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POSITIVE SELFADJOINT EXTENSIONS OF POSITIVE SYMMETRIC OPERATORS

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1. Introduction. Positive selfadjoint extensions of a symmetric operator have been investigated by many mathematicians: J. von Neumann, K. Friedrichs, M. Krein, M. Birman and others. Especially M. Krein [3] observed the class of all positive selfadjoint extensions of a given positive symmetric operator, and proved among others that, in case of a densely defined operator, the greatest and the smallest positive selfadjoint extension exist. The greatest extension is shown to coincide with the extension, established by Friedrichs, while the smallest one coincides with the extension, considered by von Neumann in case of a strongly positive operator.

In this paper, starting with the well known theorem of Friedrichs, we shall investigate the structure of the greatest extension (Friedrichs extension) T_μ and the smallest one (von Neumann extension) T_M of a given positive symmetric operator T from various points of view. Theorem 1 gives a necessary and sufficient condition for T with non-dense domain to admit positive selfadjoint extensions. If any one of such extensions exists, the von Neumann extension is shown to exist, and its domain is explicitly determined. Among many consequences of this theorem is a simple description of the von Neumann extension, when it is bounded (Theorem 2). In contrast with the identity: $(T+a)_\mu - a = T_\mu$ for all positive number a, $(T+a)_M - a$ varies largely according to a. Theorem 3 shows that $(T+a)_M - a$ converges, in a natural sense, to the von Neumann extension T_M or to the Friedrichs extension T_μ according as $a \to 0$ or $\to \infty$. This theorem permits us to determine the spectrum of T_M or T_μ, when T_M is compact or T_μ has compact resolvent.

2. Preliminaries. A linear operator T on a Hilbert space is, by definition, *symmetric* if

$$(Tf, g) = (f, Tg) \qquad (f, g \in \boldsymbol{D}(T));$$

here the domain $\boldsymbol{D}(T)$ is not assumed to be dense. T is called *positive* (resp. *strongly positive*), if $(Tf, f) \geqq 0$ (resp. $\geqq \varepsilon(f, f)$ for a constant $\varepsilon > 0$).

A positive selfadjoint operator S_1 is called *greater* than another S_2, or the latter is *smaller* than the former, in symbol $S_1 \geqq S_2$, if

$$D(S_1^{1/2}) \subseteq D(S_2^{1/2}) \quad \text{and} \quad \|S_2^{1/2}f\| \leqq \|S_1^{1/2}f\| \qquad (f \in D(S_1^{1/2}))\,.$$

For bounded positive selfadjoint operators, this definition is reduced to the usual order relation. $S_1 \geqq S_2$ implies $S_1 + a \geqq S_2 + a$ for all $a > 0$. If S_2 is invertible and $S_1 \geqq S_2$, then S_1 is also invertible and $S_1^{-1} \leqq S_2^{-1}$.

The greatest (resp. smallest) of all positive selfadjoint extensions of a given positive symmetric operator T, if exists, will be called the *Friedrichs* (resp. *von Neumann*) *extension* and denoted by T_μ (resp. T_M). Remark that M. Krein [3] used the terminology "hard (resp. soft) extension" instead of the Friedrichs (resp. von Neumann) one. The basic tool for our development is the well known result of Friedrichs (see [4] n⁰ 124):

FRIEDRICHS THEOREM. *A densely defined, positive symmetric operator T admits the Friedrichs extension T_μ. The domain of its square root $T_\mu^{1/2}$ consists of all vectors f, for which there exists a sequence $\{f_n\} \subset D(T)$ such that*

$$\lim_{n\to\infty} f_n = f \quad \textit{and} \quad \lim_{n,m\to\infty} (T(f_n - f_m), f_n - f_m) = 0\,.$$

T_μ is just the restriction of the adjoint T^* on $D(T^*) \cap D(T_\mu^{1/2})$.

An immediate consequence is the relation

$$(\hat{T} + a)_\mu = T_\mu + a \qquad \text{for all } a \geqq 0\,.$$

That T is densely defined is also a necessary condition for the existence of the Friedrichs extension. K. Friedrichs proved only that the above mentioned restriction of T^* is a positive selfadjoint extension. However, for any positive selfadjoint extension $\hat{T}$ and $\{f_n\}$ in the Friedrichs theorem, $\{\hat{T}^{1/2} f_n\}$ is a Cauchy sequence, so that f belongs to $D(\hat{T}^{1/2})$ because of the closedness of $\hat{T}^{1/2}$ and

$$\|\hat{T}^{1/2} f\|^2 = \lim_{n\to\infty} (Tf_n, f_n)\,,$$

showing that the extension by Friedrichs is the greatest one (cf. [3] and [1] n⁰ 109).

3. von Neumann extension. Throughout this section T will denote a closed positive symmetric operator on a Hilbert space $\boldsymbol{H}$. On account of the index theorem of Krasnoselskii [2 §3] T admits a selfadjoint extension, but not necessarily a positive one. To formulate a condition for the existence of positive selfadjoint extensions, let us introduce a notion. T is called *positively closable* if

$$\lim_{n\to\infty} (Tf_n, f_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} Tf_n = g$$

implies $g = 0$. If T is densely defined, it is positively closable; in fact, the inequality

$$|(Tf_n, h)|^2 \leqq (Tf_n, f_n)(Th, h) \qquad (h \in \boldsymbol{D}(T))$$

shows that $(g, h)=0$ for h in the dense set $\boldsymbol{D}(T)$, hence $g=0$. If T is strongly positive, it is positively closable; in fact, $\lim_{n\to\infty} (Tf_n, f_n) = 0$ implies $\lim_{n\to\infty} f_n = 0$, hence $g = 0$ because of the closedness of T.

THEOREM 1. *A closed positive symmetric operator T admits a positive selfadjoint extension if and only if it is positively closable. When this requirement is fulfilled, T has the von Neumann extension T_M such that*

$$\|T_M^{1/2} h\|^2 = \sup_{f \in D(T)} \frac{|(Tf, h)|^2}{(Tf, f)}$$

and $\boldsymbol{D}(T_M^{1/2})$ consists of all vectors h, for which the above right side is finite.

PROOF. If T admits a positive selfadjoint extension, it is positively closable, because a positive selfadjoint operator is always positively closable. Suppose, conversely, that T is positively closable. Consider the operator S, defined on the space $P\boldsymbol{H}$ by

$$S(Tf) = Pf \qquad (f \in \boldsymbol{D}(T)),$$

where P is the orthogonal projection onto the closure of the range of T. This definition causes no ambiguity, for $Tf=0$ implies $(f, Th) = 0$ $(h \in \boldsymbol{D}(T))$, hence $Pf = 0$. Since

$$(S(Tf), Tf) = (Pf, Tf) = (f, Tf) \geqq 0,$$

S is a densely defined, positive symmetric operator on $P\boldsymbol{H}$. Consider the Friedrichs extension S_μ on $P\boldsymbol{H}$, then it has inverse; in fact, $S_\mu g = 0$ implies $S_\mu^{1/2} g = 0$, hence on account of the Friedrichs theorem there exists a sequence $\{g_n\} \subset \boldsymbol{D}(S)$ such that

$$\lim_{n\to\infty} g_n = g \quad \text{and} \quad \lim_{n\to\infty} (Sg_n, g_n) = 0 .$$

It follows, with $g_n = Tf_n$, that

$$\lim_{n\to\infty} Tf_n = g \quad \text{and} \quad \lim_{n\to\infty} (Tf_n, f_n) = 0\,,$$

hence $g = 0$ by assumption. Consider the positive selfadjoint operator $\hat{T} = S_\mu^{-1}\cdot P$ on $\boldsymbol{H}$. Since $Pf = S(Tf)$ implies $S_\mu^{-1}\cdot(Pf) = Tf$, $\hat{T}$ is an extension of T. It follows from $\hat{T}^{1/2} = S_\mu^{-1/2}\cdot P$ that

$$\|\hat{T}^{1/2}h\|^2 = \sup_{g\in D(S_\mu^{1/2})} \frac{|(g, Ph)|^2}{\|S_\mu^{1/2}g\|^2}$$

and the domain $\boldsymbol{D}(\hat{T}^{1/2})$ consists of all vectors h, for which the above right side is finite. On the other hand, on account of the Friedrichs theorem, for any $g \in \boldsymbol{D}(S_\mu^{1/2})$ there exists a sequence $\{g_n\} \subset \boldsymbol{D}(S)$ such that

$$\lim_{n\to\infty} g_n = g \quad \text{and} \quad \lim_{n\to\infty}(Sg_n, g_n) = \|S_\mu^{1/2}g\|^2\,,$$

consequently

$$\|\hat{T}^{1/2}h\|^2 = \sup_{g\in D(S)} \frac{|(g, Ph)|^2}{(Sg, g)} = \sup_{f\in D(T)} \frac{|(Tf, h)|^2}{(Tf, f)}\,.$$

It remains to show that $\hat{T}$ is really the smallest extension. Take an arbitrary positive selfadjoint extension $\check{T}$, then

$$|(Tf, h)|^2 = |(\check{T}^{1/2}f, \check{T}^{1/2}h)|^2 \leqq (Tf, f)\|\check{T}^{1/2}h\|^2 \qquad (f \in \boldsymbol{D}(T),\, h \in \boldsymbol{D}(\check{T}^{1/2}))\,,$$

hence

$$\|\hat{T}^{1/2}h\|^2 = \sup_{f\in D(T)} \frac{|(Tf, h)|^2}{(Tf, f)} \leqq \|\check{T}^{1/2}h\|^2\,.$$

This completes the proof.

COROLLARY 1. *T admits a positive selfadjoint extension if and only if the functional*

$$\theta(h) \equiv \sup_{f\in D(T)} \frac{|(Tf, h)|^2}{(Tf, f)}$$

is finite on a dense set.

PROOF. If T admits a positive selfadjoint extension, the set of h with finite $\theta(h)$ is just the domain of $T_M^{1/2}$ by Theorem 1, hence it is a dense set. Suppose, conversely, that the functional is finite on a dense set $\boldsymbol{D}$. Since

$$|(Tf, h)|^2 \leqq (Tf, f)\cdot\theta(h) \qquad (h \in \boldsymbol{D}, f \in \boldsymbol{D}(T)),$$

T is shown to be positively closable.

COROLLARY 2. *If T is densely defined, its von Neumann extension is the restriction of T^* on the set of vectors f, for which there exists a sequence $\{f_n\} \subset \boldsymbol{D}(T)$ such that*

$$\lim_{n\to\infty} Tf_n = T^*f \quad \textit{and} \quad \lim_{n,m\to\infty} (T(f_n - f_m), f_n - f_m) = 0.$$

PROOF. With the notations in the proof of Theorem 1, $f \in \boldsymbol{D}(T_M)$ is equivalent to $Pf \in \boldsymbol{R}(S_\mu)$. On account of the Friedrichs theorem the last condition means that there exists a sequence $\{f_n\} \subset \boldsymbol{D}(T)$ such that

$$S^*(\lim_{n\to\infty} Tf_n) = Pf \quad \text{and} \quad \lim_{n,m\to\infty} (T(f_n - f_m), f_n - f_m) = 0\,.$$

The assertion follows now immediately.

COROLLARY 3. *If T is positively closable with closed range, then its von Neumann extension T_M is given by the formula:*

$$T_M(f + g) = Tf \qquad (f \in \boldsymbol{D}(T), g \in \boldsymbol{R}(T)^\perp)\,.$$

PROOF. Consider the positive symmetric operator $\widehat{T}$, defined by

$$\widehat{T}(f + g) = Tf \qquad (f \in \boldsymbol{D}(T), g \in \boldsymbol{R}(T)^\perp)\,.$$

Since the range $\boldsymbol{R}(T)$ is closed by assumption, the orthogonal complement of $\boldsymbol{D}(T) + \boldsymbol{R}(T)^\perp$ consists of vectors Tf with $(Tf, f) = 0$, so that it is reduced to $\{0\}$ because of the positive closableness of T. Now that $\boldsymbol{D}(\widehat{T})$ is dense, $\boldsymbol{R}(\widehat{T})$ is closed and $\boldsymbol{D}(\widehat{T}) \supset \boldsymbol{R}(\widehat{T})^\perp$, $\widehat{T}$ is a selfadjoint operator. Finally by Theorem 1 $\boldsymbol{R}(T)^\perp$ is contained in the kernel of $T_M^{1/2}$, hence of T_M, consequently $\boldsymbol{D}(\widehat{T}) \subset \boldsymbol{D}(T_M)$. This completes the proof.

M. Krein [3] proved this corollary in quite a different manner. That the operator $\widehat{T}$ in the above proof is a positive selfadjoint extension was firstly proved by J. von Neumann for a strongly positive operator (see [1] n°107).

COROLLARY 4. *Let T be a densely defined operator with inverse. Then T^{-1} admits a positive selfadjoint extension if and only if the Friedrichs extension T_μ has inverse. When this requirement is fulfilled, it results*

$$(T_\mu)^{-1} = (T^{-1})_M .$$

PROOF. The inverse of T_μ, if exists, is a positive selfadjoint extension of T^{-1}. On the other hand, any positive selfadjoint extension S of T^{-1} has dense range because of $\boldsymbol{R}(S)\supset\boldsymbol{D}(T)$, hence has inverse. S^{-1} is a positive selfadjoint extension of T, consequently $S^{-1}\leqq T_\mu$ by definition. But this means that $(T_\mu)^{-1}$ is the von Neumann extension of T^{-1}.

COROLLARY 5. *T admits a positive selfadjoint extension, which is smaller than a given positive selfadjotni operator S, if and only if*

$$|(Tf, h)|^2 \leqq (Tf,f)(Sh, h) \qquad (f\in\boldsymbol{D}(T), h\in\boldsymbol{D}(S)) .$$

PROOF. On account of Theorem 1, Corollary 1 and the Friedrichs theorem the above inequality is equivalent to that T_M exists and $T_M\leqq S$.

THEOREM 2. *A closed positive symmetric operator T admits a bounded positive selfadjoint extension of norm $\leqq \gamma$ if and only if*

$$\|Tf\|^2\leqq\gamma(Tf,f) \qquad (f\in\boldsymbol{D}(T)) .$$

When this inequality is fulfilled, the von Neumann extension T_M is represented in the form:

$$T_M = (T_0^{-1/2}T^*)^*(T_0^{-1/2}T^*) ,$$

where T^* is the adjoint of T as a bounded operator from the Hilbert space $\boldsymbol{D}(T)$ to $\boldsymbol{H}$, T_0 is the compression of T on $\boldsymbol{D}(T)$ and T_0^{-1} is the inverse of the restriction of T_0 to the closure of $\boldsymbol{R}(T_0)$.

PROOF. The first assertion follows from Corollary 5 with $S=\gamma$. The last assertion results from Theorem 1;

$$\|T_M^{1/2}h\|^2 = \sup_{f\in D(T)} \frac{|(Tf,h)|^2}{(Tf,f)}$$

$$= \sup_{f\in D(T)} \frac{|(f,T^*h)|^2}{\|T_0^{1/2}f\|^2} = \|T_0^{-1/2}T^*h\|^2 .$$

The first part of Theorem 2 is a variant of the Krein theorem (see [3], [1] n°108 or [4] n°125) that a symmetric operator S with $\|Sf\| \leqq \|f\|$ $(f\in \boldsymbol{D}(S))$ admits a selfadjoint extension of norm$\leqq 1$.

COROLLARY 6. *T admits a compact selfadjoint extension if and only if the set $\{Tf: (Tf,f)\leqq 1\}$ has compact closure. Under the closure compactness of the set, every positive selfadjoint extension is compact if and only if $\boldsymbol{D}(T)$ has finite codimension.*

PROOF. If the set has compact closure, then

$$\sup_{f\in D(T)} \frac{\|Tf\|^2}{(Tf,f)} < \infty ,$$

so that T_M exists as a bounded operator by Theorem 2. It follows with T_0 and T^* in Theorem 2 that

$$T = (T_0^{-1/2}T^*)^* T_0^{1/2} .$$

This implies that the image of the unit ball under the operator $(T_0^{-1/2}T^*)^*$ is contained in the closure of the set in question, so that T_M is compact by Theorem 2. The converse assertion follows from the fact that for any positive selfadjoint extension $\hat{T}$ the set is contained in the image of the unit ball under $\hat{T}^{1/2}$ and that the compactness of $\hat{T}$ implies that of $\hat{T}^{1/2}$. A bounded positive selfadjoint operator S is an extension of T if and only if $T_M \leqq S \leqq T_M + \alpha Q$ for some $\alpha \geqq 0$, where Q is the orthogonal projection onto $\boldsymbol{D}(T)^{\perp}$. The last assertion of the theorem is now immediate.

COROLLARY 7. *A densely defined operator T admits a positive selfadjoint extension with compact resolvent if and only if the set $\{f: (f,f)+(Tf,f) \leqq 1\}$ has compact closure. Every positive selfadjoint extension of T has compact resolvent if and only if the range $\boldsymbol{R}(T)$ has finite codimension and the set $\{Pf: (Tf,f)\leqq 1\}$ has compact closure, where P is the orthogonal projection onto the closure of $\boldsymbol{R}(T)$.*

PROOF. T admits a positive selfadjoint extension with compact resolvent if and only if $(T_\mu+1)^{-1}$ is compact. Thus the first assertion follows from Corollary 4 and Corollary 6. Since $\boldsymbol{R}(T)^\perp$ coincides with the kernel of the von Neumann extension T_M, the compactness of $(T_M+1)^{-1}$ is equivalent to the finite dimensionality of $\boldsymbol{R}(T)^\perp$ together with the closure compactness of the set $\{Pf: (Tf,f)\leqq 1\}$.

4. Limit representation. T is again a closed positive symmetric operator. For any $a>0$, the von Neumann extension of $T+a$ is formed according to Corollary 3. For simplicity, let us use the notation:

$$T_{(a)} = (T+a)_M - a\,.$$

$T_{(a)}$ is obviously a selfadjoint extension of T, though not positive; in fact, it is given by the formula:

$$T_{(a)}(f+g) = Tf - ag \qquad (f\in \boldsymbol{D}(T),\, g\in \boldsymbol{R}(T+a)^\perp)\,.$$

When T is positively closable, $T_{(0)}$ will have the meaning of T_M.

To formulate the asymptotic behaviour of $T_{(a)}$ as $a\to 0$ or $\to\infty$, let us introduce a notion of convergence for a sequence of unbounded selfadjoint operators. A sequence $\{T_n\}$ of selfadjoint operators is said to converge to a selfadjoint operator T *in resolvent* if

$$\lim_{n\to\infty} (T_n-\xi)^{-1} = (T-\xi)^{-1} \qquad \text{(strong convergence)}$$

for a complex number ξ, uniformly apart from the spectrum of all T_n and also of T. It is easy to see that this definition does not depend on the choice of ξ. For a uniformly bounded sequence, convergence in resolvent is equivalent to strong convergence.

LEMMA. *If T is positively closable, then*

$$(T_{(a)}+1)^{-1} = \begin{cases} 1-(1-(T+1)^{-1})_{(a/(1-a))} & (0\leqq a<1), \\ (T+1)^{-1}{}_{(1/(a-1))} & (a>1). \end{cases}$$

PROOF. Consider first the case $a=0$. Since $(T_M+1)^{-1}$ is a positive selfadjoint extension of $(T+1)^{-1}$, it follows from definition that

$$1-(T_M+1)^{-1} \geqq (1-(T+1)^{-1})_M\,.$$

This inequality implies that $1-(1-(T+1)^{-1})_M$ is a positive selfadjoint operator with inverse, and that

$$\widehat{T} \equiv (1-(1-(T+1)^{-1})_M)^{-1}-1$$

is a positive selfadjoint extension of T. Since $\widehat{T} \leqq T_M$, it follows from definition that $\widehat{T}$ coincides with T_M. This proves the assertion for the case $a=0$. The case $0<a<1$ can be reduced to the above, by considering $\frac{1}{1-a}(T+a)$ instead of T. If $a>1$, the operator $(T_{(a)}+1)^{-1}$ is given by the formula:

$$(T_{(a)}+1)^{-1}((T+1)f+(a-1)g) = f-g \quad (f \in \boldsymbol{D}(T), g \in \boldsymbol{R}(T+a)^{\perp}).$$

On the other hand, since

$$(T+1)^{-1}+\frac{1}{a-1}=\frac{1}{a-1}(T+a)(T+1)^{-1},$$

the subspace $\boldsymbol{R}((T+1)^{-1}+1/(a-1))$ coincides with $\boldsymbol{R}(T+a)$, so that $(T+1)^{-1}_{(1/(a-1))}$ is determined by the same formula as that for $(T_{(a)}+1)^{-1}$.

THEOREM 3. *If T is a closed, positively closable, positive symmetric operator, then $T_{(a)}$ converges to T_M in resolvent as $a \to 0$. If, in addition, T is densely defined, then $T_{(a)}$ converges to T_μ in resolvent as $a \to \infty$.*

PROOF. Consider first the case T_M is bounded. Since, for $0 \leqq a \leqq b$, $(T+a)_M+b-a$ is a positive selfadjoint extension of $T+b$, it follows from definition that

$$T_M \geqq T_{(a)} \geqq T_{(b)} \geqq -b,$$

so that $T_{(a)}$ converges strongly to T_M. The assertion for a general case is reduced to the above; in fact, by Lemma and the above arguments

$$\begin{aligned}(T_M+1)^{-1} &= 1-(1-(T+1)^{-1})_M \\ &= 1-\lim_{a\to 0}(1-(T+1)^{-1})_{(a/(1-a))} = \lim_{a\to\infty}(T_{(a)}+1)^{-1}.\end{aligned}$$

If T is densely defined, it follows from Corollary 4 and Lemma that

$$\begin{aligned}(T_\mu+1)^{-1} &= (T+1)^{-1}{}_M \\ &= \lim_{a\to\infty}(T+1)^{-1}{}_{(1/(a-1))} = \lim_{a\to\infty}(T_{(a)}+1)^{-1},\end{aligned}$$

COROLLARY 8. *If the von Neumann extension T_M is compact, $T_{(a)}$ converges uniformly to T_M as $a \to 0$. If the Friedrichs extension T_μ has compact resolvent, $(T_{(a)}+1)^{-1}$ converges uniformly to $(T_\mu+1)^{-1}$ as $a \to \infty$.*

PROOF. If T_M is compact, for any $\varepsilon > 0$ there exists an orthogonal projection Q_ε of finite rank such that

$$T_M Q_\varepsilon = Q_\varepsilon T_M \quad \text{and} \quad \|T_M(1-Q_\varepsilon)\| < \varepsilon .$$

Then the inequality

$$0 \leqq T_{(a)} + a \leqq T_M + a$$

implies

$$\|(1-Q_\varepsilon)T_{(a)}(1-Q_\varepsilon)\| \leqq \|T_M(1-Q_\varepsilon)\| + 2a .$$

On the other hand, it follows from Theorem 3 that for sufficiently small $a > 0$

$$\|(T_{(a)} - T_M)Q_\varepsilon\| < \varepsilon ,$$

so that $T_{(a)}$ converges uniformly to T_M as $a \to 0$. Similar arguments for $(T_\mu+1)^{-1}$ instead of T_M yield the second assertion.

COROLLARY 9. *Let the set $\{Tf: (Tf,f) \leqq 1\}$ have compact closure. Then a positive number t is a regular point of the von Neumann extension T_M if and only if, for some $\delta > 0$ and all sufficiently small $a > 0$,*

$$\|Tf - tf\|^2 \geqq \delta^2 \|P_a f\|^2 + (t+a)^2 \|(1-P_a)f\|^2 \qquad (f \in \boldsymbol{D}(T)),$$

where P_a is the orthogonal projection onto $\boldsymbol{R}(T+a)$.

PROOF. T_M is compact by Corollary 6 and $T_{(a)}$ converges uniformly to T_M as $a \to 0$ by Corollary 8. Then t is a regular point of T_M if and only if $(T_{(a)} - t)^{-1}$ is uniformly bounded for sufficiently small $a > 0$. On the other hand, on account of Corollary 3, $T_{(a)} - t$ is reduced by the subspace $\boldsymbol{R}(T+a)^\perp$, on which it coincides with $-(t+a)$, so that $T_{(a)} - t$ has inverse of norm $\leqq \delta^{-1}$, δ being small, if and only if

$$\|(T_{(a)} - t)P_a f\|^2 \geqq \delta^2 \|P_a f\|^2 \qquad (f \in \boldsymbol{D}(T)) .$$

The orthogonal sum representation

$$Tf - tf = (T_{(a)} - t)P_a f - (t + a)(1 - P_a)f$$

shows that the uniform boundedness of $(T_{(a)} - t)^{-1}$ is equivalent to the inequality of the assertion.

COROLLARY 10. *Let the set* $\{f: (f,f) + (Tf,f) \leqq 1\}$ *have compact closure. Then a positive number* t *is a regular point of the Friedrichs extension* T_μ *if and only if, for some* $\delta > 0$ *and all sufficiently large* $a > 0$,

$$\|Tf - tf\|^2 \geqq \delta^2 \|P_a f\|^2 + (t + a)^2 \|(1 - P_a)f\|^2 \qquad (f \in \boldsymbol{D}(T)),$$

where P_a *is the orthogonal projection onto* $\boldsymbol{R}(T + a)$.

PROOF. As in the proof of Corollary 9, the inequality of the assertion is equivalent to the uniform boundedness of $(T_{(a)} - t)^{-1}$ for sufficiently large $a > 0$. On the other hand, $(T_\mu + 1)^{-1}$ is compact by Corollary 7 and $(T_{(a)} + 1)^{-1}$ converges uniformly to $(T_\mu + 1)^{-1}$ by Corollary 8. Now the assertion follows from the relation:

$$((T_{(a)} + 1)^{-1} - (t + 1)^{-1})^{-1} = -(t + 1) - (t + 1)^2 (T_{(a)} - t)^{-1}.$$

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