Advances in Difference Equations
a SpringerOpen Journal

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Positive solutions for a class of fractional 3-point boundary value problems at resonance

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Abstract

In this paper, we study the nonlocal fractional differential equation:

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = 0, & u(1) = \eta u(\xi), \end{cases}$$

where $1 < \alpha < 2$, $0 < \xi < 1$, $\eta \xi^{\alpha-1} = 1$, D_{0+}^{α} is the standard Riemann-Liouville derivative, $f : [0,1] \times [0,+\infty) \to \mathbb{R}$ is continuous. The existence and uniqueness of positive solutions are obtained by means of the fixed point index theory and iterative technique.

Keywords: fractional differential equation; positive solution; resonance; fixed point index

1 Introduction

In this paper, we consider the following fractional differential equation:

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = 0, & u(1) = \eta u(\xi), \end{cases} \tag{1.1}$$

where $1 < \alpha < 2$, $0 < \xi < 1$, $\eta \xi^{\alpha-1} = 1$, D_{0+}^{α} is the standard Riemann-Liouville derivative, $f : [0,1] \times [0,+\infty) \to \mathbb{R}$ is continuous. Problem (1.1) happens to be at resonance, since $\lambda = 0$ is an eigenvalue of the linear problem

$$\begin{cases} -D_{0+}^{\alpha} u = \lambda u, & 0 < t < 1, \\ u(0) = 0, & u(1) = \eta u(\xi), \end{cases} \tag{1.2}$$

and $ct^{\alpha-1}$, $c \in \mathbb{R}$, is the corresponding eigenfunction.

Fractional differential equations occur frequently in various fields such as physics, chemistry, engineering and control of dynamical systems, etc. During the last few decades, many papers and books on fractional calculus and fractional differential equations have appeared (see [1–22] and the references therein).

When $0 < \eta\xi^{\alpha-1} < 1$, problem (1.1) is non-resonant. In [9], the author studied the existence of positive solutions for the non-resonant case by means of the fixed point index theory under sublinear conditions.

In [18], the authors investigated the existence and multiplicity results of positive solutions by using of the fixed point theorem for the fractional differential equation given by

$$
\begin{cases}
D_{0+}^{\alpha} u(t) = f(t, u(t)), & 0 < t < 1, \\
u(0) = 0, & D_{0+}^{\beta} u(1) = a D_{0+}^{\beta} u(\xi),
\end{cases}
\tag{1.3}
$$

where $1 < \alpha \le 2$, $0 \le \beta \le 1$, $0 < \xi < 1$, $0 \le a \le 1$ with $a\xi^{\alpha-\beta-2} < 1 - \beta$, $0 \le \alpha - \beta - 1$.

Recently, there are some papers dealing with the existence of solutions of fractional boundary value problem at resonance by using the coincidence degree theory due to Mawhin (see [19–22]). In [22], the authors investigated the following fractional three-point boundary value problem (BVP for short) at resonance:

$$
\begin{cases}
D_{0+}^{\alpha} u(t) + f(t, u(t), D_{0+}^{\alpha-1} u(t)) = 0, & 0 < t < 1, \\
u(0) = 0, & u(1) = \eta u(\xi),
\end{cases}
\tag{1.4}
$$

where $1 < \alpha \le 2$, $0 < \xi < 1$, $\eta\xi^{\alpha-1} = 1$, D_{0+}^{α} is the standard Riemann-Liouville derivative, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. By using the coincidence degree theory, the existence of solutions for BVP (1.4) are obtained under certain growth conditions.

To the best of our knowledge, there are only very few papers dealing with the existence of positive solutions for resonant boundary value problems since the corresponding linear operator is non-reversible. For the case that α is an integer, some work has been done dealing with the existence of positive solutions for resonant boundary value problems by using Leggett-Williams norm-type theorem for coincidence (see [23–25]). Webb [26] established existence of positive solutions for second order boundary value problems at resonance by considering equivalent non-resonant perturbed problems with the same boundary conditions.

Inspired by the work mentioned above, in this paper we aim to establish the existence of positive solutions for resonant problem (1.1). The paper is organized as follows. Firstly, we reduce non-perturbed boundary value problems at resonance to equivalent non-resonant perturbed problems with the same boundary conditions. Then we derive the corresponding Green's function and argue its properties. Finally, the existence and uniqueness results of positive solutions are obtained by using of the fixed point index and iterative technique.

2 Basic definitions and preliminaries

In this section, we present some preliminaries and lemmas. The definitions and properties of fractional derivative can be found in the literature [1–22].

Definition 2.1 The fractional integral of order $\alpha > 0$ of a function $u : (0, +\infty) \to R$ is given by

$$
I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) \, ds
$$

provided that the right-hand side is point-wise defined on $(0, +\infty)$.

Definition 2.2 The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $u : (0, +\infty) \to R$ is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} u(s) \, ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right-hand side is point-wise defined on $(0, +\infty)$.

Denote

$$g(t) = \frac{\alpha - 2}{\Gamma(\alpha - 1)} + \sum_{k=1}^{+\infty} \frac{t^k}{\Gamma((k+1)\alpha - 2)}. \tag{2.1}$$

It is easy to check that $g'(t) > 0$ on $(0, +\infty)$, and

$$g(0) = \frac{\alpha - 2}{\Gamma(\alpha - 1)} < 0, \qquad \lim_{t \to +\infty} g(t) = +\infty.$$

Therefore, there exists a unique $b^* > 0$ such that

$$g(b^*) = 0. \tag{2.2}$$

For the convenience in presentation, we here list the assumptions to be used throughout the paper.

(H$_1$) $b \in (0, b^*]$ is a constant.
(H$_2$) $f : [0,1] \times [0, +\infty) \to \mathbb{R}$ is continuous and

$$f(t,x) + bx \geq 0. \tag{2.3}$$

Set

$$G_b(t) = t^{\alpha-1} E_{\alpha,\alpha}(bt^{\alpha}), \tag{2.4}$$

where

$$E_{\alpha,\alpha}(x) = \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma((k+1)\alpha)} \tag{2.5}$$

is the Mittag-Leffler function (see [1, 2]).

Next we consider the following boundary value problem:

$$\begin{cases} -D_{0+}^{\alpha} u(t) + bu(t) = f(t, u(t)) + bu(t), & 0 < t < 1, \\ u(0) = 0, \qquad u(1) = \eta u(\xi). \end{cases} \tag{2.6}$$

It is clear that (1.1) is equivalent to (2.6).

Set

$$K_0(t,s) = \frac{1}{G_b(1)} \begin{cases} G_b(t)G_b(1-s), & 0 \le t \le s \le 1, \\ G_b(t)G_b(1-s) - G_b(t-s)G_b(1), & 0 \le s \le t \le 1; \end{cases} \tag{2.7}$$

$$q(s) = \frac{\eta K_0(\xi,s)}{G_b(1) - \eta G_b(\xi)}; \tag{2.8}$$

$$K(t,s) = K_0(t,s) + G_b(t)q(s). \tag{2.9}$$

Lemma 2.1 *Suppose that* (H_1) *holds, and* $y \in L[0,1]$. *Then the unique solution of the problem*

$$\begin{cases} -D_{0+}^{\alpha}u(t) + bu(t) = y(t), & 0 < t < 1, \\ u(0) = 0, & u(1) = \eta u(\xi), \end{cases} \tag{2.10}$$

is

$$u(t) = \int_0^1 K(t,s)y(s)\,ds.$$

Proof By [1, 2], we know that the solution of (2.10) can be expressed by

$$u(t) = -\int_0^t G_b(t-s)y(s)\,ds + c_1 G_b(t) + c_2 G_b'(t).$$

By $u(0) = 0$, we have $c_2 = 0$.

On the other hand, we have

$$u(1) = -\int_0^1 G_b(1-s)y(s)\,ds + c_1 G_b(1), \tag{2.11}$$

$$u(\xi) = -\int_0^{\xi} G_b(\xi-s)y(s)\,ds + c_1 G_b(\xi). \tag{2.12}$$

Noting that $\eta\xi^{\alpha-1} = 1$, and $0 < \xi < 1$, we have

$$G_b(1) - \eta G_b(\xi) = \sum_{k=0}^{+\infty} \frac{b^k[1 - \eta\xi^{(k+1)\alpha-1}]}{\Gamma((k+1)\alpha)} = \sum_{k=0}^{+\infty} \frac{b^k(1-\xi^{k\alpha})}{\Gamma((k+1)\alpha)} > 0. \tag{2.13}$$

Equations (2.11) and (2.12) yield

$$c_1 = \frac{\int_0^1 G_b(1-s)y(s)\,ds - \eta\int_0^{\xi} G_b(\xi-s)y(s)\,ds}{G_b(1) - \eta G_b(\xi)}.$$

Therefore, the solution of (2.10) is

$$u(t) = -\int_0^t G_b(t-s)y(s)\,ds + \frac{\int_0^1 G_b(1-s)y(s)\,ds - \eta\int_0^{\xi} G_b(\xi-s)y(s)\,ds}{G_b(1) - \eta G_b(\xi)}G_b(t)$$

$$= \frac{\int_0^1 G_b(t)G_b(1-s)y(s)\,ds - \int_0^t G_b(1)G_b(t-s)y(s)\,ds}{G_b(1)}$$

$$- \frac{\int_0^1 G_b(1-s)y(s)\,ds}{G_b(1)}G_b(t) + \frac{\int_0^1 G_b(1-s)y(s)\,ds}{G_b(1)-\eta G_b(\xi)}G_b(t)$$

$$- \frac{\eta\int_0^\xi G_b(\xi-s)y(s)\,ds}{G_b(1)-\eta G_b(\xi)}G_b(t)$$

$$= \int_0^1 K_0(t,s)y(s)\,ds + \frac{\eta\int_0^1 G_b(\xi)G_b(1-s)y(s)\,ds}{G_b(1)[G_b(1)-\eta G_b(\xi)]}G_b(t)$$

$$- \frac{\eta\int_0^\xi G_b(1)G_b(\xi-s)y(s)\,ds}{G_b(1)[G_b(1)-\eta G_b(\xi)]}G_b(t)$$

$$= \int_0^1 K(t,s)y(s)\,ds.$$

This completes the proof. □

Lemma 2.2 *Suppose that* (H_1) *holds. The function* $K(t,s)$ *has the following properties*:

(1) $K(t,s) > 0, \forall t,s \in (0,1)$;

(2) $\omega_2(s)t^{\alpha-1} \le K(t,s) \le \omega_1(s)t^{\alpha-1}, \forall t,s \in [0,1]$, *where*

$$\omega_1(s) = G_b(1-s) + G_b(1)q(s), \qquad \omega_2(s) = \frac{q(s)}{\Gamma(\alpha)}. \tag{2.14}$$

Proof It is clear that we just need to prove that (2) holds.

By (2.4), we can get

$$\frac{t^{\alpha-1}}{\Gamma(\alpha)} \le G_b(t) = t^{\alpha-1}\sum_{k=0}^{+\infty}\frac{b^k t^{\alpha k}}{\Gamma((k+1)\alpha)} \le t^{\alpha-1}G_b(1), \quad t \in [0,1], \tag{2.15}$$

$$G_b'(t) = \sum_{k=0}^{+\infty}\frac{b^k t^{(k+1)\alpha-2}}{\Gamma((k+1)\alpha-1)} > 0, \quad t \in (0,1], \tag{2.16}$$

and

$$G_b''(t) = t^{\alpha-3}\left[\frac{\alpha-2}{\Gamma(\alpha-1)} + \sum_{k=1}^{+\infty}\frac{b^k t^{k\alpha}}{\Gamma((k+1)\alpha-2)}\right]$$

$$= t^{\alpha-3}g(bt^\alpha) < t^{\alpha-3}g(b) \le t^{\alpha-3}g(b^*) = 0, \quad t \in (0,1), \tag{2.17}$$

which implies $G_b(t)$ is strictly increasing on $[0,1]$, and $G_b'(t)$ is strictly decreasing on $(0,1]$.

By (2.15), we have

$$K(t,s) = K_0(t,s) + G_b(t)q(s) \le \frac{G_b(t)G_b(1-s)}{G_b(1)} + G_b(t)q(s)$$

$$\le G_b(1-s)t^{\alpha-1} + t^{\alpha-1}G_b(1)q(s) = \omega_1(s)t^{\alpha-1}. \tag{2.18}$$

On the other hand, when $0 < t \le s < 1$, noticing $G_b(0) = 0$, and the monotonicity of $G_b(t)$, it is clear that

$$G_b(t)G_b(1-s) > 0. \tag{2.19}$$

When $0 < s < t < 1$, we have

$$
\frac{\partial}{\partial s}\big[G_b(t)G_b(1-s) - G_b(t-s)G_b(1)\big]
$$

$$
= G'_b(t-s)G_b(1) - G_b(t)G'_b(1-s)
$$

$$
\geq \big[G_b(1) - G_b(t)\big]G'_b(1-s). \tag{2.20}
$$

Integrating (2.20) with respect to s, we obtain

$$
G_b(t)G_b(1-s) - G_b(t-s)G_b(1)
$$

$$
\geq \int_0^s \big[G_b(1) - G_b(t)\big]G'_b(1-\tau)\,d\tau
$$

$$
= \big[G_b(1) - G_b(t)\big]\big[G_b(1) - G_b(1-s)\big] > 0. \tag{2.21}
$$

By (2.7), (2.19), and (2.21), we get $K_0(t,s) > 0, \forall t, s \in (0,1)$. Then

$$
K(t,s) = K_0(t,s) + G_b(t)q(s) \geq G_b(t)q(s)
$$

$$
\geq \frac{t^{\alpha-1}}{\Gamma(\alpha)}q(s) = \omega_2(s)t^{\alpha-1} > 0, \quad \forall t, s \in (0,1).
$$

This completes the proof. □

Let $E = C[0,1]$ be endowed with the maximum norm $\|u\| = \max_{0 \leq t \leq 1}|u(t)|$, θ is the zero element of E, $B_r = \{u \in E : \|u\| < r\}$. Define a cone P by

$$
P = \big\{u \in E : u(t) \geq 0, t \in [0,1]\big\}.
$$

Let

$$
Au(t) = \int_0^1 K(t,s)\big[f\big(s, u(s)\big) + bu(s)\big]\,ds. \tag{2.22}
$$

$$
Tu(t) = \int_0^1 K(t,s)u(s)\,ds. \tag{2.23}
$$

By means of Lemma 2.2 and the Arzela-Ascoli theorem, we can get $A : P \to P$ is completely continuous, $T : P \to P$ is completely continuous linear operator. By virtue of the Krein-Rutmann theorem and Lemma 2.2, we have the spectral radius $r(T) > 0$ and T has a positive eigenfunction corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$. Since $\lambda = 0$ is the eigenvalue of the linear problems (1.2), and $t^{\alpha-1}$ is the corresponding eigenfunction, we have the following lemma.

Lemma 2.3 *Suppose that* (H_1) *holds, then the first eigenvalue of T is $\lambda_1 = b$, and $\varphi_1(t) = t^{\alpha-1}$ is the positive eigenfunction corresponding to λ_1, that is, $\varphi_1 = bT\varphi_1$.*

Lemma 2.4 ([27]) *Let P be a cone in a Banach space E, and Ω be a bounded open set in E. Suppose that $A: \overline{\Omega} \cap P \to P$ is a completely continuous operator. If there exists $u_0 \in P$ with*

$u_0 \neq \theta$ *such that*

$$u - Au \neq \lambda u_0, \quad \forall \lambda \geq 0, x \in \partial\Omega \cap P,$$

then $i(A, \Omega \cap P, P) = 0$.

Lemma 2.5 ([27]) *Let P be a cone in a Banach space E, and Ω be a bounded open set in E. Suppose that $A: \overline{\Omega} \cap P \to P$ is a completely continuous operator. If*

$$Au \neq \lambda u, \quad \forall \lambda \geq 1, u \in \partial\Omega \cap P,$$

then $i(A, \Omega \cap P, P) = 1$.

3 The uniqueness result

Theorem 3.1 *Assume that there exists $\lambda \in (0, b)$ such that*

$$\left| f(t, u) + bu - f(t, v) - bv \right| \leq \lambda |u - v|, \quad for\ t \in [0, 1], u, v \in [0, \infty),$$

then (1.1) *has a unique nonnegative solution.*

Proof Firstly, we will prove A has fixed point in P.

Set

$$Q = \left\{ u \in P : \exists l_1, l_2 > 0 \text{ such that } l_1 t^{\alpha-1} \leq u(t) \leq l_2 t^{\alpha-1} \right\}. \tag{3.1}$$

For any $u \in P \setminus \{\theta\}$, let

$$l_i(u) = \int_0^1 \omega_i(s) u(s)\, ds, \quad i = 1, 2. \tag{3.2}$$

By Lemma 2.2, it is obvious that $l_1(u), l_2(u) > 0$, and

$$l_1(u) t^{\alpha-1} \leq (Tu)(t) \leq l_2(u) t^{\alpha-1}, \tag{3.3}$$

that is,

$$T : P \setminus \{\theta\} \to Q.$$

For any $u_0 \in P \setminus \{\theta\}$, let

$$u_n = A(u_{n-1}), \quad n = 1, 2, \ldots. \tag{3.4}$$

We may suppose that $u_1 - u_0 \neq \theta$ (otherwise, the proof is finished). Then there exists $l_2(|u_1 - u_0|) > 0$, such that

$$T\left(|u_1 - u_0| \right) \leq l_2\left(|u_1 - u_0| \right) t^{\alpha-1} = l_2\left(|u_1 - u_0| \right) \varphi_1. \tag{3.5}$$

Thus

$$\begin{aligned}
|u_2 - u_1| &= \left| \int_0^1 K(t,s)\big[f(s,u_1(s)) + bu_1(s) - f(s,u_0(s)) - bu_0(s)\big]\,ds \right| \\
&\leq \int_0^1 K(t,s)\big|f(s,u_1(s)) + bu_1(s) - f(s,u_0(s)) - bu_0(s)\big|\,ds \\
&\leq \lambda \int_0^1 K(t,s)\big|u_1(s) - u_0(s)\big|\,ds \\
&= \lambda T\big(|u_1 - u_0|\big) \leq \lambda l_2\big(|u_1 - u_0|\big)\varphi_1,
\end{aligned}$$

$$\begin{aligned}
|u_3 - u_2| &= \left| \int_0^1 K(t,s)\big[f(s,u_2(s)) + bu_2(s) - f(s,u_1(s)) - bu_1(s)\big]\,ds \right| \\
&\leq \int_0^1 K(t,s)\big|f(s,u_2(s)) + bu_2(s) - f(s,u_1(s)) - bu_1(s)\big|\,ds \\
&\leq \lambda \int_0^1 K(t,s)\big|u_2(s) - u_1(s)\big|\,ds \\
&\leq \lambda^2 l_2\big(|u_1 - u_0|\big) T\varphi_1 = \frac{\lambda^2}{b} l_2\big(|u_1 - u_0|\big)\varphi_1,
\end{aligned}$$

$$\cdots .$$

By induction, we can get

$$|u_{n+1} - u_n| \leq \left(\frac{\lambda}{b}\right)^{n-1} \lambda l_2\big(|u_1 - u_0|\big)\varphi_1. \tag{3.8}$$

Then, for any $n, m \in \mathbb{N}$, we have

$$\begin{aligned}
|u_{n+m} - u_m| &\leq |u_{n+m} - u_{n+m-1}| + \cdots + |u_{n+1} - u_n| \\
&\leq \left[\left(\frac{\lambda}{b}\right)^{n+m-1} + \cdots + \left(\frac{\lambda}{b}\right)^n\right]\lambda l_2\big(|u_1 - u_0|\big)\varphi_1 \\
&\leq \frac{(\frac{\lambda}{b})^n}{1 - \frac{\lambda}{b}}\lambda l_2\big(|u_1 - u_0|\big)\varphi_1 = \frac{\lambda^{n+1} l_2\big(|u_1 - u_0|\big)}{b^{n-1}(b - \lambda)}\varphi_1.
\end{aligned} \tag{3.9}$$

So,

$$\|u_{n+m} - u_m\| \leq \frac{\lambda^{n+1} l_2\big(|u_1 - u_0|\big)}{b^{n-1}(b - \lambda)} \to 0, \quad n \to \infty, \tag{3.10}$$

which implies $\{u_n\}$ is a Cauchy sequence. Therefore, there exists a $u^* \in P$, such that $\{u_n\}$ converges to u^*. Clearly, u^* is a fixed point of A.

In the following, we will prove the fixed point of A is unique.

Suppose $v \neq u^*$ is a fixed point of A. Then there exists $l_2(|u^* - v|) > 0$, such that

$$T\big(|u^* - v|\big) \leq l_2\big(|u^* - v|\big)\varphi_1. \tag{3.11}$$

Then

$$
\begin{aligned}
\left| Au^* - Av \right| &= \left| \int_0^1 K(t,s)\big[f\big(s, u^*(s)\big) + bu^*(s) - f\big(s, v(s)\big) - bv(s) \big]\, ds \right| \\
&\leq \int_0^1 K(t,s) \big| f\big(s, u^*(s)\big) + bu^*(s) - f\big(s, v(s)\big) - bv(s) \big|\, ds \\
&\leq \lambda \int_0^1 K(t,s) \big| u^*(s) - v(s) \big|\, ds \\
&= \lambda T\big(\big| u^* - v \big| \big) \leq \lambda l_2 \big(\big| u^* - v \big| \big) \varphi_1.
\end{aligned}
\tag{3.12}
$$

By induction, we can get

$$
\left| A^n u^* - A^n v \right| \leq \left(\frac{\lambda}{b} \right)^{n-1} \lambda l_2 \big(\big| u^* - v \big| \big) \varphi_1.
\tag{3.13}
$$

So,

$$
\left\| u^* - v \right\| = \left\| A^n u^* - A^n v \right\| \leq \left(\frac{\lambda}{b} \right)^{n-1} \lambda l_2 \big(\big| u^* - v \big| \big) \to 0, \quad n \to \infty.
\tag{3.14}
$$

Consequently, the fixed point of A is unique.

This completes the proof. □

Remark 3.1 The unique nonnegative solution u^* of (1.1) can be approximated by the iterative schemes: for any $u_0 \in P \setminus \{\theta\}$, let

$$
u_n = A(u_{n-1}), \quad n = 1, 2, \ldots,
$$

then $u_n \to u^*$.

Remark 3.2 If $f(t,0) \equiv 0$ on $[0,1]$, then θ is the unique solution of (1.1) in P; If $f(t,0) \not\equiv 0$ on $[0,1]$, then the unique solution u^* is a positive solution.

4 Existence of positive solutions

Theorem 4.1 *Assume that* (H_1), (H_2), *and the following assumptions hold*:

$$
\liminf_{x \to 0+} \min_{t \in [0,1]} \frac{f(t,x)}{x} > 0,
\tag{4.1}
$$

$$
\limsup_{x \to +\infty} \max_{t \in [0,1]} \frac{f(t,x)}{x} < 0.
\tag{4.2}
$$

Then (1.1) *has at least one positive solution.*

Proof It follows from (4.1) that there exists $r_1 > 0$ such that

$$
f(t,x) \geq 0, \quad \forall (t,x) \in [0,1] \times [0, r_1].
\tag{4.3}
$$

Thus, for any $u \in \partial B_{r_1} \cap P$, we have

$$Au(t) = \int_0^1 K(t,s)\big[f\big(s,u(s)\big) + bu(s)\big]\,ds \geq bTu(t). \tag{4.4}$$

We may suppose that A has no fixed points on $\partial B_{r_1} \cap P$ (otherwise, the proof is finished). Now we show that

$$u - Au \neq \mu\varphi_1, \quad \forall u \in \partial B_{r_1} \cap P, \mu > 0. \tag{4.5}$$

If otherwise, there exist $u_1 \in \partial B_{r_1} \cap P$ and $\mu_0 > 0$ such that

$$u_1 - Au_1 = \mu_0\varphi_1.$$

Then

$$u_1 = Au_1 + \mu_0\varphi_1 \geq \mu_0\varphi_1.$$

Denote

$$\mu^* = \sup\{\mu : u_1 \geq \mu\varphi_1\}. \tag{4.6}$$

It is clear that $\mu^* \geq \mu_0$ and $u_1 \geq \mu^*\varphi_1$. Since $T(P) \subset P$, we have $bTu_1 \geq \mu^*bT\varphi_1 = \mu^*\varphi_1$. Then

$$u_1 = Au_1 + \mu_0\varphi_1 \geq bTu_1 + \mu_0\varphi_1 \geq \big(\mu^* + \mu_0\big)\varphi_1,$$

contradicts the definition of μ^*. Hence (4.5) holds and we see from Lemma 2.4 that

$$i(A, B_{r_1} \cap P, P) = 0. \tag{4.7}$$

On the other hand, it follows from (4.2) that there exist $0 < \sigma < 1$ and $r_2 > r_1$ such that

$$f(t,x) \leq (\sigma - 1)bx, \quad \forall t \in [0,1], x \geq r_2. \tag{4.8}$$

Let $T_1u = \sigma bTu$. Then T_1 is a bounded linear operator and $T_1(P) \subset P$. Set

$$W = \{u \in P \mid u = \mu Au, 0 \leq \mu \leq 1\}. \tag{4.9}$$

In the following, we will prove that W is bounded.

For any $u \in W$, set $\tilde{u}(t) = \min\{u(t), r_2\}$. Then

$$f\big(t,u(t)\big) + bu(t) \leq \sigma bu(t) + f\big(t,\tilde{u}(t)\big) + b\tilde{u}(t).$$

Therefore,

$$u(t) = \mu Au(t) \leq Au(t) \leq \sigma bTu(t) + A\tilde{u}(t) \leq T_1u(t) + M,$$

where

$$M = \max_{(t,x)\in[0,1]\times[0,r_2]} \{f(t,x) + bx\} \int_0^1 \omega_1(s)\,ds. \tag{4.10}$$

Thus $(I - T_1)u(t) \le M, t \in [0,1]$. Noticing b is the first eigenvalue of T and $0 < \sigma < 1$, we have $(r(T_1))^{-1} = \sigma^{-1} > 1$. So the inverse operator of $I - T_1$ exists, and

$$(I - T_1)^{-1} = I + T_1 + T_1^2 + \cdots + T_1^n + \cdots.$$

It follows from $T_1(P) \subset P$ that $(I - T_1)^{-1}(P) \subset P$. We have

$$u(t) \le (I - T_1)^{-1}M \le M\|(I - T_1)^{-1}\|, \quad t \in [0,1],$$

which implies W is bounded.

Select $r_3 > \max\{r_2, M\|(I - T_1)^{-1}\|\}$. Then by Lemma 2.5, we have

$$i(A, B_{r_3} \cap P, P) = 1. \tag{4.11}$$

By (4.7) and (4.11), we have

$$i(A, (B_{r_3}\setminus\bar{B}_{r_1}) \cap P, P) = i(A, B_{r_3} \cap P, P) - i(A, B_{r_1} \cap P, P) = 1,$$

which implies that A has at least one fixed point on $(B_{r_3}\setminus\bar{B}_{r_1}) \cap P$. This means that BVP (1.1) has at least one positive solution.

This completes the proof. □

Theorem 4.2 *Assume that* (H_1), (H_2), *and the following assumptions hold*:

$$\limsup_{x\to 0+} \max_{t\in[0,1]} \frac{f(t,x)}{x} < 0, \tag{4.12}$$

and $f(t,0) \not\equiv 0$ *on* $[0,1]$. *Then* (1.1) *has at least one positive solution*.

Proof It follows from (4.12) that there exists $r_1 > 0$ such that

$$f(t,x) \le 0, \quad \forall(t,x) \in [0,1] \times [0,r_1]. \tag{4.13}$$

Denote $T_2u = bTu$. Obviously, $r(T_2) = 1$.

We may suppose that A has no fixed points on $\partial B_{r_1} \cap P$ (otherwise, the proof is finished). In the following, we prove that

$$Au \ne \mu u, \quad \forall u \in \partial B_{r_1} \cap P, \mu > 1. \tag{4.14}$$

If otherwise, there exist $u_1 \in \partial B_{r_1} \cap P, \mu_0 > 1$, such that $Au_1 = \mu_0 u_1$. It is clear that $\mu_0 u_1 = Au_1 \le T_2u_1$, and

$$\mu_0^n u_1 \le T_2^n u_1, \quad n = 1, 2, \ldots.$$

Therefore,

$$\mu_0^n u_1 \le T_2^n u_1 \le \| T_2^n \| \| u_1 \|.$$

Thus $r(T_2) = \lim_{n \to +\infty} \sqrt[n]{\| T_2^n \|} \ge \mu_0 > 1$, which contradicts $r(T_2) = 1$. We have from Lemma 2.5

$$i(A, B_{r_1} \cap P, P) = 1. \tag{4.15}$$

Since $f(t, 0) \not\equiv 0$ on $[0, 1]$, clearly we have $A\theta \ne \theta$, here θ is the zero element of E. So (4.15) implies that the problem (1.1) has at least one positive solution. $\square$

Remark 4.1 Suppose u is a positive solution of (1.1), then there exist $l_1, l_2 > 0$, such that

$$l_1 t^{\alpha-1} \le u(t) \le l_2 t^{\alpha-1}.$$

Example 4.1 (A 3-point boundary value problem at resonance) Consider the following problem:

$$\begin{cases} D_{0+}^{\frac{3}{2}} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = 0, & u(1) = 2u(\frac{1}{4}). \end{cases} \tag{4.16}$$

Since $\Gamma(\cdot)$ is strictly increasing on $[2, +\infty)$, for any $t \in [0, +\infty)$, we have

$$g(t) = -\frac{1}{2\sqrt{\pi}} + \sum_{k=1}^{+\infty} \frac{t^k}{\Gamma(\frac{3}{2}k - \frac{1}{2})} = -\frac{1}{2\sqrt{\pi}} + t + \sum_{k=2}^{+\infty} \frac{t^k}{\Gamma(\frac{3}{2}k - \frac{1}{2})}$$

$$\le -\frac{1}{2\sqrt{\pi}} + t + \sum_{k=2}^{+\infty} \frac{t^k}{\Gamma(k)} = -\frac{1}{2\sqrt{\pi}} + t \left[1 + \sum_{k=1}^{+\infty} \frac{t^k}{k!} \right]$$

$$= -\frac{1}{2\sqrt{\pi}} + t e^t.$$

Noticing $\frac{1}{2\sqrt{\pi}} \approx 0.282$, $\frac{1}{5} e^{\frac{1}{5}} \approx 0.243$, we have $g(\frac{1}{5}) < 0$. Therefore $b^* > b_1 := \frac{1}{5}$.

Let

$$f(t, x) = \begin{cases} \frac{1}{4}(1 + t)(1 - 2b)x, & (t, x) \in [0, 1] \times [0, 4), \\ \frac{1}{2}(1 + t)(\sqrt{x} - bx), & (t, x) \in [0, 1] \times [4, +\infty), \end{cases}$$

where $b \in (0, b_1]$. It is clear that (H_1) and (H_2) hold. Moreover,

$$\liminf_{u \to 0+} \min_{t \in [0,1]} \frac{f(t, u)}{u} = \frac{1 - 2b}{4} > 0,$$

$$\limsup_{x \to +\infty} \max_{t \in [0,1]} \frac{f(t, x)}{x} = -\frac{b}{2} < 0.$$

Therefore the assumptions of Theorem 4.1 are satisfied. Thus Theorem 4.1 ensures that (4.16) has at least one positive solution.

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