

POSITIVE SENSITIVITY ANALYSIS IN LINEAR PROGRAMMING

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Positive sensitivity analysis (PSA) is a sensitivity analysis method for linear programming that finds the range of perturbations within which positive value components of a given optimal solution remain positive. Its main advantage is that it is applicable to both an optimal basic and nonbasic optimal solution.

The first purpose of this paper is to present some properties of PSA that are useful for establishing the relationship between PSA and sensitivity analysis using optimal bases, and between PSA and sensitivity analysis using the optimal partition. We examine how the range of PSA varies according to the optimal solution used for PSA, and discuss the relationship between the ranges of PSA using different optimal solutions. The second purpose is to clarify the relationship between PSA and sensitivity analysis using an optimal basis, and the relationship between PSA and sensitivity analysis using the optimal partition. We show that sensitivity analysis using the optimal partition is a special case of PSA, and its properties can be derived from the properties of PSA. The comparison among the three sensitivity analysis methods will lead to a better understanding of the difference among sensitivity analysis methods.

Keywords: Linear programming; sensitivity analysis; positive sensitivity analysis; optimal basis; optimal partition.

1. Introduction

Sensitivity analysis in linear programming is used to acquire information about how decisions are affected as the input data are varied. For instance, when the cost of an

activity or the available amount of resources is changed, we often need information about how the total cost of the current decision is altered, in order to obtain a new optimal decision for the new situation. In this case, sensitivity analysis can be applied. Moreover, when a new constraint or a new activity is added, sensitivity analysis is also performed to analyze the effects on the current decisions.

The method of sensitivity analysis in simplex method is well developed on the foundation of optimal basis. It requires little computational effort. This method has been introduced in numerous papers and textbooks so far (see, for example: Dantzig, 1963; Gal, 1979), and has been used in many linear programming codes. However, in case of degeneracy, it may yield incomplete information due to alternative optimal bases (Evans and Baker, 1982; Knolmayer, 1984; Jansen *et al.*, 1997).

On the other hand, most interior-point methods produce a solution which converges to an optimal solution relatively interior to the optimal face. Some additional computation enables us to get an exact optimal basic or nonbasic solution (Tapia and Zhang, 1991; Mehrotra and Ye, 1993; Bixby and Salzman, 1994). However, since sensitivity analysis using an optimal basis cannot be applied to an optimal nonbasic solution, other methods for sensitivity analysis have been suggested: positive sensitivity analysis (PSA), sensitivity analysis using the optimal partition, and ϵ -sensitivity analysis (Yang, 1990; Adler and Monteiro, 1992; Kim *et al.*, 1999).

Yang (1990) introduced PSA for optimal solutions including optimal nonbasic solutions. Yang (1990) defined two types of sensitivity analysis based on Sung and Park's (1988) definition. The first type of sensitivity analysis is defined to find the characteristic region within which an optimal basis still remains optimal for a perturbed problem. The second type, called PSA, is defined to find the characteristic region within which variables having a zero and having a positive value in an optimal solution remain zero and positive in the perturbed problem, respectively. Adler and Monteiro (1992) developed a method of parametric analysis on the right-hand side by introducing the optimal partition. Monteiro and Mehrotra (1996) presented a parametric analysis by generalizing Adler and Monteiro's method, and Greenberg (2000) developed a method of sensitivity analysis using the optimal partition when cost coefficients and right-hand sides change simultaneously. To use Yang's and Adler and Monteiro's methods, we need an optimal solution or the optimal partition, which requires additional computation for interior-point methods. Kim *et al.* (1999) developed a practical sensitivity analysis method, ϵ -sensitivity analysis, which can be directly applied to interior-point solutions produced by interior-point methods.

Although PSA and sensitivity analysis using the optimal partition were developed several years ago, there have been very few studies on the relationship between the three methods: sensitivity analysis using an optimal basis, PSA, and sensitivity analysis using the optimal partition. The first purpose of this paper is to clarify the relationship between PSA and sensitivity analysis using an optimal basis, and the relationship between PSA and sensitivity analysis using the optimal partition. The comparison between the three sensitivity analysis methods will lead to a better understanding of the differences between them and will be helpful in

making a choice of sensitivity analysis methods. The second purpose is to present some properties of PSA. We examine how the range of PSA varies according to the optimal solution used for PSA, and study the relationship between the ranges of PSA using different optimal solutions. In fact, the relationship between PSA and the other sensitivity analysis methods is based on these properties of PSA.

This paper is organized as follows: In Section 2, we introduce three kinds of sensitivity analyses for linear programming, and some basic results about the relationship between PSA and other sensitivity analysis methods are presented. In Section 3, we discuss the relationship between the ranges of PSA using different optimal solutions, and present a sufficient and necessary condition that the range of PSA includes a positive or negative value. In Section 4, we study the relationship between PSA using an optimal basic solution and sensitivity analysis using an optimal basis when a given optimal basic solution is degenerate. In Section 5, some concluding remarks are given.

2. Definition of the Three Sensitivity Analysis Methods

Consider the linear programming problem (LP):

$$(P) : \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0, \end{array} \quad (D) : \begin{array}{ll} \max & b^T y \\ \text{s.t.} & A^T y + s = c \\ & s \geq 0, \end{array}$$

where $c \in \mathfrak{R}^n$, $b \in \mathfrak{R}^m$, and $A \in \mathfrak{R}^{m \times n}$ with $\text{Rank}(A) = m$. Throughout this paper, it is assumed that both (P) and (D) are feasible. For sensitivity analysis on the cost coefficient c_k that is perturbed by an amount θ , we consider another linear programming problem (LP_θ):

$$(P_\theta) : \begin{array}{ll} \min & (c + \theta e_k)^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0, \end{array} \quad (D_\theta) : \begin{array}{ll} \max & b^T y \\ \text{s.t.} & A^T y + s = c + \theta e_k \\ & s \geq 0, \end{array}$$

where $e_k \in \mathfrak{R}^n$ is the vector such that the k th element is one and the others are zero. Also, for the right-hand side b_h that is perturbed by the amount γ , we consider the linear programming problem (LP_γ):

$$(P_\gamma) : \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = (b + \gamma e_h) \\ & x \geq 0, \end{array} \quad (D_\gamma) : \begin{array}{ll} \max & (b + \gamma e_h)^T y \\ \text{s.t.} & A^T y + s = c \\ & s \geq 0, \end{array}$$

where $e_h \in \mathfrak{R}^m$ is the vector such that the h th element is one and the others are zero.

Given an index set σ of variables, let A_σ denote the submatrix of A with columns that correspond to indices in σ . Similarly, we use z_σ to denote the subvector of a vector z with components that correspond to indices in σ . For any vector x , let x_j denote the j th element of x .

Let B and N be the index sets of the basic and nonbasic variables of a basis, respectively. If $(x_B^T, x_N^T)^T = ((A_B^{-1}b)^T, 0)^T$ is an optimal solution to (P) , A_B is called a *primal-optimal basis*. Also, if $y = A_B^{-T}c_B$ and $(s_B^T, s_N^T)^T = (0, c_N^T - y^T A_N)^T$ is an optimal solution to (D) , then A_B is called a *dual-optimal basis*. If A_B is both a primal-optimal and dual-optimal basis, it is called an *optimal basis*. For a primal-optimal basis A_B , let $Tc_k(A_B)$ denote the following range of θ :

$$Tc_k(A_B) = \left\{ \theta \mid \begin{bmatrix} A_B^T \\ A_N^T \end{bmatrix} y + \begin{bmatrix} s_B \\ s_N \end{bmatrix} = \begin{bmatrix} c_B + (\theta e_k)_B \\ c_N + (\theta e_k)_N \end{bmatrix}, s_B = 0, s_N \geq 0 \right\}. \quad (2.1)$$

That is, $Tc_k(A_B)$ represents the range of θ within which a primal-optimal basis A_B is an optimal basis. Note that $Tc_k(A_B)$ may be the empty set. Similarly, for a dual-optimal basis A_B , let $Tb_h(A_B)$ denote the following range of γ :

$$Tb_h(A_B) = \{ \gamma \mid x_B = A_B^{-1}(b + \gamma e_h) \geq 0, x_N = 0 \}. \quad (2.2)$$

Also, $Tb_h(A_B)$ represents the range of γ within which a dual-optimal basis A_B is an optimal basis. Note that $Tb_h(A_B)$ may also be the empty set.

The traditional sensitivity analysis using an optimal basis, which is called *basic sensitivity analysis* later on, is defined as the following:

Definition 2.1. (Basic Sensitivity Analysis, BSA) Let B be the index set of the basic variables of an optimal basis. BSA using A_B on a cost coefficient c_k is to find the range of θ within which A_B remains an optimal basis to (LP_θ) . Similarly, BSA using A_B on a right-hand side b_h is to find the range of γ within which A_B remains an optimal basis to (LP_γ) .

By the definition of $Tc_k(A_B)$ and $Tb_h(A_B)$, the ranges found by BSA using A_B on c_k and b_h are represented as $Tc_k(A_B)$ and $Tb_h(A_B)$, respectively. To perform BSA, we need an optimal basis associated with an optimal basic solution. In fact, BSA can be applied only to an optimal basic solution.

Before defining PSA, some notation is introduced. For an arbitrary vector x whose components are nonnegative, let $\eta(x)$ and $\bar{\eta}(x)$ denote the sets of indices of variables as follows:

$$\eta(x) = \{j \mid x_j > 0\}, \quad \bar{\eta}(x) = \{j \mid x_j = 0\}.$$

In addition, $\pi(x) = (\eta(x), \bar{\eta}(x))$ is called the *induced partition* of x .

Definition 2.2. (Positive Sensitivity Analysis, PSA) Let x^* be an optimal solution to (P) . The PSA using x^* on c_k is to find the range of θ within which there exists an optimal solution to (P_θ) whose induced partition is equal to $\pi(x^*)$. Similarly, the PSA using x^* on b_h is to find the range of γ within which there exists an optimal solution to (P_γ) whose induced partition is equal to $\pi(x^*)$.

Given an optimal solution x^* to (P) , the range of PSA using x^* is calculated using the following (Yang, 1990):

$$Yc_k(x^*) = \left\{ \theta \mid \begin{bmatrix} A_\sigma^T \\ A_{\bar{\sigma}}^T \end{bmatrix} y + \begin{bmatrix} s_\sigma \\ s_{\bar{\sigma}} \end{bmatrix} = \begin{bmatrix} c_\sigma + (\theta e_k)_\sigma \\ c_{\bar{\sigma}} + (\theta e_k)_{\bar{\sigma}} \end{bmatrix}, s_\sigma = 0, s_{\bar{\sigma}} \geq 0 \right\}, \quad (2.3)$$

$$Yb_h(x^*) = \{ \gamma \mid A_\sigma x_\sigma = b + \gamma e_h, x_\sigma \geq 0, x_{\bar{\sigma}} = 0 \}, \quad (2.4)$$

where $\sigma = \eta(x^*)$ and $\bar{\sigma} = \bar{\eta}(x^*)$. Note that in equation (2.4), $x_\sigma \geq 0$ is used instead of $x_\sigma > 0$ so that $Yb_h(x^*)$ can include boundary values and consequently the comparison of PSA with other sensitivity analysis methods will be more convenient. In addition, we can find that it is the induced partition, not the values of an optimal solution, that determines the range of the PSA on c_k . PSA using a different optimal solution, which has the same induced partition with x^* , produces the same range of θ .

The main advantage of PSA is that it can be applied to any optimal solution including optimal nonbasic solutions. Most interior-point methods produce a final interior solution close to the optimal face, and some additional computation is needed to obtain an optimal solution from it. Moreover, the optimal solution may be a nonbasic solution. In this case, PSA can be applied to the nonbasic optimal solution. Furthermore, there are some cases where PSA is useful. For example, when the cost or the supply of a certain material is changed, we need to determine the optimal output of each product with the constraint that the production of products that have not been produced under the current policy should be avoided. For that case, PSA can be used to find the amount of the change as long as the constraint is satisfied.

On the other hand, sensitivity analysis using the optimal partition was suggested by Adler and Monteiro (1992). According to Goldman and Tucker (1956), there exists at least one optimal solution (x^*, y^*, s^*) to (LP) which is strictly complementary, that is,

$$x_j^* + s_j^* > 0, \quad \forall j.$$

Let $B^* = \eta(x^*)$ and $N^* = \eta(s^*)$. The partition $\pi^* = (B^*, N^*)$ of indices of variables is called the *optimal partition* of (LP) . (Throughout this paper, $\pi^* = (B^*, N^*)$ denotes the optimal partition of (LP) .) The definition of sensitivity analysis using the optimal partition is as follows.

Definition 2.3. (Optimal Partition Sensitivity Analysis, OSA) Let $\pi^* = (B^*, N^*)$ be the optimal partition of (LP) . The sensitivity analysis using the optimal partition on c_k is to find the range of θ within which the optimal partition of (LP_θ) is equal to π^* . Similarly, the sensitivity analysis using the optimal partition on b_h is to find the range of γ within which the optimal partition of (LP_γ) is equal to π^* .

The range of OSA is calculated using the following (Roos *et al.*, 1997):

$$Oc_k(B^*, N^*) = \left\{ \theta \mid \begin{bmatrix} A_{B^*}^T \\ A_{N^*}^T \end{bmatrix} y + \begin{bmatrix} s_{B^*} \\ s_{N^*} \end{bmatrix} = \begin{bmatrix} c_{B^*} + (\theta e_k)_{B^*} \\ c_{N^*} + (\theta e_k)_{N^*} \end{bmatrix}, \right. \\ \left. s_{B^*} = 0, s_{N^*} \geq 0 \right\}, \quad (2.5)$$

$$Ob_h(B^*, N^*) = \{ \gamma \mid A_{B^*} x_{B^*} = b + \gamma e_h, x_{B^*} \geq 0, x_{N^*} = 0 \}. \quad (2.6)$$

Note that $Oc_k(B^*, N^*)$ and $Ob_h(B^*, N^*)$ include the boundary values where the optimal partition of the perturbed problem differs from $\pi^* = (B^*, N^*)$.

We have defined so far three kinds of sensitivity analysis for linear programming. It is trivial that if x^* is a nondegenerate optimal basic solution to (P) , then the range of PSA using x^* is equal to that of BSA. However, if x^* is a degenerate optimal basic solution, the range of PSA may differ from that of BSA. The case will be discussed in Section 4. In addition, we know easily by definition that the range of PSA using a strictly complementary optimal solution is equal to that of OSA. Since the range of PSA using an optimal solution x^* is equal to the range of perturbations within which the partition $(\eta(x^*), \bar{\eta}(x^*))$ of indices of variables remains invariant, OSA can be regarded as a special case of PSA.

3. The Range of PSA Using Different Optimal Solutions

Let $z(\theta)$ denote the optimal value of the objective function of (LP_θ) . Also, for any optimal solution x^* to (P) , let $L_k(x^*)$ denote the range of θ such that $z(\theta) = z(0) + \theta x_k^*$. That is, $L_k(x^*) = \{ \theta \mid z(\theta) = z(0) + \theta x_k^* \}$. By the definition of PSA and Jansen *et al.*'s (1992) result, it is obvious that $Yc_k(x^*) = L_k(x^*)$ for an optimal basic solution x^* to (P) . In the next lemma, we show that Jansen *et al.*'s (1992) result holds for any optimal solution.

Lemma 3.1. *For an arbitrary optimal solution x^* to (P) , $Yc_k(x^*) = L_k(x^*)$.*

Proof. If $\theta \in Yc_k(x^*)$, then x^* is an optimal solution to (P_θ) . Hence,

$$z(\theta) = (c + \theta e_k)^T x^* = c^T x^* + \theta x_k^* = z(0) + \theta x_k^*.$$

Therefore, $\theta \in L_k(x^*)$. Conversely, if $\theta \in L_k(x^*)$, then

$$z(\theta) = z(0) + \theta x_k^* = (c + \theta e_k)^T x^*,$$

which implies that x^* is an optimal solution to (P_θ) . Therefore, $\theta \in Yc_k(x^*)$. \square

Theorem 3.1. *Let \bar{x} and \tilde{x} be two different optimal solutions to (P) . If $Yc_k(\bar{x}) \cap Yc_k(\tilde{x}) \neq \{0\}$, then $Yc_k(\bar{x}) = Yc_k(\tilde{x})$.*

Proof. Suppose that $Yc_k(\bar{x}) \cap Yc_k(\tilde{x}) \neq \{0\}$. Then, there exist θ_1 and θ_2 such that $\theta_1 \in Yc_k(\bar{x}) \cap Yc_k(\tilde{x}) - \{0\}$ and $\theta_2 \in Yc_k(\bar{x}) - \{0\}$. Since

$$z(\theta_1) = z(0) + \theta_1 \bar{x}_k = z(0) + \theta_1 \tilde{x}_k,$$

it follows that $\bar{x}_k = \tilde{x}_k$. Since

$$z(\theta_2) = z(0) + \theta_2 \bar{x}_k = z(0) + \theta_2 \tilde{x}_k = (c + \theta_2 e_k)^\top \tilde{x},$$

\tilde{x} is also an optimal solution to (P_{θ_2}) . Therefore, $\theta_2 \in Y_{c_k}(\tilde{x})$, which implies that $Y_{c_k}(\bar{x}) \subset Y_{c_k}(\tilde{x})$. (Note that for any optimal solution x^* , $0 \in Y_{c_k}(x^*)$.) By similar arguments, we can show that $Y_{c_k}(\tilde{x}) \subset Y_{c_k}(\bar{x})$. Therefore, $Y_{c_k}(\tilde{x}) = Y_{c_k}(\bar{x})$. \square

Jansen *et al.* (1992) presented a theorem, similar to Theorem 3.1, but considered only basic solutions. By Theorem 3.1, we come to a conclusion that if \bar{x} and \tilde{x} are two distinct optimal solutions to (P) , either 1 or 2, not both, is satisfied:

1. $Y_{c_k}(\bar{x}) = Y_{c_k}(\tilde{x})$.
2. $(Y_{c_k}(\bar{x}) - \{0\}) \cap (Y_{c_k}(\tilde{x}) - \{0\}) = \emptyset$.

Another important result about the relationship among the ranges of PSA using different optimal solutions is described in the next theorem.

Theorem 3.2. *Let x^* , x^1 , and x^2 be distinct optimal solutions to (P) such that $x^* = \lambda x^1 + (1 - \lambda)x^2$ for some λ with $0 < \lambda < 1$. Then, $Y_{c_k}(x^*) = Y_{c_k}(x^1) \cap Y_{c_k}(x^2)$.*

Proof. Let $\sigma^1 = \eta(x^1)$, $\sigma^2 = \eta(x^2)$, and $\sigma^* = \eta(x^*)$. Suppose that $x^* = \lambda x^1 + (1 - \lambda)x^2$ for $0 < \lambda < 1$. First, we claim that $Y_{c_k}(x^1) \cap Y_{c_k}(x^2) \subset Y_{c_k}(x^*)$. If $\theta \in Y_{c_k}(x^1) \cap Y_{c_k}(x^2)$, then both x^1 and x^2 are optimal solutions to (P_θ) . Since the following equation holds,

$$\begin{aligned} z(\theta) &= \lambda(c + \theta e_k)^\top x^1 + (1 - \lambda)(c + \theta e_k)^\top x^2 \\ &= (c + \theta e_k)^\top (\lambda x^1 + (1 - \lambda)x^2) \\ &= (c + \theta e_k)^\top x^*, \end{aligned}$$

x^* is also an optimal solution to (P_θ) . Consequently, $\theta \in Y_{c_k}(x^*)$.

Next, we show that $Y_{c_k}(x^*) \subset Y_{c_k}(x^1) \cap Y_{c_k}(x^2)$. For any $\theta \in Y_{c_k}(x^*)$, we get that

$$z(\theta) = (c + \theta e_k)^\top x^* = \lambda(c + \theta e_k)^\top x^1 + (1 - \lambda)(c + \theta e_k)^\top x^2. \quad (3.1)$$

Since x^* is an optimal solution to (P_θ) , the following inequalities are satisfied:

$$(c + \theta e_k)^\top x^1 \geq (c + \theta e_k)^\top x^*, \quad (c + \theta e_k)^\top x^2 \geq (c + \theta e_k)^\top x^*. \quad (3.2)$$

By Eq. (3.1) and inequalities (3.2), we find that $(c + \theta e_k)^\top x^1 = (c + \theta e_k)^\top x^2 = z(\theta)$. Consequently, $\theta \in Y_{c_k}(x^1)$ and $\theta \in Y_{c_k}(x^2)$. \square

Corollary 3.1. Let x^*, x^1, \dots, x^r be optimal solutions to (P) such that for some λ_i ($i = 1, \dots, r$)

$$x^* = \lambda_1 x^1 + \dots + \lambda_r x^r, \quad \sum_{i=1}^r \lambda_i = 1, \quad \lambda_i > 0, \quad \forall i.$$

Then,

$$Yc_k(x^*) = \bigcap_{1 \leq i \leq r} Yc_k(x^i).$$

Moreover, if $Yc_k(x^*) \neq \{0\}$, then $Yc_k(x^1) = \dots = Yc_k(x^r)$.

Proof. By applying Theorem 3.2 repeatedly, it is easily shown that $Yc_k(x^*) = \bigcap_{1 \leq i \leq r} Yc_k(x^i)$. Moreover, if $Yc_k(x^*) \neq \{0\}$, let $\theta \in Yc_k(x^*) - \{0\}$. Then, $\theta \in Yc_k(x^i)$ for each i . This, together with Theorem 3.1, implies that $Yc_k(x^1) = \dots = Yc_k(x^r)$. \square

If x^* in Corollary 3.1 is a strictly complementary solution, then we find that $Oc_k(B, N) = \bigcap_{1 \leq i \leq r} Yc_k(x^i)$ because $Oc_k(B, N) = Yc_k(x^*)$. That is, the range of OSA is the intersection of the ranges of PSA using optimal solutions whose convex combination leads to a strictly complementary solution.

Next, consider the case when b_h is perturbed. For an arbitrary matrix $E \in \mathfrak{R}^{m \times r}$ with r being a positive integer, let $\text{Pos}(E)$ denote a set of vectors as follows:

$$\text{Pos}(E) = \left\{ x \in \mathfrak{R}^m \mid x = \sum_{1 \leq j \leq r} \lambda_j E_{.j}, \lambda_j \geq 0 \right\},$$

where $E_{.j}$ is the j th column vector of E . In the next theorem, the relationship between the ranges of PSA using different optimal solutions is presented when b_h is changed.

Theorem 3.3. Let x^*, x^1, x^2 be optimal solutions to (P) such that $x^* = \lambda x^1 + (1 - \lambda)x^2$ for λ with $0 < \lambda < 1$. Then, $Yb_h(x^i) \subset Yb_h(x^*)$ for $i = 1, 2$.

Proof. Let $\sigma = \eta(x^*)$ and $\sigma^i = \eta(x^i)$ for $i = 1, 2$. Since $\sigma^i \subset \sigma$ by the assumption of the theorem, $\text{Pos}(A_{\sigma^i}) \subset \text{Pos}(A_{\sigma})$. This, together with Eq. (2.4), implies that $Yb_h(x^i) \subset Yb_h(x^*)$ for each $i = 1, 2$. \square

From the above theorem, we may conjecture that $Yb_h(x^*) = \bigcup_{1 \leq i \leq r} Yb_h(x^i)$ where x^* and x^i are defined in the same way with Corollary 3.1. However, from

the following example (LP1), we find that, in general, $Yb_h(x^*)$ is not equal to $\bigcup_{1 \leq i \leq r} Yb_h(x^i)$:

$$\begin{aligned}
 & \min && -x_1 - x_2 \\
 & \text{s.t.} && x_1 + x_2 + x_3 = 1 \\
 (P1) : & && x_1 + \quad \quad + x_4 = 1 \\
 & && \quad \quad x_2 \quad \quad + x_5 = 1, \\
 & && x_j \geq 0, \quad j = 1, \dots, 5, \\
 \\
 & \max && y_1 + y_2 + y_3 \\
 & \text{s.t.} && y_1 + y_2 \quad -s_1 = 1 \\
 (D1) : & && y_1 \quad + y_3 \quad -s_2 = 1 \\
 & && y_1 \quad \quad \quad -s_3 = 0 \\
 & && \quad y_2 \quad \quad \quad -s_4 = 0 \\
 & && \quad \quad y_3 \quad \quad \quad -s_5 = 0, \\
 & && s_j \geq 0, \quad j = 1, \dots, 5,
 \end{aligned}$$

The problem (P1) has two optimal basic solutions, x^1 and x^2 :

$$x^1 = (1, 0, 0, 0, 1)^T, \quad x^2 = (0, 1, 0, 1, 0)^T.$$

When b_1 is changed, the ranges of PSA using x^1 and x^2 are both $[0, 0]$. However, the range of PSA using an optimal nonbasic solution x^* is $[-1, 1]$ where $x^* = (x^1 + x^2)/2$.

In addition, if x^* is a strictly complementary solution in Theorem 3.3, then we find that $Ob_h(B^*, N^*) \supset \bigcup_{1 \leq i \leq r} Yb_h(x^i)$ because $Ob_h(B^*, N^*) = Yb_h(x^*)$.

On the other hand, under what condition does the range of PSA on c_k include a nonzero value? In the rest of this section, we present a necessary and sufficient condition that c_k can be perturbed while an optimal solution to (P) remains optimal to the perturbed problem. Let \mathbf{P}^* denote the set of all optimal solutions to (P).

Theorem 3.4. *Let x^* be an optimal solution to (P). Then, $\theta \in Yc_k(x^*)$ for some $\theta > 0$ if and only if $x_k^* \leq x_k$ for all $x \in \mathbf{P}^*$.*

Proof. First, we will show that the “only if” part holds. Suppose that $\theta \in Yc_k(x^*)$ for some $\theta > 0$. In addition, suppose that $x_k < x_k^*$ for some $x \in \mathbf{P}^*$. Then,

$$(c + \theta e_k)^T x = c^T x + \theta x_k < c^T x^* + \theta x_k^* = (c + \theta e_k)^T x^*.$$

This contradicts the assertion that x^* is an optimal solution to (P_θ) . Therefore, $x_k^* \leq x_k$ for all $x \in \mathbf{P}^*$.

Next, we will show that the “if” part holds. Let $\sigma = \eta(x^*)$ and $\bar{\sigma} = \bar{\eta}(x^*)$. Also, let $\pi^* = (B^*, N^*)$ be the optimal partition of (LP). Note that $B^* \supset \sigma$ and $\bar{\sigma} = (B^* - \sigma) \cup N^*$.

(i) In the case $k \in N^*$: If (y^*, s^*) is an optimal solution to (D) , then $(y^*, s^* + \theta e_k)$ with $\theta > 0$ is a feasible solution to the following linear equation system:

$$\begin{aligned} A_{\sigma}^T y &= c_{\sigma} + (\theta e_k)_{\sigma}, & s_{\sigma} &= 0, \\ A_{B^*-\sigma}^T y + s_{B^*-\sigma} &= c_{B^*-\sigma} + (\theta e_k)_{B^*-\sigma}, & s_{B^*-\sigma} &\geq 0, \\ A_{N^*}^T y + s_{N^*} &= c_{N^*} + (\theta e_k)_{N^*}, & s_{N^*} &\geq 0. \end{aligned} \quad (3.3)$$

Since $(x^*, y^*, s^* + \theta e_k)$ is an optimal solution to (LP_{θ}) , we get that $[0, \theta] \subset Yc_k(x^*)$ where $\theta > 0$.

(ii) In the case $k \in B^*$: Consider the following linear programming:

$$\begin{aligned} \min \quad & (e_k)_{B^*}^T x_{B^*} & \max \quad & b^T y \\ (P') : \quad & \text{s.t. } A_{B^*} x_{B^*} = b & (D') : \quad & \text{s.t. } A_{\sigma}^T y + s_{\sigma} = (e_k)_{\sigma} \\ & x_{B^*} \geq 0. & & A_{B^*-\sigma}^T y + s_{B^*-\sigma} = (e_k)_{B^*-\sigma} \\ & & & s_{\sigma} \geq 0, s_{B^*-\sigma} \geq 0. \end{aligned} \quad (3.4)$$

By the assumption, x^* is an optimal solution to (P') , and the optimal value of the object function of (P') is x_k^* . This implies that (D') , the dual problem of (P') , has at least one optimal solution. Let $(\Delta y, \Delta s_{B^*})$ be an optimal solution to (D') which satisfies the following:

$$\begin{aligned} A_{\sigma}^T \Delta y &= (e_k)_{\sigma}, \\ A_{B^*-\sigma}^T \Delta y + \Delta s_{B^*-\sigma} &= (e_k)_{B^*-\sigma}, \\ \Delta s_{\sigma} &= 0, \quad \Delta s_{B^*-\sigma} \geq 0. \end{aligned} \quad (3.5)$$

In addition, let $\Delta s_{N^*} = (e_k)_{N^*} - A_{N^*}^T \Delta y$ and let (y^*, s^*) be a strictly complementary solution to (D) . We set $\hat{\theta}$ as the following:

$$\hat{\theta} = \min_{j \in N^*} \left\{ -\frac{s_j^*}{\Delta s_j} \mid \Delta s_j < 0 \right\}.$$

Note that $\hat{\theta}$ is positive. Let $\bar{\theta}$ be a real number such that $0 < \bar{\theta} \leq \hat{\theta}$. Then, we get a solution (\tilde{y}, \tilde{s}) that satisfies the linear systems (3.3) where

$$\begin{aligned} \tilde{y} &= y^* + \bar{\theta} \Delta y, \\ \tilde{s} &= s^* + \bar{\theta} (\Delta s_{B^*}^T, \Delta s_{N^*}^T)^T. \end{aligned}$$

Since $(x^*, \tilde{y}, \tilde{s})$ is an optimal solution to $(LP_{\bar{\theta}})$, we find that $\bar{\theta} \in Yc_k(x^*)$. \square

Similarly, we obtain a sufficient and necessary condition under which the range of PSA on c_k includes a negative value as follows:

Theorem 3.5. *Let x^* be an optimal solution to (P) . Then, $\theta \in Yc_k(x^*)$ for some $\theta < 0$ if and only if $x_k^* \geq x_k$ for all $x \in \mathbf{P}^*$.*

Proof. First, we will show that the ‘‘only if’’ part holds. Suppose that $\theta \in Yc_k(x^*)$ for some $\theta < 0$. In addition, suppose that $x_k > x_k^*$ for some $x \in \mathbf{P}^*$. Then,

$$(c + \theta e_k)^T x = c^T x + \theta x_k < c^T x^* + \theta x_k^* = (c + \theta e_k)^T x^*.$$

This contradicts the assertion that x^* is an optimal solution to (P_θ) . Therefore, $x_k^* \geq x_k$ for all $x \in \mathbf{P}^*$.

Next, we show that the “if” part holds. Let $\sigma = \eta(x^*)$ and $\bar{\sigma} = \bar{\eta}(x^*)$. Also, let $\pi^* = (B^*, N^*)$ be the optimal partition to (LP) .

- (i) In the case $k \in N^*$: Let (y^*, s^*) be a strictly complementary optimal solution to (D) . (Note that $s_k^* > 0$.) Then, $(y^*, s^* + \underline{\theta}e_k)$ satisfies the linear systems (3.3) where $-s_k^* \leq \underline{\theta} < 0$. Consequently, $[\underline{\theta}, 0] \in Yc_k(x^*)$ for any $-s_k^* \leq \underline{\theta} \leq 0$.
- (ii) In the case $k \in B^*$: We easily show that $\underline{\theta} \in Yc_k(x^*)$ for some $\underline{\theta}$ by replacing vector e_k in (10) and (11) with vector $-e_k$ and applying the same technique in Theorem 3.4. □

By Theorems 3.4 and 3.5, we know that the range of PSA using x^* on c_k includes both a positive and a negative value if and only if for any optimal solution x the k th element x_k has the same value. In addition, we arrive at another interesting result about the range of OSA as follows:

Corollary 3.2. *Let $\pi^* = (B^*, N^*)$ be the optimal partition to (LP) . Then, $Oc_k(B^*, N^*) \neq [0, 0]$ if and only if $x_k^* = \alpha$ for all $x^* \in \mathbf{P}^*$, where α is a non-negative constant.*

Proof. First, suppose that $Oc_k(B^*, N^*) \neq [0, 0]$. Let \bar{x} be a strictly complementary optimal solution. Since $Yc_k(\bar{x}) = Oc_k(B^*, N^*)$, $Yc_k(\bar{x}) \neq [0, 0]$. If \bar{x} is a unique optimal solution to (P) , the corollary trivially holds. Otherwise, let x^1 be an arbitrary optimal solution to (P) such that $x^1 \neq \bar{x}$. Then, there exists an optimal solution x^2 such that

$$\bar{x} = \lambda x^1 + (1 - \lambda)x^2, \quad \text{for some } \lambda > 0. \tag{3.6}$$

By Theorems 3.4 and 3.5, $Yc_k(\bar{x}) \neq [0, 0]$ implies that $\bar{x}_k \leq x_k^i$ or $\bar{x}_k \geq x_k^i$ for $i = 1, 2$. This, together with equation (3.6), implies that $\bar{x}_k = x_k^1 = x_k^2$. Since x^1 is chosen arbitrarily, $x_k^* = \alpha$ for all $x^* \in \mathbf{P}^*$ where α is a nonnegative constant.

Next, we will show that the reverse holds. Suppose that $x_k^* = \alpha$ for all $x^* \in \mathbf{P}^*$. Then, by Theorems 3.4 and 3.5, there exist $\underline{\theta}$ and $\bar{\theta}$ such that $[\underline{\theta}, \bar{\theta}] \subset Oc_k(B^*, N^*)$, $\underline{\theta} < 0$, and $\bar{\theta} > 0$. □

4. The Relationship between PSA and BSA Under Degeneracy

In this section, we discuss the relationship between PSA and BSA by comparing PSA with BSA under degeneracy. Let x^* be an optimal basic solution to (LP) . If x^* is degenerate, there can be more than one optimal basis associated with x^* . BSA using each optimal basis may produce a different range of perturbation θ . For

example, consider the following linear programming problem (LP2):

$$\begin{aligned}
 & \min && 4x_1 + 5x_2 \\
 & \text{s.t.} && 4x_1 + 3x_2 - x_3 &= 12 \\
 (P2) : &&& 2x_1 + 5x_2 & -x_4 &= 10 \\
 &&& 3x_1 + 4x_2 & & -x_5 = 11, \\
 &&& x_j \geq 0, & j = 1, \dots, 5, \\
 \\
 & \max && 12y_1 + 10y_2 + 11y_3 \\
 & \text{s.t.} && 4y_1 + 2y_2 + 3y_3 + s_1 &= 4 \\
 (D2) : &&& 3y_1 + 5y_2 + 4y_3 & +s_2 &= 5 \\
 &&& -y_1 & & +s_3 &= 0 \\
 &&& & -y_2 & & +s_4 &= 0 \\
 &&& & & -y_3 & & +s_5 = 0, \\
 &&& s_j \geq 0, & j = 1, \dots, 5.
 \end{aligned}$$

The unique optimal solution x^* to (P2) is $(15/7, 8/7, 0, 0, 0)^T$, that is a degenerate basic solution. There are three primal-optimal bases, A_{B^1} , A_{B^2} , A_{B^3} where $B^1 = \{1, 2, 3\}$, $B^2 = \{1, 2, 4\}$, and $B^3 = \{1, 2, 5\}$. Both A_{B^2} and A_{B^3} are optimal bases, but A_{B^1} is a primal-optimal basis, not an optimal basis. When c_2 is changed, the range of BSA using A_{B^2} and A_{B^3} are $[-2, 1/3]$ and $[-2, 5]$, respectively. On the other hand, the range of PSA using x^* is $[-2, 5]$.

Ward *et al.* (1990) showed that when a cost coefficient is changed, the range of θ within which an optimal basic solution x^* remains optimal to (P_θ) is the union of the ranges of sensitivity analysis using all primal-optimal bases associated with x^* . Since $Y_{c_k}(x^*)$ is the range within which x^* remains optimal to (P_θ) , we obtain the following theorem:

Theorem 4.1. *Let x^* be an optimal degenerate basic solution. Let B^1, B^2, \dots, B^r be the index set of basic variables of all the primal-optimal bases associated with x^* . Then,*

$$Y_{c_k}(x^*) = \bigcup_{1 \leq i \leq r} T_{c_k}(A_{B^i}).$$

Proof. Since the range of PSA using x^* on c_k is equal to the range of θ within which x^* remains optimal to (P_θ) , we get that $Y_{c_k}(x^*) = \bigcup_{1 \leq i \leq r} T_{c_k}(A_{B^i})$. \square

On the other hand, the relationship between PSA and BSA when a right-hand side b_h is changed differs from the case when a cost coefficient is changed. The following theorem implies that the range of PSA using x^* on b_h is included in the range of BSA using any optimal basis associated with x^* .

Theorem 4.2. Let x^* be an optimal degenerate basic solution to (P) . Let B^1, B^2, \dots, B^r be the index sets of basic variables of all the optimal bases associated with x^* . Then,

$$Yb_h(x^*) \subset \bigcap_{1 \leq i \leq r} Tb_h(A_{B^i})$$

Moreover, if $Yb_h(x^*) \neq \{0\}$, then $Yb_h(x^*) = Tb_h(A_{B^i})$ for $1 \leq i \leq r$.

Proof. Let $\sigma = \eta(x^*)$. For each B^i , $\text{Pos}(A_\sigma) \subset \text{Pos}(A_{B^i})$ because $\sigma \subset B^i$. This, together with Eqs. (2.2) and (2.4), implies that $Yb_h(x^*) \subset Tb_h(A_{B^i})$ for each i . Therefore, $Yb_h(x^*) \subset \bigcap_{1 \leq i \leq r} Tb_h(A_{B^i})$.

Suppose that $Yb_h(x^*)$ includes any nonzero value. Then, $b \in \text{Pos}(A_\sigma)$ and $e_h \in \text{Pos}(A_\sigma)$. For an arbitrary optimal basis B^i , $\sigma \subset B^i$ and each column in A_σ is linearly independent from all columns in $A_{B^i-\sigma}$. Therefore, the range of γ such that

$$[A_\sigma, A_{B^i-\sigma}] \begin{bmatrix} x_\sigma \\ x_{B^i-\sigma} \end{bmatrix} = b + \gamma e_h, \quad x_\sigma \geq 0, \quad x_{B^i-\sigma} \geq 0$$

is the same with the range of γ such that

$$A_\sigma x_\sigma = b + \gamma e_h, \quad x_\sigma \geq 0, \quad x_{B^i-\sigma} = 0.$$

This implies that $Tb_h(A_{B^i}) = Yb_h(x^*)$. □

That is, if $Yb_h(x^*)$ includes any nonzero value, we know that $Yb_h(x^*) = \bigcap_{1 \leq i \leq r} Tb_h(A_{B^i})$, which is similar to Theorem 4.1. However, when $Yb_h(x^*)$ includes no nonzero value, i.e., $Yb_h(x^*) = \{0\}$, $Yb_h(x^*)$ may not be equal to $\bigcap_{1 \leq i \leq r} Tb_h(A_{B^i})$, which is illustrated by the following linear programming (LP3):

$$(P3) : \begin{array}{ll} \min & x_1 \\ \text{s.t.} & x_1 + x_2 + x_3 = 1 \\ & x_1 + 2x_2 + 2x_3 = 1, \\ & x_j \geq 0, \quad j = 1, 2, 3, \end{array}$$

$$(D3) : \begin{array}{llll} \max & y_1 + y_2 & & \\ \text{s.t.} & y_1 + y_2 - s_1 & & = 1 \\ & y_1 + 2y_2 & -s_2 & = 0 \\ & y_1 + 2y_2 & & -s_3 = 0, \\ & s_j \geq 0, & j = 1, 2, 3. \end{array}$$

The unique optimal solution to $(P3)$ is $x^* = (1, 0, 0)^T$, and there are two optimal bases associated with x^* :

$$A_{B^1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad A_{B^2} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},$$

Table 1. Comparison between the three sensitivity analysis methods.

Features	BSA	PSA	OSA
Prerequisite information	An optimal basis	An arbitrary optimal solution	The optimal partition
Range found	The range where an optimal basis remains optimal	The range where the induced partition of an optimal solution remains invariant	The range where the optimal partition remains invariant
Usage	Usually used for simplex method, and can be applied only to an optimal basic solution	Can be used for interior-point methods after finding any optimal solution	Can be used for interior-point methods after finding the optimal partition
Relationship with other methods	$Yc_k(x^*) = \bigcup_i Tc_k(A_{B^i}), Yb_h(x^*) \subset \bigcap_j Tb_h(A_{B^j})$, where x^* is an optimal basic solution, and B^i and B^j are a primal-optimal and an optimal basis associated with x^* , respectively		A special case of PSA

where $B^1 = \{1, 2\}$ and $B^2 = \{1, 3\}$. When b_1 is perturbed, the ranges of BSA using A_{B^1} and A_{B^2} are $Tb_1(A_{B^1}) = [-1/2, 0]$ and $Tb_1(A_{B^2}) = [-1/2, 0]$, respectively. However, the range of PSA using x^* is $Yb_h(x^*) = [0, 0]$ by the following equation:

$$Yb_1(x^*) = \left\{ \gamma \left| \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_1 = \begin{bmatrix} 1 + \gamma \\ 1 \end{bmatrix}, x_1 \geq 0, x_2 = x_3 = 0 \right. \right\}$$

Consequently, we find that $Yb_1(x^*) \neq Tb_1(A_{B^1}) \cap Tb_1(A_{B^2})$.

The features of the three sensitivity analysis methods are summarized in Table 1.

5. Concluding Remarks

In this paper, we study the properties of PSA and its relationship with two other sensitivity analysis methods, BSA and OSA. The main advantage of PSA is that it can be performed with any optimal solution which is a nonbasic or basic solution. PSA finds the range within which there exists an optimal solution to the perturbed problem whose induced partition is equal to the induced partition of a given optimal solution. PSA focuses only on the induced partition of primal-optimal solutions. That is why the properties of PSA on a cost coefficient differs from those of PSA on a right-hand side.

We presented some properties of PSA that are useful for comparing PSA with the other two sensitivity analysis methods. When a cost coefficient is perturbed, the range of PSA is equal to the interval where a given optimal solution remains optimal to the perturbed problem. On the other hand, when a right-hand side is changed, the range of PSA finds the interval where the induced partition of a given optimal solution remains the induced partition of some optimal solution to the perturbed problem. Another important property of PSA on a cost coefficient is

that the range of PSA using an optimal nonbasic solution is the intersection of the ranges of PSA using optimal basic solutions whose convex combination leads to the optimal nonbasic solution.

Finally, further studies will be needed, which deal with the computational performance and numerical experience of sensitivity analysis methods. Given an optimal basis, BSA is obviously the most efficient where the computational time concerned. However, most codes using interior-point methods often produce an optimal nonbasic solution, and in this case PSA is expected to be a good alternative because PSA can be applied without obtaining an optimal basis or the optimal partition, which may require much computational time if a problem is ill-conditioned.

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