REVISTA MATEMATICA de la Universidad Complutense de Madrid Volumen 6, número 2; 1993 http://dx.doi.org/10.5209/rev\_REMA.1993.v6.n2.17825

# Positive Radial Solutions for Semilinear Biharmonic Equations in Annular Domains

### ROBERT DALMASSO

**ABSTRACT.** We study the existence of positive radial solutions of  $\Delta^2 u = g(|x|)f(u)$  in an annulus with Dirichlet boundary conditions. We establish that the equation has at least one positive radially symmetric solution on any annulus if f and g are nonnegative,  $g \neq 0$  and f is superlinear at zero and  $+\infty$ . We also give a property of positive radial solutions.

# 1. INTRODUCTION

In this paper we consider the existence of positive radial solutions of the semilinear biharmonic equation

(1.1) 
$$\Delta^2 u = g(|x|)f(u) \qquad \text{in } \Omega(a, b)$$

(1.2) 
$$u = \frac{\partial u}{\partial v} = 0 \qquad \text{on } \partial\Omega(a, b)$$

where  $0 < a < b < +\infty$ ,  $\Omega(a, b)$  denotes the annulus  $\{x \in \mathbb{R}^n; a < |x| < b\}$   $(n \ge 2)$ ,  $\frac{\partial}{\partial v}$  is the outward normal derivative and f, g satisfy the following hypotheses

$$(H_1)$$
  $f \in C([0, +\infty))$  and  $f(u) \ge 0$  for  $u > 0$ .

$$(H_2)$$
  $g \in C([a, b]), g(r) \ge 0$  for  $r \in [a, b]$  and  $g \ne 0$  in  $[a, b]$ .

<sup>1991</sup> Mathematics Subject Classification: 35B45, 35J40, 35J65. Editorial Complutense. Madrid, 1993.

$$(H_3)$$
  $\lim_{u\to+\infty} f(u)/u = +\infty$ .

$$(H_4) \quad \lim_{u\to 0} f(u)/u = 0.$$

The analogous problem for the Laplace equation has been intensively studied in recent years (see e.g. [1]-[4], [8], [11], [12], [15]) and nearly optimal results have been obtained. In most papers, the shooting method was used to establish the existence of positive radial solutions. In contrast the result of [1] was obtained by a variational method and the use of a priori estimates, while in [15] an expansion fixed point theorem was applied.

Our main result is the following theorem.

**Theorem 1.1.** Assume  $(H_1)-(H_4)$ . Then problem (1.1), (1.2) possesses at least one positive radial solution  $u \in C^4(\overline{\Omega(a,b)})$ .

In this paper our method of proof makes use of a priori estimates and well-known properties of compact mappings taking a cone in a Banach space into itself (see [7]).

Since we are interested in positive radial solutions, the problem under consideration reduces to the one-dimensional boundary value problem

$$(1.3) \Delta^2 u(t) = g(t)f(u(t)), t \in (a, b)$$

(1.4) 
$$u^{(j)}(a) = u^{(j)}(b) = 0, j = 0, 1$$

where  $\Delta$  denotes the polar form of the Laplacian, i.e.  $\Delta = t^{1-n} \frac{d}{dt} \left( t^{n-1} \frac{d}{dt} \right)$ .

Our next result gives a property of nonnegative nontrivial solutions of (1.3), (1.4) when f and g satisfy some monotonicity conditions.

**Theorem 1.2.** Suppose that f and g in equation (1.3) satisfy the following assumptions:

$$(H_5)$$
  $f:[0,+\infty) \rightarrow [0,+\infty)$  is nondecreasing.

$$(H_6)$$
  $g:[a,b] \rightarrow [0,+\infty)$  is nonincreasing.

Let  $u \in C^4([a, b])$  be a nonnegative nontrivial solution of problem (1.3), (1.4). Then  $\Delta u(a) > \Delta u(b)$ .

**Remark 1.1.** Note that  $\Delta u(a) = u''(a)$  (resp.  $\Delta u(b) = u''(b)$ ) since u'(a) = 0 (resp. u'(b) = 0).

**Remark 1.2.** Theorems 1.1 and 1.2 can be easily extended to handle more general nonlinearities of the type f(|x|, u).

Our paper is organized as follows. In Section 2 we give a maximum principle for fourth order equations and we describe the special shape of nontrivial solutions of (1.3), (1.4) when  $f \ge 0$  and  $g \ge 0$ . In Section 3 we prove our a priori bounds for positive solutions of (1.3), (1.4). Theorem 1.1 is proved in Section 4. Finally, Section 5 contains the proof of theorem 1.2.

## 2. PRELIMINARIES

We have the following theorem.

**Theorem 2.1.** Let  $u \in C^4([a, b])$  be such that

$$\begin{cases} \Delta^2 u \ge 0 \text{ in } (a, b) \\ u^{(j)}(a) = u^{(j)}(b) = 0, j = 0, 1. \end{cases}$$

Assume that  $u \not\equiv 0$ . Then:

- (i) There exist  $r, s \in (a, b)$  such that  $r < s, \Delta u > 0$  on  $[a, r) \cup (s, b]$  and  $\Delta u < 0$  on (r, s).
- (ii) There exist  $d_1$ ,  $d_2 \in (r, s)$  such that  $d_1 \leq d_2$ ,  $(\Delta u)' < 0$  on  $[a, d_1)$ ,  $(\Delta u)' > 0$  on  $(d_2, b]$  and  $(\Delta u)' \equiv 0$  on  $[d_1, d_2]$ .
- (iii) u>0 on (a, b). Moreover there exists  $c \in (r, s)$  such that u'>0 on (a, c) and u'<0 on (c, b).

**Proof.** We first prove (i). Suppose that  $\Delta u(a) \le 0$  and  $\Delta u(b) \le 0$ . Then the one-dimensional maximum principle ([14] p. 2) implies that  $\Delta u \le 0$  on [a, b]. Since u(a) = u'(a) = u(b) = u'(b) = 0, the maximum prin-

ciple and the Hopf boundary lemma ([14] p. 4) imply that  $u \equiv 0$  on [a, b] and we reach a contradiction. Thus  $\Delta u(a) > 0$  or  $\Delta u(b) > 0$ . Suppose for instance that  $\Delta u(a) > 0$ . If  $\Delta u \ge 0$  on [a, b], we get a contradiction as before. Thus there exists  $x \in (a, b)$  such that  $\Delta u(x) < 0$  and we can define  $r \in (a, b)$  to be the first zero of  $\Delta u$  in (a, b). Since  $u''(a) = \Delta u(a) > 0$  we have u > 0 in  $(a, a + \eta]$  for some  $\eta > 0$ . Using the maximum principle and the Hopf lemma we get u' > 0 on (a, r]. Now, if  $\Delta u(b) \le 0$ , the maximum principle implies  $\Delta u < 0$  on (r, b). Since u(r) > 0 and u(b) = u'(b) = 0, we again reach a contradiction. Thus we have proved that  $\Delta u(b) > 0$ . Now we can define  $s \in (r, b)$  to be the last zero of  $\Delta u$  in (a, b). Since  $\Delta u(x) < 0$ , the maximum principle implies that  $\Delta u < 0$  on (r, s).

We now prove (ii). Denoting by m<0 the minimum value of  $\Delta u$  in [a, b], we define  $E = \{t \in (a, b)/\Delta u(t) = m\}$ . Suppose first that E contains only one point. Then with the aid of the Hopf lemma we obtain (ii). Now, if E contains at least two points, the maximum principle and the continuity of  $\Delta u$  imply that  $E = \{d_1, d_2\}$  where  $r < d_1 < d_2 < s$ . Then, using the Hopf lemma we obtain (ii).

Finally we prove (iii). We have already seen that u' > 0 on (a, r]. In the same way we show that u' < 0 on [s, b). Now let  $t_0$  (resp.  $t_1$ ) be the first (resp. the last) zero of u' in (a, b). Clearly  $r < t_0 \le t_1 < s$ . Suppose that  $t_0 < t_1$ . Then the Hopf lemma implies that either  $u'(t_0) > 0$  or  $u'(t_1) < 0$ , a contradiction. Thus  $t_0 = t_1$  and (iii) is proved.

#### 3. A PRIORI BOUNDS

**Theorem 3.1.** Assume  $(H_1)-(H_3)$ . Then there exists M>0 such that

$$||u||_{\infty} \leq M$$

for all positive solutions  $u \in C^4([a, b])$  of (1.3), (1.4).

**Proof.** We denote by S the set of all positive solutions of (1.3), (1.4) in  $C^4([a, b])$ . Let  $u \in S$ . By theorem 2.1 there exist c(u),  $d_1(u)$ ,  $d_2(u)$ , r(u) and s(u) in (a, b) such that u' > 0 on (a, c(u)), u' < 0 on (c(u), b),  $(\Delta u)' < 0$  on  $[a, d_1(u))$ ,  $(\Delta u)' > 0$  on  $(d_2(u), b]$ ,  $(\Delta u)' \equiv 0$  on  $[d_1, d_2]$ ,  $\Delta u > 0$  on  $[a, r(u)) \cup (s(u), b]$  and  $\Delta u < 0$  on (r(u), s(u)). Moreover we have a < r(u) < c(u),  $d_1(u)$ ,  $d_2(u) < s(u) < b$  and  $d_1(u) \le d_2(u)$ .

We shall divide the proof into several steps. Subsequently C will denote various generic constants which may vary from line to line.

**Step 1.** We first prove that  $\{gf(u); u \in S\}$  and  $\{gu; u \in S\}$  are bounded in  $L^1_{loc}(a, b)$ . Define

$$\varrho(t) = (t-a)^2 (t-b)^2 \qquad \text{for } a \le t \le b.$$

Let  $\varphi \in C^4([a,b])$  be the solution of the boundary problem

$$\begin{cases} \Delta^2 \varphi = g\varrho & \text{in } (a, b) \\ \varphi^{(j)}(a) = \varphi^{(j)}(b) = 0, & j = 0, 1. \end{cases}$$

By theorem 2.1  $\varphi$ >0 in (a, b) and there exist  $c_1$ >0 and  $c_2$ >0 such that

$$(3.1) c_1 \varrho \leq \varphi \leq c_2 \varrho \text{on } [a, b].$$

By  $(H_3)$ , there exists  $\lambda > c_1^{-1}$  and  $u_0 \ge 0$  such that

$$(3.2) f(u) \ge \lambda u \text{for } u \ge u_0.$$

If we multiply equation (1.3) by  $t^{n-1}\varphi$  and integrate by parts four times we obtain

(3.3) 
$$\int_{a}^{b} t^{n-1} \varphi gf(u) dt = \int_{a}^{b} t^{n-1} \varrho gu dt.$$

From (3.2) and (3.3) we deduce

$$\int_{a}^{b} t^{n-1} \varrho g u dt \ge \lambda \int_{a}^{b} t^{n-1} \varphi g u dt - C \ge \lambda c_1 \int_{a}^{b} t^{n-1} \varrho g u dt - C$$

that is

(3.4) 
$$\int_{a}^{b} t^{n-1} \varrho \, gu dt \leq \frac{C}{\lambda c_1 - 1}$$

(3.5) 
$$\int_{a}^{b} t^{n-1} \varrho gf(u) dt \leq \frac{C}{c_1(\lambda c_1 - 1)}.$$

Thus  $\{gf(u); u \in S\}$  and  $\{gu; u \in S\}$  are bounded in  $L^1_{loc}(a, b)$ .

Step 2. Now we prove the following lemma.

# **Lemma 3.1.** Let A be a subset of S. Then:

- (i) If  $\{\Delta u; u \in A\}$  is bounded in  $L^1(a, b)$ , then there exists a constant M>0 such that  $\|u\|_{\infty} \leq M$  for all  $u \in A$ .
- (ii) If there exist  $\gamma > 0$ ,  $\eta > 0$  and C > 0 such that  $\gamma + \eta \le b a$  and  $u(t) \le C$  for  $t \in [a, a + \gamma] \cup [b \eta, b]$  and  $u \in A$ , then there exists a constant  $M \ge C$  such that  $||u||_{\infty} \le M$  for all  $u \in A$ .

**Proof.** (i) follows readily from the fact that u and u' vanish at least once in [a, b]. We now prove (ii). Setting  $m = \inf_{[a+\gamma, b-\eta]} \varrho(t)$  and using (3.5) we obtain

$$\int_{a}^{b} t^{n-1} \Delta^{2} u dt = \int_{a}^{b} t^{n-1} gf(u) dt = \int_{a}^{a+y} + \int_{a+y}^{b-\eta} + \int_{b-\eta}^{b} dt$$

$$\leq C + \int_{a+y}^{b-\eta} t^{n-1} \frac{1}{m} \varrho gf(u) dt$$

$$\leq C(1 + \int_{a+y}^{b} t^{n-1} \varrho gf(u) dt) \leq C.$$

Thus  $\{\Delta^2 u; u \in A\}$  is bounded in  $L^1(a, b)$ . Since  $u, u', \Delta u$  and  $(\Delta u)'$  vanish at least once in [a, b] the result follows.

**Step 3.** Finally we prove that S is bounded in  $L^{\infty}(a, b)$ . Let  $\gamma > 0$ ,  $\eta > 0$  and  $\delta > 0$  be such that  $\gamma + \eta + 2\delta < b - a$  and g > 0 on  $[a + \gamma, b - \eta]$ . Define  $a' = a + \gamma + \delta$  and  $b' = b - \eta - \delta$ . Then by (3.4) there exists K > 0 such that

$$(3.6) \qquad \int_{a'}^{b'} u dt \leq K.$$

Let  $n_0 > 0$  be such that  $1/n_0 < (b' - a')/4$ . By (3.6) we have

(3.7) 
$$meas \{t \in [a', b']; u(t) \ge n_0 K\} \le \frac{1}{n_0} < \frac{b' - a'}{4}.$$

Now define

$$S_{+} = \{u \in S; c(u) > b'\}$$

$$S_{-} = \{u \in S; c(u) < a'\}$$

and

$$S_0 = \{u \in S; a' \le c(u) \le b'\}.$$

Thus  $S = S_- \cup S_0 \cup S_+$ . Let  $\alpha = a' + (b' - a')/4$  and  $\beta = b' - (b' - a')/4$ . Clearly by the shape of u, (3.7) implies

(3.8) 
$$u(t) < n_0 K \text{ for } t \in [a, \beta] \text{ and } u \in S_+$$

and

(3.9) 
$$u(t) < n_0 K \text{ for } t \in [\alpha, b] \text{ and } u \in S_-.$$

**Lemma 3.2.**  $S_0$  is bounded in  $L^{\infty}(a, b)$ .

**Proof.** Let  $u \in S_0$ . From (3.4) we get

$$C \ge \int_{a+\gamma}^{b-\eta} u dt \ge \int_{a+\gamma}^{a'} u dt + \int_{b'}^{b-\eta} u dt$$

which implies

$$u(t) \leq C/\delta$$
 for  $t \in [a, a+\gamma] \cup [b-\eta, b]$ 

and we get the conclusion by using lemma 3.1 (ii).

Clearly Theorem 3.1 follows from Lemma 3.2 and the next lemma.

**Lemma 3.3.**  $\underline{S}_+$  and  $\underline{S}_-$  are bounded in  $L^{\infty}(a, b)$ .

**Proof.** We shall show that  $S_{-}$  is bounded in  $L^{\infty}(a, b)$ . We first prove that  $F = \{u \in S_{-}; s(u) > \beta\}$  is bounded in  $L^{\infty}(a, b)$ . We claim that for  $u \in F$ 

$$||u||_{\infty} \leq M = n_0 K + T(b-a)$$

where  $T = (2/(\beta - \alpha)) n_0 K(b/a)^{n-1}$ . Indeed, suppose this is not the case. Then let  $u \in F$  be such that u(c(u)) > M. We have  $c(u) < \alpha$ . Let  $t \in [\alpha, \beta]$  be arbitrary. By the mean value theorem there exists  $x \in (c(u), t)$  such that, by virtue of (3.9)

$$u'(x) = \frac{u(t) - u(c(u))}{t - c(u)} < -T.$$

Since  $r(u) < c(u) < \alpha < \beta < s(u)$ ,  $y^{n-1}u'(y)$  is nonincreasing on [x, t], thus we get

$$u'(t) \le \left(\frac{x}{t}\right)^{n-1} u'(x) < -\left(\frac{a}{b}\right)^{n-1} T$$

with  $t \in [\alpha, \beta]$ . This and (3.9) imply

$$u(t) < n_0 K - T \left(\frac{a}{b}\right)^{n-1} (t-\alpha)$$
 for  $t \in [\alpha, \beta]$ .

For  $t \in [(\alpha + \beta)/2, \beta]$ , we deduce

$$u(t) < n_0 K - T \left(\frac{a}{h}\right)^{n-1} \frac{\beta - \alpha}{2} = 0$$

and we reach a contradiction. Thus our assertion is proved

Now we prove that  $G = \{u \in S_-; s(u) \le \beta\}$  is bounded in  $L^{\infty}(a, b)$ . Let  $t \in [\beta, b)$ . We first show that  $H = \{\Delta u(t); u \in G\}$  is bounded. Suppose  $n \ge 3$ . Since  $\Delta u(s) \ge \Delta u(t) \ge 0$  for  $s \in [t, b]$ , we have

$$u(t) = \frac{1}{n-2} \int_{t}^{b} \left( \left( \frac{s}{t} \right)^{n-2} - 1 \right) s \Delta u(s) ds \ge C(t) \Delta u(t)$$

where C(t) > 0 does not depend on u. Thus, if H is not bounded we get a contradiction with (3.9). When n = 2 the argument is the same.

Now let  $\chi \in C^4([\beta, b])$  be the solution of the boundary problem

$$\begin{cases} \Delta^{2} \chi = 1 \text{ in } (\beta, b) \\ \chi^{(j)}(\beta) = \chi^{(j)}(b) = 0, j = 0, 1. \end{cases}$$

Multiplying equation (1.3) by  $t^{n-1}\chi$  and integrating by parts four times we get

$$\int_{t}^{b} s^{n-1} \chi g f(u) ds = \int_{t}^{b} s^{n-1} \chi \Delta^{2} u ds = -t^{n-1} \chi(t) (\Delta u)'(t)$$

$$+ t^{n-1} \chi'(t) \Delta u(t) - t^{n-1} \Delta \chi(t) u'(t)$$

$$+ t^{n-1} (\Delta \chi)'(t) u(t) + \int_{t}^{b} s^{n-1} u ds$$

for all  $t \in [\beta, b)$ . Setting  $t = \beta$  in (3.10) we obtain

$$(3.11) \int_{\beta}^{b} s^{n-1} \chi g f(u) ds = -\beta^{n-1} \Delta \chi(\beta) u'(\beta) + \beta^{n-1} (\Delta \chi)'(\beta) u(\beta) + \int_{\beta}^{b} s^{n-1} u ds.$$

Since by Theorem 2.1  $\Delta \chi(\beta) > 0$ , we deduce from (3.9) and (3.11) that  $\{u'(\beta); u \in G\}$  is bounded. Since  $\Delta u > 0$  on  $(\beta, b]$  when  $u \in G$ , we have  $\beta^{n-1}u'(\beta) < t^{n-1}u'(t) < 0$  for  $t \in (\beta, b)$ . Hence  $\{u'(s); u \in G, s \in [\beta, b]\}$  is bounded. From this, (3.9), (3.10), Theorem 2.1 and the fact that  $H = \{\Delta u(t); u \in G\}$  is bounded for each fixed  $t \in [\beta, b)$  we deduce that  $\{(\Delta u)'(t); u \in G\}$  is bounded for each fixed  $t \in (\beta, b)$ . Since for  $t \in (\beta, b)$  we have

$$b^{n-1}(\Delta u)'(b) - t^{n-1}(\Delta u)'(t) = \int_{t}^{b} s^{n-1}gf(u)ds$$

we obtain that  $\{(\Delta u)'(b); u \in G\}$  is bounded. Now let d be such that  $d_1(u) \le d \le d_2(u)$ . Using the fact that  $t^{n-1}(\Delta u)'$  is nondecreasing on [a, b] we can write

$$0 = d^{n-1}(\Delta u)'(d) \le t^{n-1}(\Delta u)'(t) \le b^{n-1}(\Delta u)'(b)$$

for  $t \in [d, b]$ , from which we deduce that  $\{(\Delta u)'(t); u \in G, t \in [d, b]\}$  is bounded. Since

$$\Delta u(b) = \int_{s(u)}^{b} (\Delta u)' ds$$

we obtain that  $\{\Delta u(b); u \in G\}$  is bounded. Now we write

$$\Delta u(d) = \Delta u(b) - \int_{d}^{b} (\Delta u)' ds$$

and we finally obtain that  $\{\Delta u(t); u \in G, t \in [d, b]\}$  is bounded. Using the fact that

$$\int_{a}^{b} t^{n-1} \Delta u dt = 0$$

for all  $u \in S$ , we easily deduce that  $\{\Delta u; u \in G\}$  is bounded in  $L^1(a, b)$  and the conclusion follows from lemma 3.1 (i).

It remains to prove that  $S_+$  is bounded in  $L^{\infty}(a, b)$ . The proof is similar. Using analogous arguments we show that  $\{u \in S_+; r(u) < \alpha\}$  and  $\{u \in S_+; r(u) \ge \alpha\}$  are bounded in  $L^{\infty}(a, b)$ . The proof of lemma 3.3 is complete.

**Remark 3.1.** Note that the constant M in theorem 3.1 can be chosen independently of the parameter  $x \in [0, x_0]$  for each fixed  $x_0 \in (0, +\infty)$  if we consider positive solutions of (1.3), (1.4) for the family of nonlinearities  $f_x(t) = f(t+x)$ ,  $t \ge 0$ .

## 4. PROOF OF THEOREM 1.1

We shall prove that problem (1.3), (1.4) has at least one positive solution  $u \in C^4([a, b])$ . The proof makes use of the Kranosel'skii type fixed point theorem [7] (proposition 2.1 and remark 2.1).

The homogeneous Dirichlet problem

$$\begin{cases} \Delta^2 v = 0 \text{ in } (a, b) \\ v^{(j)}(a) = v^{(j)}(b) = 0, j = 0, 1 \end{cases}$$

has only the trivial solution. Then it is well-known (see e.g. [13] p. 29) that the operator  $\Delta^2$  with Dirichlet boundary conditions has one and only one Green's function G(t, s). Define the closed cone

$$Z = \{u \in C([a, b]); u \ge 0\}.$$

For  $(u, x) \in \mathbb{Z} \times [0, +\infty)$  we define

$$F(u, x)(t) = \int_{a}^{b} G(t, s)g(s)f(u(s) + x)ds$$

and

$$\Phi(u) = F(u, 0)$$
.

By Theorem 2.1 F maps  $Z \times [0, +\infty)$  into Z. Since G is continuous, it is well-known that F is compact. Now the following properties hold:

(i) By Theorem 2.1 and the properties of the Green's function any nontrivial solution of the fixed point equation

$$\Phi(u) = u, \qquad u \in \mathbb{Z}$$

yields a positive solution of (1.3), (1.4) in  $C^4([a, b])$ .

(ii)  $u \neq \theta \Phi(u)$  for all  $\theta \in [0, 1]$  and  $u \in \mathbb{Z}$  such that  $||u||_{\infty} = r$  for sufficiently small r > 0. Indeed, let  $\alpha \in (0, c_2^{-1})$ , where  $c_2$  is the constant in (3.1). By  $(H_4)$  we can choose r > 0 such that  $f(s) \leq \alpha s$  for  $0 \leq s \leq r$ . Now suppose that there exist  $\theta \in [0, 1]$  and  $u \in \mathbb{Z}$  such that  $u = \theta \Phi(u)$  with  $||u||_{\infty} = r$ . Then  $\Delta^2 u = \theta gf(u)$ . With the notations of step 1 of the proof of Theorem 3.1, we have

$$\int_{a}^{b} t^{n-1} \varrho \, gudt = \int_{a}^{b} t^{n-1} u \Delta^{2} \varphi \, dt = \int_{a}^{b} t^{n-1} \varphi \Delta^{2} u dt$$

$$= \theta \int_{a}^{b} t^{n-1} \varphi \, gf(u) \, dt \le \alpha c_{2} \theta \int_{a}^{b} t^{n-1} \varrho \, gudt$$

and we reach a contradiction.

(iii) By  $(H_3)$ , there exists  $\lambda > c_1^{-1}$  (where  $c_1$  is the constant in (3.1)) and  $x_0 > 0$  such that  $f(t+x) \ge \lambda t$  for all  $t \ge 0$  and  $x \ge x_0$ . Then using the same arguments as in step 1 of the proof of Theorem 3.1, we can show that the equation F(u, x) = u has no solutions  $u \in \mathbb{Z}$  for  $x \ge x_0$ .

(iv) Finally, using Remark 3.1 and (iii) above we can find a constant R > r such that  $F(u, x) \neq u$  for all  $x \ge 0$  and  $u \in Z$  with  $||u||_{\infty} = R$ .

Now we can apply Proposition 2.1 and Remark 2.1 stated in [7] to conclude that  $\Phi$  has a nontrivial fixed point. The proof of the theorem is complete.

# 5. PROOF OF THEOREM 1.2

The proof of Theorem 1.2 is based on the maximum principle and the technique of moving parallel planes as in [9], [16] for second order equations and [5], [6] for fourth order equations. Subsequently  $\triangle$  denotes equally the cartesian form and the polar form of the Laplacian. In the same way we write indifferently u(x) or u(|x|).

Assume that  $\Delta u(a) \leq \Delta u(b)$  for some nonnegative nontrivial solution  $u \in C^4([a, b])$  of problem (1.3), (1.4). Then by Theorem 2.1 we have u > 0 in  $\Omega(a, b)$  and  $\Delta u < \Delta u(b)$  in  $\Omega(a, b)$ . Let  $\lambda \in \left[\frac{a+b}{2}, b\right]$  and de-

fine  $\Sigma(\lambda) = \Omega(a, b) \cap \{x = (x_1, x') \in \mathbb{R}^n; x_1 > \lambda\}$ . Let  $\Sigma'(\lambda)$  denote the reflection of  $\Sigma(\lambda)$  in the plane  $T_{\lambda} = \{x = (x_1, x') \in \mathbb{R}^n; x_1 = \lambda\}$ . Define the function

$$u_{\lambda}(x) = u(2\lambda - x_1, x')$$
 for  $x \in \Sigma'(\lambda)$ .

We have the following Lemma.

**Lemma 5.1.** u (resp.  $\Delta u$ ) is strictly increasing (resp. strictly decreasing) as one enters  $\Omega(a, b)$  from  $\{x \in \mathbb{R}^n; |x| = b\}$  along any nontangential direction  $\overrightarrow{s}$ , for some positive distance d > 0 into  $\Omega(a, b)$ .

**Proof.** We have  $u = \frac{\partial u}{\partial v} = 0$  on  $\partial \Omega(a, b)$ . Since by Theorem 2.1  $\Delta u$  is a positive constant on  $\{x \in \mathbb{R}^n; |x| = b\}$  the proof is immediate.

From Lemma 5.1 we deduce the existence of  $\eta \in \left(0, \frac{a+b}{2}\right)$  such that, for  $\lambda \in [b-\eta, b)$ , we have

(5.1) 
$$\begin{cases} u_{\lambda} - u < 0 \text{ in } \Sigma'(\lambda) \text{ and } \frac{\partial u}{\partial x_{1}} < 0 \text{ in } \Sigma(\lambda) \\ \Delta(u_{\lambda} - u) > 0 \text{ in } \Sigma'(\lambda) \text{ and } \frac{\partial \Delta u}{\partial x_{1}} > 0 \text{ in } \Sigma(\lambda). \end{cases}$$

Decrease  $\lambda$  until a critical value  $\mu \ge (a+b)/2$  is reached, beyond which (5.1) is no longer true. Then (5.1) holds for  $\lambda \in (\mu, b)$  while for  $\lambda = \mu$  we have by continuity

$$\begin{cases} u_{\mu} - u \leq 0 \text{ in } \Sigma'(\mu) \text{ and } \frac{\partial u}{\partial x_{1}} < 0 \text{ in } \Sigma(\mu) \\ \Delta(u_{\mu} - u) \geq 0 \text{ in } \Sigma'(\mu) \text{ and } \frac{\partial \Delta u}{\partial x_{1}} > 0 \text{ in } \Sigma(\mu). \end{cases}$$

Suppose  $\mu > (a+b)/2$ . We have  $u_{\mu} \not\equiv u$  in  $\Sigma'(\mu)$  since u > 0 in  $\Omega(a, b)$ . The maximum principle ([10] p. 15) and the Hopf boundary lemma ([10] p. 33) imply that

(5.2) 
$$u_{\mu} - u < 0 \text{ in } \Sigma'(\mu) \text{ and } \frac{\partial u}{\partial x_1} < 0 \text{ on } T_{\mu} \cap \Omega(a, b)$$

where the second inequality follows from the fact that  $\frac{\partial}{\partial x_1}(u_\mu - u) = -2\frac{\partial u}{\partial x_1}$  on  $T_\mu \cap \Omega(a, b)$ . Now  $(H_5)$  and  $(H_6)$  imply that  $\Delta^2(u_\mu - u) \leq 0$  in  $\Sigma'(\mu)$ . From our assumption we have  $\Delta u < \Delta u(b)$  in  $\Omega(a, b)$ . Thus  $\Delta(u_\mu - u) \not\equiv 0$  in  $\Sigma'(\mu)$ . The maximum principle and the Hopf boundary lemma imply that

(5.3) 
$$\Delta(u_{\mu} - u) > 0 \text{ in } \Sigma'(\mu) \text{ and } \frac{\partial \Delta u}{\partial x_1} > 0 \text{ on } T_{\mu} \cap \Omega(a, b)$$

where the second inequality follows from the fact that  $\frac{\partial}{\partial x_1}(\Delta(u_\mu - u)) = -2\frac{\partial \Delta u}{\partial x_1}$  on  $T_\mu \cap \Omega(a, b)$ . (5.2) and (5.3) show that (5.1) holds for  $\lambda = \mu$ .

Now our definition of  $\mu$  implies that either there is a strictly increasing sequence  $(\lambda_j)$  with  $\lim_{j\to\infty} \lambda_j = \mu(\lambda_j > (a+b)/2 \,\forall j)$  such that for each j there is a point  $x_j \in \Sigma'(\lambda_i)$  for which

$$(5.4) u_{\lambda_i}(x_j) - u(x_j) \ge 0 \forall j$$

or that there is a strictly increasing sequence  $(\mu_j)$  with  $\lim_{j\to\infty} \mu_j = \mu$   $(\mu_j > (a+b)/2 \ \forall j)$  such that for each j there is a point  $z_j \in \Sigma'(\mu_j)$  for which

$$(5.5) \Delta u_{\mu_i}(z_j) - \Delta u(z_j) \leq 0 \forall j.$$

In the situation (5.4), a subsequence which we still call  $x_j$  will converge to some point  $x \in \overline{\Sigma'(\mu)}$ ; then  $u_{\mu}(x) - u(x) \ge 0$ . Since (5.1) holds for  $\lambda = \mu$  we must have  $x \in \partial \Sigma'(\mu)$ ; If  $x \in \partial \Sigma'(\mu) \setminus T_{\mu}$  then  $0 = u_{\mu}(x) < u(x)$ , a contradiction. Therefore  $x \in T_{\mu}$ . Using Lemma 5.1, (5.2) and (5.1) with  $\lambda = \mu$  we see that for some  $\varepsilon > 0$  we have

(5.6) 
$$\frac{\partial u}{\partial x_1} < 0 \text{ in } \Omega(a, b) \cap \{x = (x_1, x') \in \mathbb{R}^n; x_1 > \mu - \varepsilon\}.$$

The straight segment joining  $x_j$  to its symmetric with respect to  $T_{\lambda_j}$  belongs to  $\Omega(a, b)$  and by the theorem of the mean it contains a point  $y_i$  such that

$$\frac{\partial u}{\partial x_1}(y_j) \ge 0.$$

Since  $\lim_{j\to\infty} y_j = x$ , we obtain a contradiction to (5.6).

In the situation (5.5), a subsequence which we still call  $z_j$  will converge to some point  $z \in \overline{\Sigma'(\mu)}$ ; then  $\Delta u_{\mu}(z) - \Delta u(z) \le 0$ . Since (5.1) holds for  $\lambda = \mu$  we must have  $z \in \partial \Sigma'(\mu)$ ; If  $z \in \partial \Sigma'(\mu) \setminus T_{\mu}$  then  $\Delta u(b) = \Delta u_{\mu}(z) > \Delta u(z)$ , a contradiction. Therefore  $z \in T_{\mu}$ . Using Lemma 5.1, (5.3) and (5.1) with  $\lambda = \mu$  we see that for some  $\varepsilon > 0$  we have

(5.7) 
$$\frac{\partial \underline{\Delta}\underline{u}}{\partial x_1} > 0 \text{ in } \Omega(a, b) \cap \{x = (x_1, x') \in \mathbb{R}^n; x_1 > \mu - \varepsilon\}.$$

The straight segment joining  $z_j$  to its symmetric with respect to  $T_{\mu_j}$  belongs to  $\Omega(a, b)$  and by the theorem of the mean it contains a point  $t_j$  such that

$$\frac{\partial \underline{\Delta} \underline{u}}{\partial x_1}(t_j) \leq 0.$$

Since  $\lim_{j\to\infty} t_j = z$ , we obtain a contradiction to (5.7).

Thus we have proved that  $\mu = (a+b)/2$  and that (5.1) holds for  $\lambda \in \left(\frac{a+b}{2}, b\right)$ . By continuity we have

$$u_{\varrho} - u \le 0$$
 in  $\Sigma'(\varrho)$  and  $\frac{\partial u}{\partial x_1} < 0$  in  $\Sigma(\varrho)$ 

and

$$\Delta(u_{\varrho} - u) \ge 0$$
 in  $\Sigma'(\varrho)$  and  $\frac{\partial \Delta u}{\partial x_1} > 0$  in  $\Sigma(\varrho)$ 

where  $\varrho = \frac{a+b}{2}$ . Now let x = (a, 0) then

$$(u_{\varrho}-u)(x) = \frac{\partial}{\partial v}(u_{\varrho}-u)(x) = 0$$

and the Hopf lemma implies that  $u_{\varrho} - u \equiv 0$  in  $\Sigma'(\varrho)$ , but this is impossible. The proof of the theorem is complete.

#### References

- [1] D. ARCOYA: Positive solutions for semilinear Dirichlet problems in an annulus, J. Differential Equations 94 (1991), 217-227.
- [2] C. BANDLE and M. KWONG: Semilinear elliptic problems in annular domains, J. Appl. Math. Phys. 40 (1989), 245-257.
- [3] C. BANDLE, C. V. COFFMAN and M. MARCUS: Nonlinear elliptic problems in annular domains, J. Differential Equations 69 (1987), 322-345.
- [4] C. V. COFFMAN and M. MARCUS: Existence and uniqueness results for semilinear Dirichlet problems in annuli, Arch. Ration. Mech. Analysis 108 (1989), 293-307.
- [5] R. DALMASSO: Symmetry properties of solutions of some fourth order ordinary differential equations, Bull. Sc. Math., 2° série, 117 (1993), 441-462.
- [6] R. DALMASSO: Un problème de symétrie pour une équation biharmonique, Annales de la Faculté des Sciences de Toulouse, vol. XI, n.º 3, (1990) 45-53
- [7] D. DE FIGUEIREDO, P. L. LIONS and R. D. NUSSBAUM: A priori estimates and existence of positive solutions of semilinear elliptic equations, J. Math. Pures Appl. 61 (1982), 41-63.
- [8] X. GARAIZAR: Existence of positive radial solutions for semilinear elliptic equations in the annulus, J. Differential Equations 70 (1987), 69-92.
- [9] B. GIDAS, W.-M. NI and L. NIRENBERG: Symmetry and related properties via the maximum principle, Comm. Math. Phys., 68 (1979), 209-243.

- [10] D. GILBARG and N. S. TRUDINGER: Elliptic partial differential equations of second order, Springer-Verlag, Vol. 224, Berlin-Heidelberg-New York, 1977.
- [11] S. S. LIN: On the existence of positive radial solution for nonlinear elliptic equations in annular domains, J. Differential Equations 81 (1989), 221-233.
- [12] S. S. Lin and F. M. Pai: Existence and multiplicity of positive radial solutions for semilinear elliptic equations in annular domains, SIAM J. Math. Anal., Vol. 22, n.º 6 (1991), 1500-1515.
- [13] M. A. NAIMARK: Elementary theory of linear differential operators Part I, F. UNGAN, New York, 1967.
- [14] M. PROTTER and H. WEINBERGER: Maximum principles in differential equations, Prentice Hall, 1967.
- [15] J. SANTANILLA: Existence and nonexistence of positive radial solutions for some semilinear elliptic problems in annular domains, Nonlinear Anal. 16 (1991) 861-879.
- [16] J. SERRIN: A symmetry problem in potential theory, Arch. Ration. Mech. Analysis 43 (1971), 304-318.

Recibido: 11 de diciembre de 1992

Laboratoire LMC-IMAG Equipe EDP Tour IRMA-BP 53 F. 38041 Grenoble Cedex 9 France

E-mail: Dalmasso @ imag. fr