langle \nabla\Psi, \nabla\eta\rangle + \frac{1}{4}\int_U \mathcal{R}\langle\Psi,\eta\rangle,$$

where $\mathcal{R}$ is the scalar curvature of N.

Proof. Let $T = \partial\Omega \cap U$. Since the metric g is smooth up to the boundary on $\overline{\Omega}$, by the standard Lichnerowicz formula applied to $\Omega\cap U$, we have:

$$\int_{\Omega\cap U} \langle \mathcal{D}\Psi, \mathcal{D}\eta\rangle + \int_T \langle \nu\cdot D\Psi + \nabla_\nu\Psi, \eta\rangle = \int_{\Omega\cap U} \langle \nabla\Psi, \nabla\eta\rangle + \frac{1}{4}\int_{\Omega\cap U} \mathcal{R}\langle\Psi,\eta\rangle, \tag{3.2}$$

here ν is the outer normal unit vector of $\partial\Omega$, see [1, p. 689] for example.

Let $\{e_i\}$ be an adapted frame near a point $x \in T$. Direct computations show that

$$\begin{aligned}&\nu \cdot \mathcal{D}\Psi + \nabla_\nu \Psi \\ &= \nu \cdot \sum_{i=1}^{n-1} e_i \cdot (\nabla_{e_i} \Psi^I)\sigma_I - \frac{1}{4}\nu \cdot \sum_{i=1}^{n-1}\sum_{s,t=1}^{n-1} \omega_{st}(e_i)e_i \cdot e_s \cdot e_t \cdot \Psi + \frac{H}{4}\nu \cdot \Psi.\end{aligned}$$

where H is the mean curvature of the level set ρ=constant with respect to $\frac{\partial}{\partial\rho}$. Hence, we have:

$$\begin{aligned}&\int_{\Omega\cap U} \langle \mathcal{D}\Psi, \mathcal{D}\eta\rangle + \int_T \nu \cdot \sum_{i=1}^{n-1} e_i \cdot (\nabla_{e_i}\Psi^I)\sigma_I \\ &- \frac{1}{4}\int_T \nu \cdot \sum_{i=1}^{n-1}\sum_{s,t=1}^{n-1} \omega_{st}(e_i)e_i \cdot e_s \cdot e_t \cdot \Psi + \int_T \frac{H}{4}\nu \cdot \Psi \\ &= \int_{\Omega\cap U} \langle \nabla\Psi, \nabla\eta\rangle + \frac{1}{4}\int_{\Omega\cap U} \mathcal{R}\langle \Psi, \eta\rangle.\end{aligned}$$

By the same reasoning, we have the formula on $U \setminus \overline{\Omega}$,

$$\begin{aligned}&\int_{U\setminus\overline{\Omega}} \langle \mathcal{D}\Psi, \mathcal{D}\eta\rangle - \int_T \nu \cdot \sum_{i=1}^{n-1} e_i \cdot (\nabla_{e_i}\Psi^I)\sigma_I \\ &+ \frac{1}{4}\int_T \nu \cdot \sum_{i=1}^{n-1}\sum_{s,t=1}^{n-1} \omega_{st}(e_i)e_i \cdot e_s \cdot e_t \cdot \Psi - \int_T \frac{H}{4}\nu \cdot \Psi \\ &= \int_{U\setminus\overline{\Omega}} \langle \nabla\Psi, \nabla\eta\rangle + \frac{1}{4}\int_{U\setminus\overline{\Omega}} \mathcal{R}\langle \Psi, \eta\rangle.\end{aligned}$$

In the above, we have used the fact that the mean curvatures of T in Ω are equal to that of $N \setminus \Omega$, and the unit outward normals are in opposite directions. Adding these two equalities, we see the integrals on T are canceled. Hence, the proof of the lemma is completed. q.e.d.

Let Ψ be a spinor in $W^{1,2}_{\mathrm{loc}}(U)$. Ψ is said to satisfy

$$\mathcal{D}^2\Psi = 0$$

in the weak sense in an open set U if for any spinor $\Phi \in W^{1,2}_0(U)$,

$$\int_U \langle \mathcal{D}\Psi, \mathcal{D}\Phi\rangle = 0.$$

Even though g is not smooth, however the coefficients of a weak solution Ψ of $\mathcal{D}^2\Psi = 0$ with respect to an adapted frame behave well. Namely, we have:

Lemma 3.3. *Suppose $\Psi \in W^{1,2}_{\text{loc}}(U)$ satisfies $\mathcal{D}^2\Psi = 0$ weakly in an open set U. Then Φ is locally Hölder continuous and Ψ is in $W^{2,2}_{\text{loc}}(U)$ in the following sense:*

(a) *If $x \notin \partial\Omega$, then $\Psi \in W^{2,2}$ near x.*

(b) *If $x \in \partial\Omega$, and if $\{e_i\}$ is an adapted orthonormal frame near x so that $\{\sigma_I\}$ is an orthonormal basis for $\mathcal{S}$ with respect to $\{e_i\}$ and that $\Psi = \sum_I \Psi^I \sigma_I$, then Ψ^I is in $W^{2,2}$ and is Hölder continuous near x.*

Proof. It is sufficient to study the behavior of Ψ near a point in $\partial\Omega$. Let $\Psi = \sum_I \Psi^I \sigma_I$ as in case (b). We may assume U is small enough so that there is an adapted orthonormal frame e_i in U. We claim that Ψ^I satisfies the following equations:

$$\triangle\Psi^I + \sum_{J,i} A^I_{Ji} e_i \Psi^J + \sum_J B^I_J \Psi^J = 0,$$

in the weak sense, where, $\|A^I_{Ji}\|_{L^\infty} + \|B^I\|_{L^\infty} \leq C < \infty$ locally, and Δ is the Laplacian for function on N. In particular, for each fixed I, Ψ^I satisfies the following equation in the weak sense:

$$\triangle\Psi^I = f$$

where $f = -\sum_{J,i} A^I_{Ji} e_i \Psi^J - \sum_J B^J_I \Psi^J$. Since $\Psi \in W^{1,2}_{\text{loc}}(U)$, $f \in L^2_{\text{loc}}(U)$. Since the metric is Lipschitz, in local coordinates Δ is of divergence form with coefficients being Lipschitz. Then, by the standard theory in elliptic equations, we know that $\Psi^I \in W^{2,2}_{\text{loc}}(U)$, see [8, Theorem 8.8]. Hence by Sobolev embedding theorem, $\Delta\Psi^I$ is in $L^p_{\text{loc}}(U)$ for $p = 2n/(n-p)$, and $\Psi^I \in W^{2,p}_{\text{loc}}(U)$, see [8, Lemma 9.16]. We can then iterate by using the Sobolev embedding theorem to conclude that the lemma is true.

To prove the claim, let $\Phi = \sum_I \Phi^I \sigma_I \in W^{1,2}_0(U)$. Then

$$\nabla_{e_i}\Psi = \sum_I e_i(\Psi^I)\sigma_I - \frac{1}{4}\sum_{k,l,I} \Psi^I \omega_{kl}(e_i) e_k \cdot e_l \cdot \sigma_I$$

and

$$\nabla_{e_i}\Phi = \sum_I e_i(\Phi^I)\sigma_I - \frac{1}{4}\sum_{k,l,I}\Phi^I\omega_{kl}(e_i)e_k \cdot e_l \cdot \sigma_I$$

where ω_{kl} are the connection forms with respect to the adapted frame e_i. By Lemma 3.1

$$\begin{aligned}
&\langle \nabla\Psi, \nabla\Phi\rangle \\
&= \sum_I \langle \nabla\Psi^I, \nabla\Phi^I\rangle - \frac{1}{4}\sum_{i=1}^{n-1}\sum_{j,k,l,I,J}\langle \Psi^I\omega_{kl}(e_i)e_k \cdot e_l \cdot \sigma_I, e_i(\Phi^J)\sigma_J\rangle \\
&\quad - \frac{1}{4}\sum_{i=1}^{n-1}\sum_{j,k,l,I,J}\langle e_i(\Psi^I)\sigma_I, \Phi^J\omega_{kl}(e_i)e_k \cdot e_l \cdot \sigma_J\rangle + \sum_{I,J} a_{IJ}\Psi^I\overline{\Phi}^J
\end{aligned}$$

where a_{IJ} is a bounded function. Since $e_i(\omega_{kl}(e_i))$ is smooth up to boundary in $N \setminus \Omega$ and in Ω, and for $1 \le i \le n-1$ we can perform integration by parts to conclude that

$$\begin{aligned}
&\sum_{i=1}^{n-1}\int_U \sum_{j,k,l,I,J}\langle \Psi^I\omega_{kl}(e_i)e_k \cdot e_l \cdot \sigma_I, e_i(\Phi^J)\sigma_J\rangle \\
&= \sum_{i=1}^{n-1}\sum_{IJ}\int_U \left(e_i(\Psi^I)\overline{\Phi}^J b_{iIJ} + \Psi^I\overline{\Phi}^J c_{iIJ}\right),
\end{aligned}$$

where b_{iIJ} and c_{iIJ} are L^∞ functions in U. For simplicity, we set:

$$\begin{aligned}
&\sum_{i=1}^{n-1}\int_U \sum_{j,k,l,I,J}\langle e_i(\Psi^I)\sigma_I, \Phi^J\omega_{kl}(e_i)e_k \cdot e_l \cdot \sigma_I\rangle \\
&= \sum_{i=1}^{n-1}\sum_{IJ}\int_U \left(e_i(\Psi^I)\overline{\Phi}^J d_{iIJ}\right),
\end{aligned}$$

here, d_{iIJ} are also L^∞ functions in U. By Lemma 3.2 and the fact that Ψ is a weak solution of $\mathcal{D}^2\Psi = 0$, it is easy to see that the claim is true with

$$\begin{aligned}
{A^J}_{Ii} &= -\frac{1}{4}(b_{iIJ} + d_{iIJ}), \\
{B^J}_I &= a_{IJ} + \sum_{i=1}^{n-1} c_{iIJ} + \frac{\mathcal{R}}{4}\delta_{IJ}
\end{aligned}$$

where $\delta_{II} = 1$ and $\delta_{IJ} = 0$ if $I \neq J$. This completes the proof of the lemma. q.e.d.

As a corollary, we have:

Corollary 3.1. *Suppose Ψ is $W^{1,2}$ weak solution of*

$$\mathcal{D}^2\Psi = 0$$

in an open set U in N. Then $\mathcal{D}\Psi \in W^{1,2}_{\text{loc}}(U)$.

Proof. It is sufficient to consider the behavior of Φ near a point x in $\partial\Omega$. We choose an adapted orthonormal frame near x as before. With the notations as in the proof of the previous lemma, we have

$$\mathcal{D}\Psi = \sum_I \nabla\Psi^I \cdot \sigma_I - \frac{1}{4}\sum_{i,j,k=1}^{n-1}\sum_I \omega_{kl}(e_i)e_i \cdot e_k \cdot e_l \cdot \sigma_I - He_n \cdot \Psi$$

where we have used Lemma 3.1. Here H is the mean curvature of the level surface ρ=constant. By Lemma 3.3, first term in the above equality is in $W^{1,2}_{\text{loc}}$. By Lemma 3.1, by the assumption of the smoothness of g, we see for $1 \leq i, k, l \leq n-1$, $\omega_{kl}(e_i)$ is Lipschitz. By the assumption of the mean curvature on $\partial\Omega$, we see H is also Lipschitz on N. The corollary follows because being in $W^{1,2}_{\text{loc}}$ does not depend on the choice of orthonormal frame. q.e.d.

Lemma 3.4. *Suppose $\sigma \in W^{1,2}_{\text{loc}}(N)$, $\int_N \|\sigma\|^2 < \infty$ and $\mathcal{D}\sigma = 0$. Then $\sigma = 0$.*

Proof. By the assumption and Lemma 3.2, for any $\xi \in W^{1,2}_0(N)$, we have:

$$0 = \int_N \langle\, D\sigma, \mathcal{D}\xi\rangle = \int_N \langle\nabla\sigma, \nabla\xi\rangle + \frac{1}{4}\int_N \mathcal{R}\langle\sigma, \xi\rangle$$

Let $\xi = \eta^2\sigma$, here η is a cut-off function such that for $\rho > 0$,

$$\eta = \begin{cases} 1 & \text{in } B_\rho(o) \\ 0 & \text{outside } B_{2\rho}(o), \end{cases}$$

and

$$|\nabla\eta| \leq \frac{C}{\rho}.$$

Then, we get:

$$\int_{B_\rho} \|\nabla\sigma\|^2 \leq \frac{C}{\rho}\int_N \|\sigma\|^2 \to 0.$$

Hence, σ is parallel inside and outside Ω. Thus $\sigma = 0$ outside Ω because $\int_N ||\sigma||^2 < \infty$. $\sigma = 0$ inside Ω because σ is continuous on N by Lemma 3.3 and is parallel. This completes the proof of the lemma.

q.e.d.

Now one can proceed as in the case that g is smooth to prove the positive mass theorem.

Proof of Theorem 3.1 Let us first prove that $m_E \geq 0$ for all end E. We assume that N has only one end, the proof for the general case is similar. Let η be a parallel spinor outside $\mathbb{R}^n$ with respect to the Euclidean metric. We may extend η so that it is zero on a neighborhood of $\overline{\Omega}$. By the asymptotic conditions on g, $||\eta||$ is asymptotically constant,

$$\|\mathcal{D}\eta\|(x) = O\left(r^{1-n}(x)\right), \tag{3.3}$$

and

$$||\mathcal{D}^2\eta||(x) = O\left(r^{-n}(x)\right). \tag{3.4}$$

Here r is the geodesic distance function with respect to g. Let $R > 0$ be large enough, then one can find spinor $\Psi_R \in W^{1,2}(B_o(R))$ where $o \in N$ is a fixed point, so that

$$\mathcal{D}^2\Psi_R = 0$$

in the weak sense in $B_o(R)$ such that $\Phi_R = \eta$ on $\partial B_o(R)$. This is equivalent to solve the following:

$$\begin{cases} \mathcal{D}^2\sigma_R & = -\mathcal{D}^2\eta \ \text{ in } B_o(R), \\ \sigma_R|_{\partial B_o(R)} & = 0 \end{cases} \tag{$*$}$$

One may use Lax-Milgram theorem to solve $(*)$. Indeed, in the Hilbert space consisting of all spinors in ${W_0}^{1,2}(B_o(R))$, define the sesqui-bilinear form:

$$a(\Phi, \Psi) = \int_{B_o(R)} \langle \mathcal{D}\Phi, \mathcal{D}\Psi \rangle.$$

Consider the linear functional

$$F(\Psi) = -\int_{B_o(R)} \langle \mathcal{D}\Psi, \mathcal{D}\eta \rangle.$$

It is easy to see that:

(i) a is bounded, i.e., there is $C > 0$ such that:

$$|a(\Phi, \Psi)| \leq C|||\Phi|||\,|||\Psi|||.$$

(ii) It is positive by Lemma 3.2, the fact that $\mathcal{R} \geq 0$ and the Poincaré inequality. That is, there is $\delta > 0$ such that:

$$a(\Psi, \Psi) \geq \delta|||\Psi|||^2.$$

(iii) F is bounded.

Then by Lax-Milgram theorem [31, Sec. 7, Chap. 3], we conclude that $(*)$ has a solution. By Lemma 3.3, Ψ_R is bounded. Hence $||\Psi_R||^2$ is in $W^{1,2}(B_o(R))$. Moreover, if $f \in C_0^\infty(B_o(R))$ with $f \geq 0$, then

$$\begin{aligned}
\int_{B_o(R)} \langle \nabla ||\Psi_R||^2, \nabla f \rangle &= \int_{B_o(R)} \left(\langle \nabla \Psi_R, \nabla (f\Psi_R) \rangle + \langle \nabla (f\Psi_R), \nabla \Psi_R \rangle \right) \\
&\quad - 2\int_{B_o(R)} f||\nabla \Psi_R||^2 \\
&= -\frac{1}{2}\int_{B_o(R)} f\left(\mathcal{R}||\Psi_R||^2 + 4||\nabla \Psi_R||^2\right) \\
&\leq 0
\end{aligned}$$

where we have used the Lichnerowicz formula in Lemma 3.2. Hence $||\Psi_R||^2$ is subharmonic in the weak sense. Since η is uniformly bounded, we conclude that Ψ_R are uniformly bounded by the maximum principle. Hence there is $R_i \to \infty$ such that $\Psi_i = \Psi_{R_i}$ converges in $W^{1,2}_{\text{loc}}(N)$ to Ψ with $\mathcal{D}^2\Psi = 0$ in the weak sense.

We claim that Ψ is asymptotically close to η in the following sense:

$$||\Psi - \eta||(x) \leq Cr^{2-n}(x)\log r(x) \tag{3.5}$$

for some constant if $r(x)$ is large enough, and

$$\int_N ||\nabla(\Psi - \eta)||^2 + ||\mathcal{D}(\Psi - \eta)||^2 < \infty. \tag{3.6}$$

To prove the claim, let us assume $\overline{\Omega} \subset B_o(R_0)$ and that $R_i > R_0$ for all i. Then for any i, since $\Psi_i - \sigma = 0$ on $\partial B_o(R_i)$, we have

$$\int_{B_o(R_i)} \langle \mathcal{D}\Psi_i, \mathcal{D}(\Psi_i - \sigma) \rangle = 0.$$

From this, it is easy to see that

$$\int_{B_o(R_i)} ||\mathcal{D}(\Psi_i - \eta)||^2 \leq \int_N ||\mathcal{D}\sigma||^2.$$

Moreover, by Lemma 3.2 and the fact that $\mathcal{R} \geq 0$, it is easy to see that

$$\int_{B_o(R_i)} ||\nabla(\Psi_i - \eta)||^2 \leq \int_{B_o(R_i)} ||\mathcal{D}(\Psi_i - \eta)||^2 \leq \int_N ||\mathcal{D}\sigma||^2.$$

Using (3.3) and the fact that Ψ_i converge to Ψ in $W^{1,2}_{\text{loc}}(N)$ weakly, we conclude that (3.6) is true.

On the other hand, since Ψ_i are uniformly bounded, there is a constant C_1 such that

$$||\Psi_i - \eta|| \leq C_1$$

on $\partial B_o(R_0)$. Let $u \geq 0$ be a solution of $\Delta u \leq -||\mathcal{D}^2\eta||$ outside $B_o(R_0)$ such that $u \geq C_1$ on $\partial B_o(R_1)$ and such that $u(x) \leq C_2 r^{2-n}(x) \log r(x)$ for some constant $C_2 > 0$. Such u can be found, see for example [26]. Note that outside $B_o(R_0)$, Ψ_i is smooth and by the usual Lichnerowicz formula,

$$\Delta||\Psi_i - \eta|| \geq -||\mathcal{D}^2\eta||$$

in the weak sense on $B_o(R_i) \setminus B_o(R_0)$. Hence

$$||\Psi_i - \eta||(x) \leq u(x)$$

on $B_o(R_i) \setminus B_o(R_0)$. Note that Ψ_i converges pointwisely to Ψ outside $B_o(R_0)$, hence (3.5) is true.

By Corollary 3.1, $\mathcal{D}\Psi \in W^{1,2}_{\text{loc}}(N)$. Apply Lemma 3.4 to $\sigma = \mathcal{D}\Psi$ and using (3.6), we conclude that $\mathcal{D}\Psi = 0$.

Now choose η to be nonzero constant spinor (with respect to the Euclidean metric) near infinity normalized so that $||\eta||$ is asymptotically 1 at infinity. Let $S(r)$ be the Euclidean sphere with radius r near infinity of N and let $U(r)$ be the interior of $S(r)$, then by Lemma 3.2,

$$\int_{S(r)} \langle \nu \cdot \mathcal{D}\Psi + \nabla_\nu \Psi, \Psi \rangle = \int_{U(r)} ||\nabla\Psi||^2 + \frac{1}{4}\int_{U(r)} \mathcal{R}||\Psi||^2. \tag{3.7}$$

where ν is the outward normal of $S(r)$. On the other hand,

$$
\begin{aligned}
(3.8)\qquad &\int_{S(r)} \langle \nu\cdot\mathcal{D}\Psi + \nabla_\nu\Psi, \Psi\rangle \\
&= \int_{S(r)} \langle \mathcal{D}_T\Psi, \Psi\rangle \\
&= \int_{S(r)} \langle \mathcal{D}_T(\Psi-\eta), \Psi-\eta\rangle + \int_{S(r)} \langle \mathcal{D}_T(\Psi-\eta), \eta\rangle \\
&\quad + \int_{S(r)} \langle \mathcal{D}_T\eta, \Psi-\eta\rangle + \int_{S(r)} \langle \mathcal{D}\eta, \eta\rangle \\
&= \int_{S(r)} \langle \mathcal{D}_T(\Psi-\eta), \Psi-\eta\rangle + \int_{S(r)} \langle \Psi-\eta, \mathcal{D}_T\eta\rangle \\
&\quad + \int_{S(r)} \langle \mathcal{D}_T\eta, \Psi-\eta\rangle + \int_{S(r)} \langle \mathcal{D}\eta, \eta\rangle
\end{aligned}
$$

where $\mathcal{D}_T = \sum_{i=1}^{n-1} e_i\nabla_{e_i}$, with e_i to be orthonormal and tangential to $S(r)$. Here we have used the fact that D_T is self-adjoint on the boundary.

By (3.5) and (3.6), for each r large enough, we may choose $r'\in(r,2r)$, such that:

$$
\begin{aligned}
&\left|\int_{S(r')} \langle \mathcal{D}_T(\Psi-\eta), \Psi-\eta\rangle\right| \\
&\le \int_{\partial B'_r} \|\langle \mathcal{D}_T(\Psi-\eta), \Psi-\eta\rangle\| \\
&= \frac{1}{r}\int_{U(2r)\setminus U(r)} \|\langle \mathcal{D}_T(\Psi-\eta), \Psi-\eta\rangle\| \\
&\le \frac{1}{r}\left(\int_{U(2r)\setminus U(r)} |\langle \mathcal{D}_T(\Psi-\eta)|^2\right)^{\frac{1}{2}} \left(\int_{U(2r)\setminus U(r)} ||\Psi-\eta||^2\right)^{\frac{1}{2}} \\
&\le Cr^{1-\frac{n}{2}}\log r.
\end{aligned}
$$

Thus, we see that we can find $r_i\to\infty$ such that

$$
(3.9)\qquad \lim_{i\to\infty}\int_{S(r_i)} \langle \mathcal{D}_T(\Psi-\eta), \Psi-\eta\rangle = 0.
$$

By (3.5) and (3.3), we also have

$$
(3.10)\qquad \lim_{i\to\infty}\int_{S(r_i)} \langle \Psi-\eta, \mathcal{D}_T\eta\rangle + \int_{S(r_i)} \langle \mathcal{D}_T\eta, \Psi-\eta\rangle = 0.
$$

Finally, by the argument in [1, p. 691–692], we can prove that

$$\lim_{i\to\infty}\int_{S(r_i)}\langle\mathcal{D}\eta,\eta\rangle = C(n)m_E$$

for some positive constant $C(n)$ depending only on n. Combining (3.7)-(3.10), we conclude that $m_E \geq 0$.

Suppose the mass of some end E_1 is zero. Suppose N has at least two ends E_1, E_2. Then we may choose η such that η is almost parallel but nonzero in E_1 and η is zero on E_2 and other ends. By the above arguments, we conclude that there is a spinor Ψ which is asymptotically close to η near infinity. Moreover, Ψ is parallel, namely Ψ is smooth and parallel in the usual sense in the interior and exterior of Ω. By Lemma 3.3, Ψ is continuous. This is impossible. Therefore N has only one end. By choosing enough linearly independent η and by constructing continuous parallel spinors from η, we can conclude as in [1] that the curvature of N is zero both inside and outside Ω. This completes the proof of the theorem. q.e.d.

4. Compact manifolds with boundary and with nonnegative scalar curvature

In this section, we will use the results in Sections 2 and 3 to study the boundary behaviors of a compact Riemannian manifold (Ω^n, g) of dimension n with smooth boundary $\partial\Omega$ and with nonnegative scalar curvature. First we need the following lemma.

Lemma 4.1. *Let M be a smooth differentiable manifold and Ω be a domain in M with smooth boundary. Suppose g is a Riemannian metric on M satisfying the following:*

(a) *$g_{M\setminus\overline{\Omega}}$ and g_Ω are smooth up to the boundary $\partial\Omega$ and g is Lipschitz near any point on $\partial\Omega$.*

(b) *The sectional curvature of $M\setminus\overline{\Omega}$ and Ω are zero near $\partial\Omega$.*

(c) *$\partial\Omega$ has the same second fundamental form with respect to $g_{M\setminus\overline{\Omega}}$ and g_Ω (with respect to the same normal direction).*

Then g is C^2 in a neighborhood of $\partial\Omega$.

Proof. Let $p \in \partial\Omega$ and let ρ be the signed distance function from $\partial\Omega$. Near p, the metric g can be expressed in the form:

$$g = d\rho^2 + \sum_{i,j=1}^{n-1} g_{ij} d\theta_i d\theta_j,$$

where $\sum_{i,j=1}^{n-1} g_{ij}(\rho,\theta) d\theta_i d\theta_j$ is the induced metric on the level sets of ρ and $(\theta_1, \ldots, \theta_{n-1}, \rho)$ are the local coordinates. It is sufficient to prove that the components g_{ij} are in C^2. Note that partial derivatives of g_{ij} of all order with respect to θ exist.

Let $\sum_{ij}^{n-1} h_{ij} d\theta_i d\theta_j$ be the second fundamental form on the level surface ρ=constant with respect to the unit normal $\partial/\partial\rho$. Then for $\rho \neq 0$, we have that

$$\begin{aligned} h_{ij} &= -\left\langle \nabla_{\frac{\partial}{\partial\theta_i}} \frac{\partial}{\partial\theta_j}, \frac{\partial}{\partial\rho} \right\rangle \\ &= -\Gamma_{ij}^n \\ &= \frac{1}{2}\frac{\partial g_{ij}}{\partial\rho} \end{aligned}$$

where Γ_{ab}^c are the Christoffel symbols and ρ is considered to be the n-th coordinate. Hence

$$\frac{\partial g_{ij}}{\partial\rho} = 2h_{ij}. \tag{4.1}$$

By the assumption that h_{ij} agrees on $\partial\Omega$, we see that $\partial g_{ij}/\partial\rho$ is continuous up to $\partial\Omega$. Hence g_{ij} is C^1 near p.

By (4.1), for $\rho \neq 0$

$$\begin{aligned} \frac{\partial^2 g_{ik}}{\partial\rho\partial\theta_j} &= \frac{\partial^2 g_{ik}}{\partial\theta_j\partial\rho} \\ &= 2\frac{\partial h_{ik}}{\partial\theta_j}. \end{aligned}$$

Since h_{ij} agrees on $\partial\Omega$ and g is smooth up to the boundary when restricted on Ω or on $M \setminus \overline{\Omega}$, we conclude that $\partial^2 g_{ik}/\partial\rho\partial\theta_j$ is continuous near p.

Next, we want to show $\frac{\partial^2 g_{ik}}{\partial\rho^2}$ is also continuous. For $\rho \neq 0$, using the fact that the sectional curvature is zero near p, we have

$$\begin{aligned}\frac{\partial h_{ij}}{\partial\rho} &= -\frac{\partial}{\partial\rho}\left\langle \nabla_{\frac{\partial}{\partial\theta_i}}\frac{\partial}{\partial\theta_j},\frac{\partial}{\partial\rho}\right\rangle \\ &= -\left\langle \nabla_{\frac{\partial}{\partial\rho}}\nabla_{\frac{\partial}{\partial\theta_i}}\frac{\partial}{\partial\theta_j},\frac{\partial}{\partial\rho}\right\rangle \\ &= -\left\langle \nabla_{\frac{\partial}{\partial\theta_i}}\nabla_{\frac{\partial}{\partial\rho}}\frac{\partial}{\partial\theta_j},\frac{\partial}{\partial\rho}\right\rangle \\ &= -\frac{\partial}{\partial\theta_i}\left\langle \nabla_{\frac{\partial}{\partial\rho}}\frac{\partial}{\partial\theta_j},\frac{\partial}{\partial\rho}\right\rangle + \left\langle \nabla_{\frac{\partial}{\partial\rho}}\frac{\partial}{\partial\theta_j},\nabla_{\frac{\partial}{\partial\theta_i}}\frac{\partial}{\partial\rho}\right\rangle \\ &= \left\langle \nabla_{\frac{\partial}{\partial\theta_j}}\frac{\partial}{\partial\rho},\nabla_{\frac{\partial}{\partial\theta_i}}\frac{\partial}{\partial\rho}\right\rangle \\ &= g^{ks}g^{ls}h_{ik}h_{is}\end{aligned}$$

where (g^{ij}) is the inverse of (g_{ij}). Combining this with (4.1), we see that $\frac{\partial^2 g_{ij}}{\partial\rho^2}$ is continuous. Hence g is in C^2. q.e.d.

Remark. We can prove g is actually C^∞ by the same argument.

Let (Ω^n, g) be a Riemannian manifold of dimension n with compact closure and with smooth boundary. Let us first consider the case that $n > 3$. We assume the following:

(i) $\partial\Omega$ has finitely many components Σ_i, $1 \le i \le k$.

(ii) The mean curvature H of Σ_i with respect to the outward normal is positive.

(iii) There is an isometric embedding $\iota_i : \Sigma_i \to \mathbb{R}^n$ such that Σ_i is a strictly convex closed hypersurface in $\mathbb{R}^n$. Here we identify (Σ_i, g) with its image with the metric induced by the Euclidean metric in $\mathbb{R}^n$.

(iv) Ω is spin.

We may extend Ω across $\partial\Omega$ to a smooth manifold $\widetilde{\Omega}$ which contains $\overline{\Omega}$. By the embedding of Σ_i, we can define a diffeomorphism from a neighborhood of Σ_i in $\widetilde{\Omega}$ to a neighborhood of Σ_i in $\mathbb{R}^n$ by mapping the set with distance r from Σ_i in $\widetilde{\Omega}$ to the set with distance r from Σ_i in $\mathbb{R}^n$, so that the part near Σ_i which is outside of Ω in $\widetilde{\Omega}$ will be mapped into an open set which is outside of Σ_i in $\mathbb{R}^n$.

Theorem 4.1. *Let* (Ω^n, g) *be a compact manifold with smooth boundary and with nonnegative scalar curvature and* $n > 3$. *Suppose* Ω *satisfies conditions* (i)–(iv). *Then for each boundary component* Σ_i,

$$\int_{\Sigma_i} H d\sigma \leq \int_{\Sigma_i} H_0^{(i)} d\sigma \tag{4.2}$$

where $H_0^{(i)}$ *is the mean curvature of* Σ_i *in* $\mathbb{R}^n$ *with respect to the outward normal. Moreover, if equality holds in* (4.2) *for some* i, *then* $\partial\Omega$ *has only one boundary component and* Ω *is a domain in* $\mathbb{R}^n$.

To prove the theorem, let us fix some notations. For each i, we may suppose Σ_i is a strictly convex hypersurface in $\mathbb{R}^n$. For simplicity, let us denote Σ_i by Σ_0 and $H_0^{(i)}$ by H_0. In the setting as in Section 2, let u be the solution of (2.1) with initial data $u(x,0) = H_0(x)/H(x)$ which is positive by (ii) and (iii).

Lemma 4.2. *The function*

$$m(r) = \int_{\Sigma_r} H_0(1 - u^{-1}) d\sigma_r$$

is nonincreasing in r, *where* H_0 *is the mean curvature of* Σ_r *in* $\mathbb{R}^n$.

Proof. Let h_{ij}^0 be the second fundamental form of Σ_r with respect to the Euclidean metric. Then by the Gauss equations, it is easy to see that

$$\frac{\partial H_0}{\partial r} = -\sum_{i,j=1}^{n-1} \left(h_{ij}^0\right)^2.$$

Since u satisfies (2.1), we get:

$$\begin{aligned}
&\frac{d}{dr}\int_{\Sigma_r} H_0(1-u^{-1})d\sigma_r \\
&= -\int_{\Sigma_r} \sum_{i,j=1}^{n-1} \left(h_{ij}^0\right)^2 (1-u^{-1})d\sigma_r + \int_{\Sigma_r} u^{-2} H_0 \frac{\partial u}{\partial r} \\
&\quad + \int_{\Sigma_r} H_0^2 (1-u^{-1}) d\sigma_r \\
&= \int_{\Sigma_r} \left[\left(H_0^2 - \sum_{i,j} \left(h_{ij}^0\right)^2 \right) (1-u^{-1}) + \Delta_r u + \frac{1}{2}(u^{-1}-u)\mathcal{R}^r \right] d\sigma_r \\
&= -\frac{1}{2}\int_{\Sigma_r} \mathcal{R}^r u^{-1}(1-u)^2 \\
&\leq 0.
\end{aligned}$$

where Δ_r is the Laplacian on Σ_r and $\mathcal{R}^r$ is the scalar curvature of Σ_r with respect to the induced metric in $\mathbb{R}^n$ (which is the same as the metric induced by g). Here we have used the fact

$$H_0^2 - \sum_{i,j}(h_{ij}^0)^2 = \mathcal{R}^r.$$

Thus, we see that $m(r)$ is nonincreasing. q.e.d.

We are ready to prove the theorem.

Proof of Theorem 4.1. Using the above method, we can attach each boundary component Σ_i to the exterior of a convex hypersurface in $\mathbb{R}^n$, which is denoted by E_i. On each E_i, we construct the metric given by $g_i = u_i^2 dr^2 + g_r$ with initial data $H_0^{(i)}/H$ as in Theorem 2.1. Denote the resulting manifold by N. Let g_N be the metric on N defined by $g_N = g$ in Ω and $g_N = g_i$ on each E_i. Since Ω is spin, N is spin. By Theorem 2.1 and (1.6), N satisfy the assumptions in Theorem 3.1. Hence the mass m_{E_i} is nonnegative for each i. By Theorem 2.1(c) and Lemma 4.1, we conclude that (4.2) is true for all i.

Suppose equality holds in (4.2) for some i, then the mass of E_i must be zero. Hence N has only one end and N is flat by Theorem 3.1. Therefore $\partial\Omega$ has only one component and Ω is flat. By (1.7), we have $u \equiv 1$. On the other hand, we note that $n > 3$ and the boundary is strictly convex in $\mathbb{R}^n$. By the proof in [6, §60], we know that the second fundamental forms of the boundary of Ω with respect to g and the Euclidean metric (in the same normal direction) are equal. By Lemma 4.1, we see the metric on N is actually C^2. Since $u \equiv 1$, N is the Euclidean space outside a compact set. By volume comparison theorem, N is isometrically to $\mathbb{R}^n$ which implies that Ω is a domain in $\mathbb{R}^n$. This completes the proof of the theorem. q.e.d.

In case $n = 3$ then condition (iv) mentioned above is automatically satisfied. Also, by a well-known result, see [19] for example, condition (iii) is equivalent to the condition that Σ_i has positive Gaussian curvature. It is also well-known that the embedding is unique up to an isometry in $\mathbb{R}^3$. Hence in this case we have:

Theorem 4.2. (Ω^3, g) *be a Riemannian manifold of dimension* 3 *with compact closure with smooth boundary and with nonnegative scalar curvature. Suppose* Ω *satisfies conditions* (i)–(ii). *Moreover, suppose each boundary component of* Σ_i *has positive Gaussian curvature. For* *each boundary component Σ_i, we have*

$$\int_{\Sigma_i} H d\sigma \leq \int_{\Sigma_i} H_0^{(i)} d\sigma \tag{4.3}$$

where $H_0^{(i)}$ is the mean curvature of Σ_i with respect to the outward normal when it is isometrically embedded in $\mathbb{R}^3$. Moreover, if equality holds in (4.3) *for some i, then $\partial\Omega$ has only one component and Ω is a domain in $\mathbb{R}^3$.*

Proof. The proof of (4.3) is the same as before. Using the same notations as in the proof of Theorem 4.1, by the same argument as in the proof of Theorem 4.1, if equality holds in (4.3) for some end, then $\partial\Omega$ has only one component and N is flat with $u \equiv 1$. It remains to prove that Ω is actually a domain in $\mathbb{R}^3$.

Since $u \equiv 1$, the mean curvatures of $\partial\Omega$ with respect to g and the Euclidean metric are equal. Since $\partial\Omega$ is now a strictly convex surface in $\mathbb{R}^3$, it is easy to see that the proof in [14, Theorem 6.2.8] can be carried over and we can conclude that the second fundamental forms of $\partial\Omega$ with respect to g and the Euclidean metric are equal. Hence we can conclude from Lemma 4.1 as before that Ω is a domain in $\mathbb{R}^3$. This finishes the proof of the theorem. q.e.d.

By the result of Weyl [29], the boundary component Σ_i of Ω in Theorem 4.2 satisfies

$$4K \leq \left(H_0^{(i)}\right)^2 \leq \sup_{\Sigma_i} \left(4K - K^{-1}\Delta K\right)$$

where Δ is the Laplacian of Σ_i with metric induced by g, K is the Gaussian curvature of Σ_i and $H_0^{(i)}$ is the mean curvature of Σ_i when it is embedded in $\mathbb{R}^3$. Hence we have the following corollary.

Corollary 4.1. *Let (Ω^3, g) be a compact manifold of dimension* 3 *with boundary and with nonnegative scalar curvature. Suppose Ω satisfies conditions* (i)–(ii). *Moreover, suppose each boundary component of Σ_i has positive Gaussian curvature K. Then*

$$\frac{1}{\text{Area}\,(\Sigma_i)} \int_{\Sigma_i} H d\sigma \leq \left[\sup_{\Sigma_i} \left(4K - K^{-1}\Delta K\right)\right]^{\frac{1}{2}}.$$

Moreover, if equality holds for some Σ_i, then $\partial\Omega$ has only one component and Ω is a domain in $\mathbb{R}^3$.

5. An equivalent statement of the positive mass theorem

In the previous section, we obtain Theorem 4.2 from the positive mass theorem: Theorem 3.1. In this section, we want to show that one can obtain the first part of the positive mass theorem by assuming that Theorem 4.2 is true. More precisely, let (N, g) be a complete noncompact manifold with finitely many ends with the following properties:

(i) N has nonnegative scalar curvature $\mathcal{R}$ which is in $L^1(N)$.

(ii) Each end E is diffeomorphic to the exterior of some compact set in $\mathbb{R}^3$, so that the metric $g = \sum_{i,j=1}^3 g_{ij} dx_i dx_j$ is asymptotically flat in the sense that

$$g_{ij} = \delta_{ij} + \widetilde{b}_{ij}$$

such that

$$|\widetilde{b}_{ij}| + r|\nabla_0 \widetilde{b}_{ij}| + r^2|\nabla_0\nabla_0\widetilde{b}_{ij}| = O(r^{-1}).$$

where r is the Euclidean distance from the origin and ∇_0 is the derivatives with respect to the Euclidean metric.

Then we have:

Theorem 5.1. *Suppose* (4.2) *is true for any compact Riemannian three manifold* Ω *with boundary satisfying the assumptions in Theorem* 4.2. *Let* (N, g) *be as above, then the* ADM *mass of each end of* N *is nonnegative.*

Proof. By the result of [25], it is sufficient to prove the theorem under the stronger assumption that at each end E,

$$g_{ij} = \left(1 + \frac{2m_E}{r}\right)\delta_{ij} + b_{ij}, \tag{5.1}$$

$$|b_{ij}| + r|\nabla_0 b_{ij}| + r^2|\nabla_0\nabla_0 b_{ij}| + r^3|\nabla_0\nabla_0\nabla_0 b_{ij}| + r^4|\nabla_0\nabla_0\nabla_0\nabla_0 b_{ij}| = O(r^{-2})$$

where m_E is a constant, and r is the Euclidean distance from the origin. Namely, it is sufficient to prove that $m_E \geq 0$ under the additional condition (5.1).

In fact, by [25], given any $\epsilon > 0$, one can construct a new metric $\widetilde{g}$ on N with zero scalar curvature such that near infinity at each end E, the metric $\widetilde{g}$ is of the form:

$$\widetilde{g}_{ij} = \varphi^4 \delta_{ij},$$

where φ satisfies

$$\varphi = 1 + \frac{\widetilde{m}_E}{r} + h.$$

Here $\widetilde{m}_E$ is a constant and $|h| = O(r^{-2})$. Moreover, $\widetilde{m}_E \leq m_E + \epsilon$. Since the scalar curvature of $\widetilde{g}$ is zero, φ and hence h is harmonic outside a compact set of the Euclidean space. By the gradient estimates of harmonic functions on Euclidean space, we conclude that $\widetilde{g}$ satisfies (5.1).

We assume that N has only one end, and denote m_E by m. The general case can be proved similarly. The proof of Theorem 5.1 are divided into several steps.

Step 1. We want to compute the Gaussian curvature K of $\mathbb{S}(r)$ with respect to the metric g. Here $\mathbb{S}(r)$ is the Euclidean sphere of radius r.

Let $\mathbf{Y} = \mathbf{Y}(\zeta_1, \zeta_2)$ be local parametrization of the standard unit sphere $\mathbb{S} = \mathbb{S}(1)$, here $\mathbf{Y} = (y_1, y_2, y_3)$. Then local parametrization for the standard $\mathbb{S}(r)$ is given by

$$\mathbf{X} = r\mathbf{Y}.$$

Here and below, $i, j \ldots$ are from 1 to 3, and $\alpha, \beta \ldots$ are from 1 to 2. Now

$$\frac{\partial x_i}{\partial r} = y_i \tag{5.2}$$

and

$$\frac{\partial x_i}{\partial \zeta_\alpha} = r\frac{\partial y_i}{\partial \zeta_\beta}. \tag{5.3}$$

Let us first compute the metric on $\mathbb{S}(r)$. Since

$$\frac{\partial}{\partial \zeta_\alpha} = \frac{\partial x_i}{\partial \zeta_a}\frac{\partial}{\partial x_i},$$

we have

$$\begin{aligned} \tau_{\alpha\beta} &= g\left(\frac{\partial}{\partial \zeta_\alpha}, \frac{\partial}{\partial \zeta_\beta}\right) \\ &= g_{ij}\frac{\partial x_i}{\partial \zeta_\alpha}\frac{\partial x_j}{\partial \zeta_\beta} \\ &= r^2\left[\left(1 + \frac{2m}{r}\right)\delta_{ij} + b_{ij}\right]\frac{\partial y_i}{\partial \zeta_\alpha}\frac{\partial y_j}{\partial \zeta_\beta} \\ &= r^2\left[\left(1 + \frac{2m}{r}\right)a_{\alpha\beta} + b_{ij}\frac{\partial y_i}{\partial \zeta_\alpha}\frac{\partial y_j}{\partial \zeta_\beta}\right] \end{aligned} \tag{5.4}$$

where $a_{\alpha\beta}$ is the standard metric on $\mathbb{S}(1)$ in the coordinates (ζ_1, ζ_2). Hence

$$\tau = \tau_{11}\tau_{22} - \tau_{12}^2 = ar^4\left(1+\frac{2m}{r}\right)^2(1+f) \tag{5.5}$$

where $a = \det(a_{\alpha\beta})$ and f is a smooth function which satisfies

$$|f| + |\partial f| + |\partial^2 f| + |\partial^3 f| + |\partial^4 f| = O\left(r^{-2}\right). \tag{5.6}$$

∂ denotes the partial derivatives with respect to ζ_α. Moreover, $|\frac{\partial f}{\partial r}| = O(r^{-3})$. Here and below f always denotes a smooth function, but the meaning of f may vary from line to line. The function f in (5.5) satisfies (5.6) because of the assumptions on b_{ij} and (5.3). The inverse of $(\tau_{\alpha\beta})$ is given by

$$\tau^{\alpha\beta} = r^{-2}\left(1+\frac{2m}{r}\right)^{-1}\left(a^{\alpha\beta}+f\right) \tag{5.7}$$

where $(a^{\alpha\beta}) = (a_{\alpha\beta})^{-1}$ and f also satisfies (5.6). Let $\Gamma^\gamma_{\alpha\beta}$ and $\widetilde{\Gamma}^\gamma_{\alpha\beta}$ be the Christoffel symbols of $\mathbb{S}(r)$ with induced metric $\tau_{\alpha\beta}$ and that of the standard unit sphere $\mathbb{S}$ in the coordinates ζ. Then

$$\Gamma^\gamma_{\alpha\beta} = \widetilde{\Gamma}^\gamma_{\alpha\beta} + f \tag{5.8}$$

with

$$|f| + |\partial f| + |\partial^2 f| + |\partial^3 f| = O\left(r^{-2}\right),$$

and

$$\frac{\partial \Gamma^\gamma_{\alpha\beta}}{\partial \zeta_\delta} = \frac{\partial \widetilde{\Gamma}^\gamma_{\alpha\beta}}{\partial \zeta_\delta} + f \tag{5.9}$$

with

$$|f| + |\partial f| + |\partial^2 f| = O\left(r^{-2}\right).$$

Hence the Gaussian curvature K of $\mathbb{S}(r)$ with metric induced by g is

(5.10)

$$\begin{aligned} K &= -\tau_{11}^{-1}\left[\left(\Gamma^2_{12}\right)_1 - \left(\Gamma^2_{11}\right)_2 + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{11}\Gamma^2_{12}\right] \\ &= -\tau_{11}^{-1}\left(-a_{11}+f\right) \\ &= r^{-2}\left(1+\frac{2m}{r}\right)^{-1}(1+f) \end{aligned}$$

with

$$|f| + |\partial f| + |\partial^2 f| = O\left(r^{-2}\right).$$

Step 2. We want to show that the mean curvature H on $\mathbb{S}(r)$ is positive for r large enough. Moreover, we want to compute the integral of H over $\mathbb{S}(r)$.

Let H be the mean curvature of $\mathbb{S}(r)$ with respect to the outward normal $\mathbf{N}$ in the metric g. Let $A(r)$ be the area of $\mathbb{S}(r)$. Then by the first variational formula and (5.5)

$$\begin{aligned}\int_{\mathbb{S}(r)} \left\langle \frac{\partial}{\partial r}, \mathbf{N} \right\rangle H d\sigma_r &= A'(r) \\ &= (2r + 2m) \int_{\mathbb{S}(1)} (1 + f)^{\frac{1}{2}} d\sigma_0 + O(r^{-1}) \\ &= 8\pi(r + m) + O(r^{-1})\end{aligned}$$

where $d\sigma_r$ is the volume form of $\mathbb{S}(r)$ with metric induced by g and $d\sigma_0$ is the volume form of the standard unit sphere.

Note that

$$\frac{\partial}{\partial r} = \frac{\partial x_i}{\partial r} \frac{\partial}{\partial x_i} = \frac{x_i}{r} \frac{\partial}{\partial x_i}$$

and the gradient of r with respect to g is

$$\nabla r = g^{ij} \frac{\partial r}{\partial x_i} \frac{\partial}{\partial x_j} = g^{ij} \frac{x_i}{r} \frac{\partial}{\partial x_j}$$

one obtains

$$\left\langle \frac{\partial}{\partial r}, \nabla r \right\rangle = 1 \tag{5.11}$$

and

$$|\nabla r|^2 = g^{ij} \frac{\partial r}{\partial x_i} \frac{\partial r}{\partial x_j} = 1 - \frac{2m}{r} + h \tag{5.12}$$

where $h = O(r^{-2})$. Since $\mathbb{S}(r)$ is the level surface of the function r, $\mathbf{N} = |\nabla r|^{-1} \nabla r$. We have

(5.13)

$$\int_{\mathbb{S}(r)} |\nabla r|^{-1} H d\sigma_r = \int_{\mathbb{S}(r)} \left\langle \frac{\partial}{\partial r}, \mathbf{N} \right\rangle H d\sigma_r = 8\pi(r + m) + O(r^{-1}).$$

For any point x on $\mathbb{S}(r)$, choose an orthonormal frame e_i with respect to g such that e_1, e_2 are tangential and e_3 is the unit outward normal. Moreover, assume the second fundamental form $h_{\alpha\beta}$ is diagonalized at x. By the Gauss equations,

$$h_{11}h_{22} = K - R_{1212}$$

where R_{ijij} is the curvature tensor of N. By the asymptotic behavior of g, we have $|R_{ijij}| = O(r^{-3})$. By (5.9), we conclude that

$$h_{11}h_{22} > 0,$$

if r is large enough and so h_{11} and h_{22} are of the same sign. Hence H is either everywhere positive or everywhere negative when r is large. By (5.12), we must have $H > 0$.

Since $H > 0$ for r large enough, we may apply mean value theorem and use (5.11), we have

$$\int_{\mathbb{S}(r)} H d\sigma_r = 8\pi r + O(r^{-1}). \tag{5.14}$$

Since the Gaussian curvature of $\mathbb{S}(r)$ is positive for r large enough, $(\mathbb{S}(r), g)$ can be embedded isometrically in $\mathbb{R}^3$. Let H_0 be its mean curvature in $\mathbb{R}^3$.

Step 3. We want to estimate H_0. Note that $H_0^2 \geq 4K$, and by [29], we have

$$4K \leq H_0^2 \leq \sup_{\mathbb{S}(r)} \left(4K - K^{-1}\Delta K\right), \tag{5.15}$$

where Δ is the Laplacian of $\mathbb{S}(r)$ with metric induced by g.

$$\begin{aligned} K^{-1}\Delta K &= K^{-1}\left[\tau^{\alpha\beta}\frac{\partial^2 K}{\partial\zeta_\alpha\partial\zeta_\beta} + \frac{1}{\sqrt{\tau}}\frac{\partial}{\partial\zeta_\alpha}\left(\sqrt{\tau}\tau^{\alpha\beta}\right)\frac{\partial K}{\partial\zeta_\beta}\right] \\ &= O(r^{-4}) \end{aligned} \tag{5.16}$$

because by (5.9)

$$\frac{\partial^2 K}{\partial\zeta_\alpha\partial\zeta_\beta} = r^{-2}\left(1 + \frac{2m}{r}\right)^{-1}\frac{\partial^2 f}{\partial\zeta_\alpha\partial\zeta_\beta} = O(r^{-4}).$$

Combining (5.9), (5.14) and (5.15) we have

$$H_0 = \frac{2}{r} - \frac{2m}{r^2} + O(r^{-3}).$$

Hence by (5.5)

(5.17)

$$\begin{aligned}\int_{S(r)} H_0 d\sigma_r &= \left(\frac{2}{r} - \frac{2m}{r^2} + O(r^{-3})\right) \cdot r^2 \left(1 + \frac{2m}{r}\right) \left(4\pi + O\left(r^{-2}\right)\right) \\ &= 8\pi(r+m) + O(r^{-1}).\end{aligned}$$

Step 4. We can now conclude the proof of the theorem. Let Ω_r be the domain in N so that $\partial\Omega_r$ is $\mathbb{S}(r)$. Then Ω_r has nonnegative scalar curvature, $\partial\Omega_r$ has positive mean curvature and positive Gaussian curvature. By assumptions, (4.2) is true for $\partial\Omega_r$. Combining with (5.13) and (5.16) we have

$$\begin{aligned}0 &\le \int_{\mathbb{S}(r)} (H_0 - H)\, d\sigma_r \\ &= 8\pi m + O(r^{-1}).\end{aligned}$$

Let $r \to \infty$, we have $m \ge 0$. q.e.d.

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