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Positive solutions for a system of second-order discrete boundary value problems

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Abstract

We study the existence and multiplicity of positive solutions for a system of nonlinear second-order difference equations subject to multi-point boundary conditions, under some assumptions on the nonlinearities of the system which contains concave functions. In the proofs of our main results we use some theorems from the fixed point index theory.

MSC: 39A10; 39A12

Keywords: Difference equations; Multi-point boundary conditions; Positive solutions; Existence; Multiplicity

1 Introduction

In this paper, we consider the system of nonlinear second-order difference equations

$$\begin{cases} \Delta^2 u_{n-1} + f(n, u_n, v_n) = 0, & n = \overline{1, N-1}, \\ \Delta^2 v_{n-1} + g(n, u_n, v_n) = 0, & n = \overline{1, N-1}, \end{cases} \tag{S}$$

subject to the multi-point boundary conditions

$$u_0 = \sum_{i=1}^{p} a_i u_{\xi_i}, \qquad u_N = \sum_{i=1}^{q} b_i u_{\eta_i}, \qquad v_0 = \sum_{i=1}^{r} c_i v_{\zeta_i}, \qquad v_N = \sum_{i=1}^{l} d_i v_{\rho_i}, \tag{BC}$$

where $N \in \mathbb{N}$, $N \geq 2$, $p, q, r, l \in \mathbb{N}$, Δ is the forward difference operator with stepsize 1, $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_{n-1} = u_{n+1} - 2u_n + u_{n-1}$, and $n = \overline{k, m}$ means that $n = k, k+1, \ldots, m$ for $k, m \in \mathbb{N}$, $\xi_i \in \mathbb{N}$ for all $i = \overline{1, p}$, $\eta_i \in \mathbb{N}$ for all $i = \overline{1, q}$, $\zeta_i \in \mathbb{N}$ for all $i = \overline{1, r}$, $\rho_i \in \mathbb{N}$ for all $i = \overline{1, l}$, $1 \leq \xi_1 < \cdots < \xi_p \leq N-1$, $1 \leq \eta_1 < \cdots < \eta_q \leq N-1$, $1 \leq \zeta_1 < \cdots < \zeta_r \leq N-1$, and $1 \leq \rho_1 < \cdots < \rho_l \leq N-1$.

Under sufficient conditions on the nonnegative nonlinearities f and g which contain some concave functions, we investigate the existence and multiplicity of positive solutions of problem (S)–(BC) by using the fixed point index theory. By a positive solution of (S)–(BC), we mean a pair of sequences $(u, v) = ((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}})$ satisfying (S) and

(BC) with $u_n \geq 0$ and $v_n \geq 0$ for all $n = \overline{0,N}$, and $u_n > 0$ for all $n = \overline{1,N}$ or $v_n > 0$ for all $n = \overline{1,N}$. The existence and nonexistence of nonnegative and nontrivial solutions (u,v) ($u_n \geq 0$, $v_n \geq 0$ for all $n = \overline{0,N}$ and $(u,v) \neq (0,0)$) of problem (S)–(BC) with some positive parameters in system (S) were studied in the papers [14] and [15] by using the Guo–Krasnosel'skii fixed point theorem. We also mention the paper [19], where the authors investigated the existence and multiplicity of positive solutions for problem (S)–(BC) under some assumptions on the functions f and g which are different than those we use in this paper. The existence, nonexistence, and multiplicity of positive solutions for system (S) with parameters or without parameters, subject to the multi-point coupled boundary conditions

$$u_0 = 0, \qquad u_N = \sum_{i=1}^{p} a_i v_{\xi_i}, \qquad v_0 = 0, \qquad v_N = \sum_{i=1}^{q} b_i u_{\eta_i}, \tag{BC$_1$}$$

were studied in the papers [16] and [18].

The mathematical modeling of many nonlinear problems from computer science, economics, mechanical engineering, control systems, biological neural networks, and others leads to the consideration of nonlinear difference equations (see [2, 4, 21, 23]). In the last decades, many authors have investigated such problems by using various methods, such as fixed point theorems, the critical point theory, upper and lower solutions, the fixed point index theory, and the topological degree theory (see, for example, [1, 6–12, 17, 20, 24–28]).

The paper is organized as follows. In Sect. 2, we investigate a system of second-order linear difference equations subject to the boundary conditions (BC), and we present the properties of the corresponding Green functions. In Sect. 3, we prove the main theorems for the existence and multiplicity of positive solutions of problem (S)–(BC) which are based on some theorems from the fixed point index theory, and we present two examples to support our results.

2 Preliminary results

We begin this section with a result from [14] related to the following system of second-order difference equations:

$$\Delta^2 u_{n-1} + y_n = 0, \quad n = \overline{1,N-1}, \tag{1}$$

subject to the multi-point boundary conditions

$$u_0 = \sum_{i=1}^{p} a_i u_{\xi_i}, \qquad u_N = \sum_{i=1}^{q} b_i u_{\eta_i}, \tag{2}$$

where $p,q \in \mathbb{N}$, $\xi_i \in \mathbb{N}$ for all $i = \overline{1,p}$, $\eta_i \in \mathbb{N}$ for all $i = \overline{1,q}$, $1 \leq \xi_1 < \cdots < \xi_p \leq N-1$, $1 \leq \eta_1 < \cdots < \eta_q \leq N-1$, and $y_n \in \mathbb{R}$ for all $n = \overline{1,N-1}$.

We denote $\Delta_1 = (1 - \sum_{i=1}^{q} b_i) \sum_{i=1}^{p} a_i \xi_i + (1 - \sum_{i=1}^{p} a_i)(N - \sum_{i=1}^{q} b_i \eta_i)$.

Lemma 2.1 ([14]) *If $\Delta_1 \neq 0$, then the solution $(u_n)_{n=\overline{0,N}}$ of problem (1)–(2) is given by $u_n = \sum_{j=1}^{N-1} G_1(n,j)y_j$ for all $n = \overline{0,N}$, where the Green function G_1 is defined by*

$$G_1(n,j) = g_0(n,j) + \frac{1}{\Delta_1}\left[(N-n)\left(1-\sum_{k=1}^{q} b_k\right) + \sum_{i=1}^{q} b_i(N-\eta_i)\right]\sum_{i=1}^{p} a_i g_0(\xi_i,j)$$
$$+ \frac{1}{\Delta_1}\left[n\left(1-\sum_{k=1}^{p} a_k\right) + \sum_{i=1}^{p} a_i\xi_i\right]\sum_{i=1}^{q} b_i g_0(\eta_i,j), \quad n = \overline{0,N}, j = \overline{1,N-1}, \quad (3)$$

and

$$g_0(n,j) = \frac{1}{N}\begin{cases} j(N-n), & 1 \le j \le n \le N, \\ n(N-j), & 0 \le n \le j \le N-1. \end{cases} \quad (4)$$

Next we will present some properties of the function g_0 and the Green function G_1.

Lemma 2.2 *The function g_0 given by (4) has the following properties*:

(a) *$g_0(n,j) \ge 0$ for all $n = \overline{0,N}$, $j = \overline{1,N-1}$;*
(b) *$g_0(n,j) \le h(j)$ for all $n = \overline{0,N}$, $j = \overline{1,N-1}$, where $h(j) = g_0(j,j) = \frac{1}{N}j(N-j)$ for all $j = \overline{1,N-1}$;*
(c) *$g_0(n,j) \ge k(n)h(j)$ for all $n = \overline{0,N}$, $j = \overline{1,N-1}$, where $k(n) = \frac{1}{N(N-1)}n(N-n)$ for all $n = \overline{0,N}$.*

Proof For the proofs of (a) and (b), see [7].

For (c), if $1 \le j \le n \le N$, then we have

$$\frac{1}{N}j(N-n) \ge \frac{1}{N(N-1)}n(N-n)\frac{1}{N}j(N-j) \quad \Leftrightarrow \quad N(N-1) \ge n(N-j),$$

which is satisfied for all $j = \overline{1,N-1}$ and $n = \overline{0,N}$.

If $0 \le n \le j \le N-1$, then we obtain

$$\frac{1}{N}n(N-j) \ge \frac{1}{N(N-1)}n(N-n)\frac{1}{N}j(N-j) \quad \Leftrightarrow \quad N(N-1) \ge j(N-n),$$

which is satisfied for all $j = \overline{1,N-1}$ and $n = \overline{0,N}$. □

Lemma 2.3 *If $a_i \ge 0$ for all $i = \overline{1,p}$, $\sum_{i=1}^{p} a_i < 1$, $b_i \ge 0$ for all $i = \overline{1,q}$, $\sum_{i=1}^{q} b_i < 1$, then the Green function G_1 of problem (1)–(2) given by (3) satisfies the inequalities*

(a) *$G_1(n,j) \le Ah(j)$ for all $n = \overline{0,N}$, $j = \overline{1,N-1}$, where*

$$A = 1 + \frac{1}{\Delta_1}\left(N - \sum_{i=1}^{q} b_i\eta_i\right)\left(\sum_{i=1}^{p} a_i\right) + \frac{1}{\Delta_1}\left(N - \sum_{i=1}^{p} a_i(N-\xi_i)\right)\left(\sum_{i=1}^{q} b_i\right) > 0.$$

(b) *$G_1(n,j) \ge k(n)h(j)$ for all $n = \overline{0,N}$, $j = \overline{1,N-1}$.*

Proof By the assumptions on the coefficients a_i, $i=\overline{1,p}$ and b_j, $j=\overline{1,q}$, we can easily see that $\Delta_1>0$ and $A>0$. By using Lemma 2.2, for all $n=\overline{0,N}$ and $j=\overline{1,N-1}$, we deduce

$$\begin{aligned} G_1(n,j) &\le h(j)\left\{1+\frac{1}{\Delta_1}\left[N\left(1-\sum_{k=1}^{q}b_k\right)+\sum_{i=1}^{q}b_i(N-\eta_i)\right]\left(\sum_{i=1}^{p}a_i\right)\right.\\ &\quad\left.+\frac{1}{\Delta_1}\left[N\left(1-\sum_{k=1}^{p}a_k\right)+\sum_{i=1}^{p}a_i\xi_i\right]\left(\sum_{i=1}^{q}b_i\right)\right\}\\ &= Ah(j), \end{aligned}$$

and

$$G_1(n,j)\ge g_0(n,j)\ge k(n)h(j),$$

that is, we obtain inequalities (a) and (b). □

Lemma 2.4 *Assume that $a_i\ge 0$ for all $i=\overline{1,p}$, $\sum_{i=1}^{p}a_i<1$, $b_i\ge 0$ for all $i=\overline{1,q}$, $\sum_{i=1}^{q}b_i<1$, and $y_n\ge 0$ for all $n=\overline{1,N-1}$. Then the solution $(u_n)_{n=\overline{0,N}}$ of problem (1)–(2) satisfies the inequality $u_n\ge\frac{1}{A}k(n)u_m$ for all $n,m=\overline{0,N}$.*

Proof By using Lemmas 2.1–2.3, we deduce

$$\begin{aligned} u_n &= \sum_{j=1}^{N-1}G_1(n,j)y_j\ge\sum_{j=1}^{N-1}k(n)h(j)y_j\ge\sum_{j=1}^{N-1}\frac{1}{A}G_1(m,j)k(n)y_j\\ &= \frac{1}{A}k(n)\sum_{j=1}^{N-1}G_1(m,j)y_j=\frac{1}{A}k(n)u_m,\quad \forall n,m=\overline{0,N}. \end{aligned}$$

□

We can also formulate similar results as Lemmas 2.1–2.4 for the discrete boundary value problem

$$\Delta^2 v_{n-1}+\widetilde{y}_n=0,\quad n=\overline{1,N-1},\tag{5}$$

$$v_0=\sum_{i=1}^{r}c_iv_{\zeta_i},\qquad v_N=\sum_{i=1}^{l}d_iv_{\rho_i},\tag{6}$$

where $r,l\in\mathbb{N}$, $c_i\ge 0$ for all $i=\overline{1,r}$, $\sum_{i=1}^{r}c_i<1$, $\zeta_i\in\mathbb{N}$ for all $i=\overline{1,r}$, $d_i\ge 0$ for all $i=\overline{1,l}$, $\sum_{i=1}^{l}d_i<1$, $\rho_i\in\mathbb{N}$ for all $i=\overline{1,l}$, $1\le\zeta_1<\cdots<\zeta_r\le N-1$, $1\le\rho_1<\cdots<\rho_l\le N-1$, and $\widetilde{y}_n\ge 0$ for all $n=\overline{1,N-1}$.

We denote by

$$\Delta_2=\left(1-\sum_{i=1}^{l}d_i\right)\sum_{i=1}^{r}c_i\zeta_i+\left(1-\sum_{i=1}^{r}c_i\right)\left(N-\sum_{i=1}^{l}d_i\rho_i\right)>0,$$

$$G_2(n,j) = g_0(n,j) + \frac{1}{\Delta_2}\left[(N-n)\left(1-\sum_{k=1}^{l} d_k\right) + \sum_{i=1}^{l} d_i(N-\rho_i)\right]\sum_{i=1}^{r} c_i g_0(\zeta_i,j)$$

$$+ \frac{1}{\Delta_2}\left[n\left(1-\sum_{k=1}^{r} c_k\right) + \sum_{i=1}^{r} c_i\zeta_i\right]\sum_{i=1}^{l} d_i g_0(\rho_i,j), \quad n=\overline{0,N}, j=\overline{1,N-1},$$

$$B = 1 + \frac{1}{\Delta_2}\left(N-\sum_{i=1}^{l} d_i\rho_i\right)\left(\sum_{i=1}^{r} c_i\right) + \frac{1}{\Delta_2}\left(N-\sum_{i=1}^{r} c_i(N-\zeta_i)\right)\left(\sum_{i=1}^{l} d_i\right) > 0.$$

Then we deduce the inequalities $G_2(n,j) \le Bh(j)$ and $G_2(n,j) \ge k(n)h(j)$ for all $n = \overline{0,N}$, $j = \overline{1,N-1}$. In addition the solution $(v_n)_{n=\overline{0,N}}$ of problem (5)–(6) satisfies the inequality $v_n \ge \frac{1}{B}k(n)v_m$ for all $n, m = \overline{0,N}$.

We recall now some theorems concerning the fixed point index theory. Let E be a real Banach space with the norm $\|\cdot\|$, $P \subset E$ be a cone, "$\le$" be the partial ordering defined by P, and 0 be the zero element in E. For $\varrho > 0$, let $B_\varrho = \{u \in E, \|u\| < \varrho\}$ be the open ball of radius ϱ centered at 0, and its boundary $\partial B_\varrho = \{u \in E, \|u\| = \varrho\}$. The proofs of our results are based on the following fixed point index theorems (see [3, 5, 13, 22]).

Theorem 2.1 *Let $A : \overline{B}_\varrho \cap P \to P$ be a completely continuous operator which has no fixed points on $\partial B_\varrho \cap P$. If $\|Au\| \le \|u\|$ for all $u \in \partial B_\varrho \cap P$, then $i(A, B_\varrho \cap P, P) = 1$.*

Theorem 2.2 *Let $A : \overline{B}_\varrho \cap P \to P$ be a completely continuous operator. If there exists $u_0 \in P \setminus \{0\}$ such that $u - Au \ne \lambda u_0$ for all $\lambda \ge 0$ and $u \in \partial B_\varrho \cap P$, then $i(A, B_\varrho \cap P, P) = 0$.*

Theorem 2.3 *Let $\Omega \subset E$ be a bounded open set with $0 \in \Omega$. Assume that $A : \overline{\Omega} \cap P \to P$ is a completely continuous operator.*

(a) *If $u \not\le Au$ for all $u \in \partial\Omega \cap P$, then the fixed point index $i(A, \Omega \cap P, P) = 1$.*

(b) *If $Au \not\le u$ for all $u \in \partial\Omega \cap P$, then the fixed point index $i(A, \Omega \cap P, P) = 0$.*

3 Existence and multiplicity of positive solutions

In this section we present sufficient conditions on the functions f and g such that problem (S)–(BC) has positive solutions with respect to a cone.

We present the assumptions that we shall use in the sequel.

$(H1)$ $a_i \ge 0$, $\xi_i \in \mathbb{N}$ for all $i = \overline{1,p}$, $1 \le \xi_1 < \cdots < \xi_p \le N-1$,
$b_i \ge 0$, $\eta_i \in \mathbb{N}$ for all $i = \overline{1,q}$, $1 \le \eta_1 < \cdots < \eta_q \le N-1$,
$c_i \ge 0$, $\zeta_i \in \mathbb{N}$ for all $i = \overline{1,r}$, $1 \le \zeta_1 < \cdots < \zeta_r \le N-1$,
$d_i \ge 0$, $\rho_i \in \mathbb{N}$ for all $i = \overline{1,l}$, $1 \le \rho_1 < \cdots < \rho_l \le N-1$, and
$\sum_{i=1}^{p} a_i < 1$, $\sum_{i=1}^{q} b_i < 1$, $\sum_{i=1}^{r} c_i < 1$, $\sum_{i=1}^{l} d_i < 1$.

$(H2)$ The functions $f, g : \{1,\dots,N-1\} \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous, ($\mathbb{R}_+ = [0,\infty)$).

$(H3)$ There exist functions $a, b \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that

(a) $a(\cdot)$ is concave and strictly increasing on $\mathbb{R}_+$ with $a(0) = 0$;

(b)

$$\begin{cases} f_0^i = \liminf_{v\to 0+} \frac{f(n,u,v)}{a(v)} \in (0,\infty], \\ \quad \text{uniformly with respect to } (n,u) \in \{1,\dots,N-1\} \times \mathbb{R}_+, \quad \text{and} \\ g_0^i = \liminf_{u\to 0+} \frac{g(n,u,v)}{b(u)} \in (0,\infty], \\ \quad \text{uniformly with respect to } (n,v) \in \{1,\dots,N-1\} \times \mathbb{R}_+; \end{cases}$$

(c) $\lim_{u\to 0+} \frac{a(Cb(u))}{u} = \infty$ exists for any constant $C > 0$.

$(H4)$ There exist $\alpha_1, \alpha_2 > 0$ with $\alpha_1\alpha_2 \le 1$ such that

$$\begin{cases} f_\infty^s = \limsup_{v\to\infty} \frac{f(n,u,v)}{v^{\alpha_1}} \in [0,\infty), \\ \quad \text{uniformly with respect to } (n,u) \in \{1,\ldots,N-1\} \times \mathbb{R}_+, \quad \text{and} \\ g_\infty^s = \lim_{u\to\infty} \frac{g(n,u,v)}{u^{\alpha_2}} = 0 \\ \quad \text{exists uniformly with respect to } (n,v) \in \{1,\ldots,N-1\} \times \mathbb{R}_+. \end{cases}$$

$(H5)$ There exist the functions $c, d \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that

(a) $c(\cdot)$ is concave and strictly increasing on $\mathbb{R}_+$;

(b)

$$\begin{cases} f_\infty^i = \liminf_{v\to\infty} \frac{f(n,u,v)}{c(v)} \in (0,\infty], \\ \quad \text{uniformly with respect to } (n,u) \in \{1,\ldots,N-1\} \times \mathbb{R}_+, \quad \text{and} \\ g_\infty^i = \liminf_{u\to\infty} \frac{g(n,u,v)}{d(u)} \in (0,\infty], \\ \quad \text{uniformly with respect to } (n,v) \in \{1,\ldots,N-1\} \times \mathbb{R}_+; \end{cases}$$

(c) $\lim_{u\to\infty} \frac{c(Cd(u))}{u} = \infty$ exists for any constant $C > 0$.

$(H6)$ There exist $\beta_1, \beta_2 > 0$ with $\beta_1\beta_2 \ge 1$ such that

$$\begin{cases} f_0^s = \limsup_{v\to 0+} \frac{f(n,u,v)}{v^{\beta_1}} \in [0,\infty), \\ \quad \text{uniformly with respect to } (n,u) \in \{1,\ldots,N-1\} \times \mathbb{R}_+, \quad \text{and} \\ g_0^s = \lim_{u\to 0+} \frac{g(n,u,v)}{u^{\beta_2}} = 0 \\ \quad \text{exists uniformly with respect to } (n,v) \in \{1,\ldots,N-1\} \times \mathbb{R}_+. \end{cases}$$

$(H7)$ The functions $f(n,u,v)$ and $g(n,u,v)$ are nondecreasing with respect to u and v, and there exists $N_0 > 0$ such that

$$f(n,N_0,N_0) < \frac{3N_0}{(N^2-1)\max\{A,B\}} \quad \text{and} \quad g(n,N_0,N_0) < \frac{3N_0}{(N^2-1)\max\{A,B\}}$$

for all $n \in \{1,\ldots,N-1\}$.

By using the Green functions G_1 and G_2 from Sect. 2, our problem (S)–(BC) can be written equivalently as the following system:

$$\begin{cases} u_n = \sum_{i=1}^{N-1} G_1(n,i) f(i,u_i,v_i), \quad n = \overline{0,N}, \\ v_n = \sum_{i=1}^{N-1} G_2(n,i) g(i,u_i,v_i), \quad n = \overline{0,N}. \end{cases} \tag{7}$$

Then $(u,v) = ((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}})$ is a solution of problem (S)–(BC) if and only if (u,v) is a solution of system (7).

We consider the Banach space $X = \mathbb{R}^{N+1} = \{u = (u_n)_{n=\overline{0,N}}, u_i \in \mathbb{R}, i = \overline{0,N}\}$ with the maximum norm $\|\cdot\|$, $\|u\| = \max_{i=\overline{0,N}} |u_i|$, and the Banach space $Y = X \times X$ with the norm

$\|(u,v)\|_Y = \|u\| + \|v\|$. We define the cones

$$P_1 = \left\{ u \in X, u = (u_n)_{n=\overline{0,N}}, u_n \geq \frac{1}{A} k(n)\|u\|, \forall n = \overline{0,N} \right\} \subset X,$$
$$P_2 = \left\{ v \in X, v = (v_n)_{n=\overline{0,N}}, v_n \geq \frac{1}{B} k(n)\|v\|, \forall n = \overline{0,N} \right\} \subset X,$$

and $P = P_1 \times P_2 \subset Y$.

We introduce the operators $Q_1, Q_2 : Y \to X$ and $Q : Y \to Y$ defined by

$$Q_1(u,v) = \big(Q_{1n}(u,v)\big)_{n=\overline{0,N}}, \qquad Q_2(u,v) = \big(Q_{2n}(u,v)\big)_{n=\overline{0,N}},$$
$$Q_{1n}(u,v) = \sum_{i=1}^{N-1} G_1(n,i) f(i,u_i,v_i), \quad n = \overline{0,N},$$
$$Q_{2n}(u,v) = \sum_{i=1}^{N-1} G_2(n,i) g(i,u_i,v_i), \quad n = \overline{0,N},$$
$$Q(u,v) = \big(Q_1(u,v), Q_2(u,v)\big), \quad (u,v) = \big((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}}\big) \in Y.$$

The pair (u,v) is a solution of problem (S)–(BC) if and only if (u,v) is a fixed point of operator Q in the space Y. So, we will investigate the existence of fixed points of operator Q. Under assumptions $(H1)$ and $(H2)$ and by using Lemma 2.4, we can easily prove that $Q(P) \subset P$ and the operator $Q : P \to P$ is completely continuous.

Theorem 3.1 *Assume that* $(H1)$, $(H2)$, $(H3)$, *and* $(H4)$ *hold. Then problem* (S)–(BC) *has at least one positive solution.*

Proof By $(H3)$, there exist $C_1 > 0$, $C_2 > 0$, and a sufficiently small $r_1 > 0$ such that

$$\begin{aligned} &f(n,u,v) \geq C_1 a(v), \quad \forall (n,u) \in \{1,\ldots,N-1\} \times \mathbb{R}_+, v \in [0,r_1], \\ &g(n,u,v) \geq C_2 b(u), \quad \forall (n,v) \in \{1,\ldots,N-1\} \times \mathbb{R}_+, u \in [0,r_1], \end{aligned} \tag{8}$$

and

$$a\big(C_3 b(u)\big) \geq \frac{72 C_3 \max\{A,B\} N^3 u}{C_1 C_2 (N^2-1)^2}, \quad \forall u \in [0,r_1], \tag{9}$$

where $C_3 = \max\{\frac{(N-1)C_2}{N} h(j), j = \overline{1,N-1}\}$.

We will show that $(Q_1(u,v), Q_2(u,v)) \nleq (u,v)$ for all $(u,v) \in \partial B_{r_1} \cap P$. We suppose that there exists $(u,v) \in \partial B_{r_1} \cap P$, that is, $\|(u,v)\|_Y = r_1$, such that $(Q_1(u,v), Q_2(u,v)) \leq (u,v)$. Then $u \geq Q_1(u,v)$ and $v \geq Q_2(u,v)$. By using the monotonicity and concavity of $a(\cdot)$, the Jensen inequality, Lemma 2.3, relations (8) and (9), we obtain

$$\begin{aligned} u_n \geq Q_{1n}(u,v) &= \sum_{i=1}^{N-1} G_1(n,i) f(i,u_i,v_i) \geq C_1 \sum_{i=1}^{N-1} h(i) k(n) a(v_i) \\ &\geq C_1 k(1) \sum_{i=1}^{N-1} h(i) a(v_i) = \frac{C_1}{N} \sum_{i=1}^{N-1} h(i) a\left(\sum_{j=1}^{N-1} G_2(i,j) g(j,u_j,v_j) \right) \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{C_1}{N}\sum_{i=1}^{N-1} h(i) a\left(C_2 \sum_{j=1}^{N-1} G_2(i,j) b(u_j)\right) \geq \frac{C_1}{N}\sum_{i=1}^{N-1} h(i) a\left(C_2 \sum_{j=1}^{N-1} h(j)k(i) b(u_j)\right) \\
&\geq \frac{C_1}{N}\sum_{i=1}^{N-1} h(i) a\left(C_2 k(1) \sum_{j=1}^{N-1} h(j) b(u_j)\right) = \frac{C_1}{N}\left(\sum_{i=1}^{N-1} h(i)\right) a\left(\frac{C_2}{N} \sum_{j=1}^{N-1} h(j) b(u_j)\right) \\
&\geq \frac{C_1(N^2-1)}{6N(N-1)} \sum_{j=1}^{N-1} a\left(\frac{(N-1)C_2}{N} h(j) b(u_j)\right) \\
&= \frac{C_1(N+1)}{6N} \sum_{j=1}^{N-1} a\left(\frac{(N-1)C_2}{NC_3} h(j)\cdot C_3 b(u_j)\right) \\
&\geq \frac{C_1(N+1)}{6N} \sum_{j=1}^{N-1} \frac{(N-1)C_2}{NC_3} h(j) a\big(C_3 b(u_j)\big) \\
&\geq \frac{C_1 C_2(N^2-1)}{6N^2 C_3} \sum_{j=1}^{N-1} h(j) \frac{72 C_3 \max\{A,B\} N^3}{C_1 C_2 (N^2-1)^2} u_j \\
&\geq \frac{12N\max\{A,B\}}{N^2-1} \sum_{j=1}^{N-1} h(j) \frac{1}{A} k(j) \|u\| \\
&\geq \frac{12N\max\{A,B\}}{N^2-1} \sum_{j=1}^{N-1} h(j) \frac{1}{AN} \|u\| \\
&\geq 2\|u\|, \quad \forall n = \overline{1,N-1}.
\end{aligned}
$$

So, $\|u\| \geq \max_{n=\overline{1,N-1}} u_n \geq 2\|u\|$, and then

$$
\|u\| = 0. \tag{10}
$$

In a similar manner, we deduce

$$
\begin{aligned}
a(v_i) &\geq a\big(Q_{2i}(u,v)\big) = a\left(\sum_{j=1}^{N-1} G_2(i,j) g(j,u_j,v_j)\right) \\
&\geq \frac{1}{N-1}\sum_{j=1}^{N-1} a\big((N-1)G_2(i,j) g(j,u_j,v_j)\big) \\
&\geq \frac{1}{N-1}\sum_{j=1}^{N-1} a\big((N-1)h(j)k(i) C_2 b(u_j)\big) \\
&\geq \frac{1}{N-1}\sum_{j=1}^{N-1} a\left(\frac{C_2(N-1)}{N} h(j) b(u_j)\right) \\
&= \frac{1}{N-1}\sum_{j=1}^{N-1} a\left(\frac{C_2(N-1)}{NC_3} h(j)\cdot C_3 b(u_j)\right) \\
&\geq \frac{1}{N-1}\sum_{j=1}^{N-1} \frac{C_2(N-1)}{NC_3} h(j) a\big(C_3 b(u_j)\big)
\end{aligned}
$$

$$\begin{aligned}
&\geq \frac{C_2}{NC_3}\sum_{j=1}^{N-1} h(j)\frac{72C_3\max\{A,B\}N^3}{C_1C_2(N^2-1)^2}u_j \\
&= \frac{72N^2\max\{A,B\}}{(N^2-1)^2C_1}\sum_{j=1}^{N-1} h(j)\left(\sum_{\theta=1}^{N-1} G_1(j,\theta)f(\theta,u_\theta,v_\theta)\right) \\
&\geq \frac{72N^2\max\{A,B\}}{(N^2-1)^2C_1}\sum_{j=1}^{N-1} h(j)\left(\sum_{\theta=1}^{N-1} h(\theta)k(j)C_1a(v_\theta)\right) \\
&\geq \frac{72N\max\{A,B\}}{(N^2-1)^2}\left(\sum_{j=1}^{N-1} h(j)\right)\left(\sum_{\theta=1}^{N-1} h(\theta)a(v_\theta)\right) \\
&= \frac{12N\max\{A,B\}}{N^2-1}\sum_{\theta=1}^{N-1} h(\theta)a(v_\theta) \\
&\geq \frac{12N\max\{A,B\}}{N^2-1}\sum_{\theta=1}^{N-1} h(\theta)a\left(\frac{1}{B}k(\theta)\|v\|\right) \\
&\geq \frac{12N\max\{A,B\}}{N^2-1}\left(\sum_{\theta=1}^{N-1} h(\theta)\right)a\left(\frac{1}{BN}\|v\|\right) \\
&\geq 2N\max\{A,B\}\frac{1}{BN}a\bigl(\|v\|\bigr) \\
&\geq 2a\bigl(\|v\|\bigr), \quad \forall i=\overline{1,N-1}.
\end{aligned}$$

Then we conclude that $a(\|v\|) = a(\sup_{i=\overline{0,N}} v_i) \geq a(v_1) \geq 2a(\|v\|)$, and hence $a(\|v\|) = 0$. By $(H3)$(a), we obtain

$$\|v\| = 0. \tag{11}$$

Therefore, by (10) and (11), we deduce that $\|(u,v)\|_Y = 0$, which is a contradiction. Hence $(Q_1(u,v), Q_2(u,v)) \not\leq (u,v)$ for all $(u,v) \in \partial B_{r_1} \cap P$. By Theorem 2.3(b), we conclude that the fixed point index

$$i(Q, B_{r_1} \cap P, P) = 0. \tag{12}$$

On the other hand, by $(H4)$ we deduce that there exist $C_4 > 0$, $C_5 > 0$, and $C_6 > 0$ such that

$$\begin{aligned}
&f(n,u,v) \leq C_4v^{\alpha_1} + C_5, \quad \forall (n,u,v) \in \{1,\dots,N-1\} \times \mathbb{R}_+ \times \mathbb{R}_+, \\
&g(n,u,v) \leq \varepsilon_1u^{\alpha_2} + C_6, \quad \forall (n,u,v) \in \{1,\dots,N-1\} \times \mathbb{R}_+ \times \mathbb{R}_+,
\end{aligned} \tag{13}$$

with

$$\varepsilon_1 = \min\left\{\frac{6}{B(N^2-1)}\left(\frac{3}{4AC_4(N^2-1)}\right)^{\alpha_2}, \frac{6^{\alpha_2+1}}{8B(AC_4)^{\alpha_2}(N^2-1)^{\alpha_2+1}}\right\}.$$

Then, by (13), we have

$$
\begin{aligned}
Q_{1n}(u,v) &= \sum_{i=1}^{N-1} G_1(n,i) f(i,u_i,v_i) \le \sum_{i=1}^{N-1} A h(i)\left(C_4 v_i^{\alpha_1} + C_5\right) \\
&= AC_4 \sum_{i=1}^{N-1} h(i) v_i^{\alpha_1} + AC_5 \frac{N^2-1}{6}, \quad \forall n = \overline{0,N}, \\
Q_{2n}(u,v) &= \sum_{i=1}^{N-1} G_2(n,i) g(i,u_i,v_i) \le \sum_{i=1}^{N-1} B h(i)\left(\varepsilon_1 u_i^{\alpha_2} + C_6\right) \\
&= B\varepsilon_1 \sum_{i=1}^{N-1} h(i) u_i^{\alpha_2} + BC_6 \frac{N^2-1}{6}, \quad \forall n = \overline{0,N}.
\end{aligned}
\tag{14}
$$

We consider now the functions $\widetilde{p}, \widetilde{q} : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$
\begin{aligned}
\widetilde{p}(w) &= \frac{AC_4(N^2-1)}{6}\left[\left(\frac{3w}{4AC_4(N^2-1)}\right)^{\alpha_2} + \frac{BC_6(N^2-1)}{6}\right]^{\alpha_1} + \frac{AC_5(N^2-1)}{6}, \\
\widetilde{q}(w) &= \frac{6^{\alpha_2}}{8(AC_4(N^2-1))^{\alpha_2}}\left(\frac{AC_4(N^2-1)}{6} w^{\alpha_1} + \frac{AC_5(N^2-1)}{6}\right)^{\alpha_2} + \frac{BC_6(N^2-1)}{6}.
\end{aligned}
$$

Because

$$
\lim_{w\to\infty} \frac{\widetilde{p}(w)}{w} = \lim_{w\to\infty} \frac{\widetilde{q}(w)}{w} = \begin{cases} 0, & \text{if } \alpha_1\alpha_2 < 1, \\ 1/8, & \text{if } \alpha_1\alpha_2 = 1, \end{cases}
$$

we conclude that there exists $R_1 > r_1$ such that

$$
\widetilde{p}(w) \le \frac{1}{4} w, \qquad \widetilde{q}(w) \le \frac{1}{4} w, \quad \forall w \ge R_1. \tag{15}
$$

We will show that $(u,v) \not\le (Q_1(u,v), Q_2(u,v))$ for all $(u,v) \in \partial B_{R_1} \cap P$. We suppose that there exists $(u,v) \in \partial B_{R_1} \cap P$, that is, $\|(u,v)\|_Y = R_1$, such that $(u,v) \le (Q_1(u,v), Q_2(u,v))$. So, by (14), we obtain

$$
\begin{aligned}
u_n &\le Q_{1n}(u,v) \le AC_4 \sum_{i=1}^{N-1} h(i) v_i^{\alpha_1} + AC_5 \frac{N^2-1}{6}, \quad \forall n = \overline{0,N}, \\
v_n &\le Q_{2n}(u,v) \le B\varepsilon_1 \sum_{i=1}^{N-1} h(i) u_i^{\alpha_2} + BC_6 \frac{N^2-1}{6}, \quad \forall n = \overline{0,N}.
\end{aligned}
$$

Then, for all $n = \overline{0,N}$, we deduce

$$
\begin{aligned}
u_n &\le AC_4 \sum_{i=1}^{N-1} h(i)\left(B\varepsilon_1 \sum_{j=1}^{N-1} h(j) u_j^{\alpha_2} + BC_6 \frac{N^2-1}{6}\right)^{\alpha_1} + AC_5 \frac{N^2-1}{6} \\
&= AC_4 \frac{N^2-1}{6}\left(B\varepsilon_1 \sum_{j=1}^{N-1} h(j) u_j^{\alpha_2} + BC_6 \frac{N^2-1}{6}\right)^{\alpha_1} + AC_5 \frac{N^2-1}{6}
\end{aligned}
$$

$$
\begin{aligned}
&\leq AC_4\frac{N^2-1}{6}\left(B\varepsilon_1\sum_{j=1}^{N-1}h(j)\|u\|^{\alpha_2}+BC_6\frac{N^2-1}{6}\right)^{\alpha_1}+AC_5\frac{N^2-1}{6}\\
&=AC_4\frac{N^2-1}{6}\left(B\varepsilon_1\frac{N^2-1}{6}\|u\|^{\alpha_2}+BC_6\frac{N^2-1}{6}\right)^{\alpha_1}+AC_5\frac{N^2-1}{6}\\
&\leq AC_4\frac{N^2-1}{6}\left[\left(\frac{3\|u\|}{4AC_4(N^2-1)}\right)^{\alpha_2}+BC_6\frac{N^2-1}{6}\right]^{\alpha_1}+AC_5\frac{N^2-1}{6}\\
&\leq AC_4\frac{N^2-1}{6}\left[\left(\frac{3\|(u,v)\|_Y}{4AC_4(N^2-1)}\right)^{\alpha_2}+BC_6\frac{N^2-1}{6}\right]^{\alpha_1}+AC_5\frac{N^2-1}{6},
\end{aligned}
\tag{16}
$$

and

$$
\begin{aligned}
v_n&\leq B\varepsilon_1\sum_{i=1}^{N-1}h(i)u_i^{\alpha_2}+BC_6\frac{N^2-1}{6}\\
&\leq B\varepsilon_1\sum_{i=1}^{N-1}h(i)\left(AC_4\sum_{j=1}^{N-1}h(j)v_j^{\alpha_1}+AC_5\frac{N^2-1}{6}\right)^{\alpha_2}+BC_6\frac{N^2-1}{6}\\
&\leq B\varepsilon_1\frac{N^2-1}{6}\left(AC_4\frac{N^2-1}{6}\|v\|^{\alpha_1}+AC_5\frac{N^2-1}{6}\right)^{\alpha_2}+BC_6\frac{N^2-1}{6}\\
&\leq \frac{6^{\alpha_2}}{8(AC_4(N^2-1))^{\alpha_2}}\left(AC_4\frac{N^2-1}{6}\|v\|^{\alpha_1}+AC_5\frac{N^2-1}{6}\right)^{\alpha_2}+BC_6\frac{N^2-1}{6}\\
&\leq \frac{6^{\alpha_2}}{8(AC_4(N^2-1))^{\alpha_2}}\left(AC_4\frac{N^2-1}{6}\|(u,v)\|_Y^{\alpha_1}+AC_5\frac{N^2-1}{6}\right)^{\alpha_2}+BC_6\frac{N^2-1}{6}.
\end{aligned}
\tag{17}
$$

By using (16), (17), and (15), we conclude that $u_n\leq\frac{1}{4}\|(u,v)\|_Y$ and $v_n\leq\frac{1}{4}\|(u,v)\|_Y$ for all $n=\overline{0,N}$. Therefore we obtain that $\|(u,v)\|_Y\leq\frac{1}{2}\|(u,v)\|_Y$, and so $\|(u,v)\|_Y=0$, which is a contradiction because $\|(u,v)\|_Y=R_1>0$. So, $(u,v)\not\leq(Q_1(u,v),Q_2(u,v))$ for all $(u,v)\in\partial B_{R_1}\cap P$. By Theorem 2.3(a), we deduce that the fixed point index

$$
i(Q,B_{R_1}\cap P,P)=1.
\tag{18}
$$

Because Q has no fixed points on $\partial B_{r_1}\cup\partial B_{R_1}$, by (12) and (18), we conclude that

$$
i\big(Q,(B_{R_1}\setminus\overline{B}_{r_1})\cap P,P\big)=i(Q,B_{R_1}\cap P,P)-i(Q,B_{r_1}\cap P,P)=1.
$$

So the operator Q has at least one fixed point $(u^1,v^1)\in(B_{R_1}\setminus\overline{B}_{r_1})\cap P$, with $r_1<\|(u^1,v^1)\|_Y<R_1$, that is, $\|u^1\|>0$ or $\|v^1\|>0$. Because $u^1\in P_1$ and $v^1\in P_2$, we obtain $u_n^1>0$ for all $n=\overline{1,N}$ or $v_n^1>0$ for all $n=\overline{1,N}$. □

Theorem 3.2 *Assume that* $(H1)$, $(H2)$, $(H5)$, *and* $(H6)$ *hold. Then problem* (S)–(BC) *has at least one positive solution.*

Proof By $(H5)$ there exist $C_i>0$, $i=7,\dots,11$, such that

$$
\begin{aligned}
&f(n,u,v)\geq C_7c(v)-C_8,\qquad g(n,u,v)\geq C_9d(u)-C_{10},\\
&\quad\forall(n,u,v)\in\{1,\dots,N-1\}\times\mathbb{R}_+\times\mathbb{R}_+,
\end{aligned}
\tag{19}
$$

and

$$c\big(C_{12}d(u)\big) \geq \frac{72C_{12}N^3\max\{A,B\}u}{C_7C_9(N^2-1)^2} - C_{11}, \quad \forall u \in \mathbb{R}_+, \tag{20}$$

where $C_{12} = \max\{\frac{C_9(N-1)}{N}h(i), i = \overline{1,N-1}\} > 0$. Then we obtain

$$\begin{aligned} Q_{1n}(u,v) &= \sum_{i=1}^{N-1} G_1(n,i)f(i,u_i,v_i) \geq \sum_{i=1}^{N-1} G_1(n,i)\big(C_7c(v_i) - C_8\big) \\ &\geq \sum_{i=1}^{N-1} h(i)k(n)\big(C_7c(v_i) - C_8\big) \geq \sum_{i=1}^{N-1} h(i)k(1)\big(C_7c(v_i) - C_8\big) \\ &= \frac{1}{N}\sum_{i=1}^{N-1} h(i)\big(C_7c(v_i) - C_8\big), \quad \forall n = \overline{1,N-1}, \\ Q_{2n}(u,v) &= \sum_{i=1}^{N-1} G_2(n,i)g(i,u_i,v_i) \geq \sum_{i=1}^{N-1} G_2(n,i)\big(C_9d(u_i) - C_{10}\big) \\ &\geq \sum_{i=1}^{N-1} h(i)k(n)\big(C_9d(u_i) - C_{10}\big) \geq \sum_{i=1}^{N-1} h(i)k(1)\big(C_9d(u_i) - C_{10}\big) \\ &= \frac{1}{N}\sum_{i=1}^{N-1} h(i)\big(C_9d(u_i) - C_{10}\big), \quad \forall n = \overline{1,N-1}. \end{aligned} \tag{21}$$

We will prove that the set $U = \{(u,v) \in P, (u,v) = Q(u,v) + \lambda(\varphi^1,\varphi^2), \lambda \geq 0\}$ is bounded, where $(\varphi^1,\varphi^2) \in P \setminus \{(0,0)\}$. Indeed, $(u,v) \in U$ implies that $u \geq Q_1(u,v)$, $v \geq Q_2(u,v)$ for some $\varphi^1,\varphi^2 \geq 0$. By (21), we obtain

$$u_n \geq Q_{1n}(u,v) \geq \frac{C_7}{N}\sum_{i=1}^{N-1} h(i)c(v_i) - C_{13}, \quad \forall n = \overline{1,N-1}, \tag{22}$$

$$v_n \geq Q_{2n}(u,v) \geq \frac{C_9}{N}\sum_{i=1}^{N-1} h(i)d(u_i) - C_{14}, \quad \forall n = \overline{1,N-1}, \tag{23}$$

where $C_{13} = C_8(N^2-1)/(6N)$, $C_{14} = C_{10}(N^2-1)/(6N)$.

By the monotonicity and concavity of $c(\cdot)$ and the Jensen inequality, inequality (23) implies that

$$\begin{aligned} c(v_n + C_{14}) &\geq c\left(\frac{C_9}{N}\sum_{i=1}^{N-1} h(i)d(u_i)\right) \\ &\geq \frac{1}{N-1}\sum_{i=1}^{N-1} c\left(\frac{C_9(N-1)}{N}h(i)d(u_i)\right) \\ &= \frac{1}{N-1}\sum_{i=1}^{N-1} c\left(\frac{C_9(N-1)}{NC_{12}}h(i)\cdot C_{12}d(u_i)\right) \end{aligned}$$

$$\geq \frac{1}{N-1}\sum_{i=1}^{N-1}\frac{C_9(N-1)}{NC_{12}}h(i)c\big(C_{12}d(u_i)\big)$$
$$= \frac{C_9}{NC_{12}}\sum_{i=1}^{N-1}h(i)c\big(C_{12}d(u_i)\big), \quad \forall n=\overline{1,N-1}. \tag{24}$$

Since $c(v_n) \geq c(v_n + C_{14}) - c(C_{14})$, by relations (22), (23), and (24), we deduce

$$\begin{aligned}
u_n &\geq \frac{C_7}{N}\sum_{i=1}^{N-1}h(i)c(v_i) - C_{13} \\
&\geq \frac{C_7}{N}\sum_{i=1}^{N-1}h(i)\big[c(v_i+C_{14}) - c(C_{14})\big] - C_{13} \\
&= \frac{C_7}{N}\sum_{i=1}^{N-1}h(i)c(v_i+C_{14}) - C_{15} \\
&\geq \frac{C_7}{N}\sum_{i=1}^{N-1}h(i)\left[\frac{C_9}{NC_{12}}\sum_{j=1}^{N-1}h(j)c\big(C_{12}d(u_j)\big)\right] - C_{15} \\
&= \frac{C_7C_9(N^2-1)}{6N^2C_{12}}\sum_{j=1}^{N-1}h(j)c\big(C_{12}d(u_j)\big) - C_{15} \\
&\geq \frac{C_7C_9(N^2-1)}{6N^2C_{12}}\sum_{j=1}^{N-1}h(j)\left(\frac{72C_{12}N^3\max\{A,B\}}{C_7C_9(N^2-1)^2}u_j - C_{11}\right) - C_{15} \\
&= \frac{12N\max\{A,B\}}{N^2-1}\sum_{j=1}^{N-1}h(j)u_j - C_{16} \geq 2\|u\| - C_{16}, \quad \forall n=\overline{1,N-1},
\end{aligned}$$

where $C_{15} = \frac{C_7c(C_{14})(N^2-1)}{6N} + C_{13}$, $C_{16} = \frac{C_7C_9C_{11}(N^2-1)^2}{36N^2C_{12}} + C_{15}$.

Therefore $\|u\| \geq u_1 \geq 2\|u\| - C_{16}$, and then

$$\|u\| \leq C_{16}. \tag{25}$$

Since $c(v_n) \geq c(\frac{1}{B}k(n)\|v\|) \geq c(\frac{1}{BN}\|v\|) \geq \frac{1}{BN}c(\|v\|)$ for all $n = \overline{1,N-1}$, then by relations (19), (22), (23), and (24), we obtain

$$\begin{aligned}
c(v_n) &\geq c(v_n + C_{14}) - c(C_{14}) \\
&\geq \frac{C_9}{NC_{12}}\sum_{i=1}^{N-1}h(i)c\big(C_{12}d(u_i)\big) - c(C_{14}) \\
&\geq \frac{C_9}{NC_{12}}\sum_{i=1}^{N-1}h(i)\left(\frac{72C_{12}N^3\max\{A,B\}}{C_7C_9(N^2-1)^2}u_i - C_{11}\right) - c(C_{14}) \\
&= \frac{72N^2\max\{A,B\}}{C_7(N^2-1)^2}\sum_{i=1}^{N-1}h(i)u_i - C_{17} \\
&\geq \frac{72N^2\max\{A,B\}}{C_7(N^2-1)^2}\sum_{i=1}^{N-1}h(i)\left(\frac{C_7}{N}\sum_{j=1}^{N-1}h(j)c(v_j) - C_{13}\right) - C_{17}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{12N\max\{A,B\}}{N^2-1}\sum_{j=1}^{N-1} h(j)c(v_j) - C_{18} \\
&\geq \frac{12N\max\{A,B\}}{N^2-1}\sum_{j=1}^{N-1} h(j)\frac{1}{BN}c(\|v\|) - C_{18} \\
&\geq 2c(\|v\|) - C_{18}, \quad \forall n = \overline{1,N-1},
\end{aligned}
$$

where $C_{17} = \frac{C_9C_{11}(N^2-1)}{6NC_{12}} + c(C_{14})$, $C_{18} = \frac{12C_{13}N^2\max\{A,B\}}{C_7(N^2-1)} + C_{17}$.

Then $c(\|v\|) \geq c(v_1) \geq 2c(\|v\|) - C_{18}$, and so $c(\|v\|) \leq C_{18}$. By $(H5)$(a) and (c), we deduce that $\lim_{v\to\infty} c(v) = \infty$. Thus there exists $C_{19} > 0$ such that

$$\|v\| \leq C_{19}. \tag{26}$$

By (25) and (26), we conclude that $\|(u,v)\|_Y \leq C_{16} + C_{19}$ for all $(u,v) \in U$. That is the set U is bounded. Then there exists a sufficiently large $R_2 > 0$ such that $(u,v) \neq Q(u,v) + \lambda(\varphi^1,\varphi^2)$ for all $(u,v) \in \partial B_{R_2} \cap P$ and $\lambda \geq 0$. By Theorem 2.2 we deduce that

$$i(Q, B_{R_2} \cap P, P) = 0. \tag{27}$$

On the other hand, by $(H6)$ there exist $C_{20} > 0$ and a sufficiently small $r_2 > 0$, ($r_2 < R_2$, $r_2 \leq 1$) such that

$$
\begin{aligned}
&f(n,u,v) \leq C_{20}v^{\beta_1}, \quad \forall (n,u) \in \{1,\ldots,N-1\} \times \mathbb{R}_+, v \in [0,r_2], \\
&g(n,u,v) \leq \varepsilon_2 u^{\beta_2}, \quad \forall (n,v) \in \{1,\ldots,N-1\} \times \mathbb{R}_+, u \in [0,r_2],
\end{aligned}
\tag{28}
$$

where $\varepsilon_2 = (2AB^{\beta_1}C_{20}(\frac{N^2-1}{6})^{\beta_1+1})^{-1/\beta_1} > 0$.

We will show that $(u,v) \not\leq Q(u,v)$ for all $(u,v) \in \partial B_{r_2} \cap P$. We suppose that there exists $(u,v) \in \partial B_{r_2} \cap P$, that is, $\|(u,v)\|_Y = r_2 \leq 1$, such that $(u,v) \leq (Q_1(u,v), Q_2(u,v))$, or $u \leq Q_1(u,v)$ and $v \leq Q_2(u,v)$. Then by (28) we obtain

$$
\begin{aligned}
u_n &\leq Q_{1n}(u,v) = \sum_{i=1}^{N-1} G_1(n,i)f(i,u_i,v_i) \leq AC_{20}\sum_{i=1}^{N-1} h(i)v_i^{\beta_1} \\
&\leq AC_{20}\sum_{i=1}^{N-1} h(i)\left(\sum_{j=1}^{N-1} G_2(i,j)g(j,u_j,v_j)\right)^{\beta_1} \\
&\leq AC_{20}\sum_{i=1}^{N-1} h(i)\left(B\sum_{j=1}^{N-1} h(j)\varepsilon_2 u_j^{\beta_2}\right)^{\beta_1} \\
&\leq \frac{AB^{\beta_1}C_{20}(N^2-1)\varepsilon_2^{\beta_1}}{6}\left(\sum_{j=1}^{N-1} h(j)\right)^{\beta_1} \|u\|^{\beta_1\beta_2} \\
&= AB^{\beta_1}C_{20}\left(\frac{N^2-1}{6}\right)^{\beta_1+1} \varepsilon_2^{\beta_1}\|u\|^{\beta_1\beta_2} \\
&\leq AB^{\beta_1}C_{20}\varepsilon_2^{\beta_1}\left(\frac{N^2-1}{6}\right)^{\beta_1+1}\|u\| = \frac{1}{2}\|u\|, \quad \forall n = \overline{0,N}.
\end{aligned}
$$

Therefore $\|u\| \le \frac{1}{2}\|u\|$, so

$$\|u\| = 0. \tag{29}$$

In addition

$$\begin{aligned} v_n \le Q_{2n}(u,v) &= \sum_{i=1}^{N-1} G_2(n,i)g(i,u_i,v_i) \\ &\le B\sum_{i=1}^{N-1} h(i)\varepsilon_2 u_i^{\beta_2} \le \frac{B\varepsilon_2(N^2-1)}{6}\|u\|^{\beta_2}, \quad \forall n=\overline{0,N}. \end{aligned} \tag{30}$$

By (29) and (30) we deduce that $\|v\| = 0$, and then $\|(u,v)\|_Y = 0$, which is a contradiction because $\|(u,v)\|_Y = r_2 > 0$. Then $(u,v) \not\le Q(u,v)$ for all $(u,v) \in \partial B_{r_2} \cap P$. By Theorem 2.3(a), we conclude that

$$i(Q, B_{r_2} \cap P, P) = 1. \tag{31}$$

Because Q has no fixed points on $\partial B_{r_2} \cup \partial B_{R_2}$, by (27) and (31), we deduce that

$$i\bigl(Q,(B_{R_2} \setminus \overline{B}_{r_2}) \cap P, P\bigr) = i(Q, B_{R_2} \cap P, P) - i(Q, B_{r_2} \cap P, P) = -1.$$

So the operator Q has at least one fixed point $(u^2,v^2) \in (B_{R_2} \setminus \overline{B}_{r_2}) \cap P$, with $r_2 < \|(u^2,v^2)\|_Y < R_2$, which is a positive solution for our problem (S)–(BC). □

Theorem 3.3 *Assume that assumptions* $(H1)$, $(H2)$, $(H3)$, $(H5)$, *and* $(H7)$ *hold. Then problem* (S)–(BC) *has at least two positive solutions.*

Proof By using $(H7)$, for any $(u,v) \in \partial B_{N_0} \cap P$, we obtain

$$\begin{aligned} Q_{1n}(u,v) &\le A\sum_{i=1}^{N-1} h(i)f(i,N_0,N_0) < \frac{3AN_0}{(N^2-1)\max\{A,B\}}\sum_{i=1}^{N-1} h(i) \le \frac{N_0}{2}, \quad \forall n=\overline{0,N}, \\ Q_{2n}(u,v) &\le B\sum_{i=1}^{N-1} h(i)g(i,N_0,N_0) < \frac{3BN_0}{(N^2-1)\max\{A,B\}}\sum_{i=1}^{N-1} h(i) \le \frac{N_0}{2}, \quad \forall n=\overline{0,N}. \end{aligned}$$

Then we deduce

$$\bigl\|Q(u,v)\bigr\|_Y = \bigl\|Q_1(u,v)\bigr\| + \bigl\|Q_2(u,v)\bigr\| < N_0 = \bigl\|(u,v)\bigr\|_Y, \quad \forall (u,v) \in \partial B_{N_0} \cap P.$$

Because Q has no fixed points on ∂B_{N_0}, by Theorem 2.1 we conclude that

$$i(Q, B_{N_0} \cap P, P) = 1. \tag{32}$$

On the other hand, from $(H3)$ and $(H5)$, and the proofs of Theorems 3.1 and 3.2, we know that there exist a sufficiently $r_1 > 0$ $(r_1 < N_0)$ and a sufficiently large $R_2 > N_0$ such that

$$i(Q, B_{r_1} \cap P, P) = 0, \qquad i(Q, B_{R_2} \cap P, P) = 0. \tag{33}$$

Because Q has no fixed points on $\partial B_{r_1} \cup \partial B_{R_2} \cup \partial B_{N_0}$, by relations (32) and (33), we obtain

$$i\big(\mathcal{Q}, (B_{R_2} \setminus \bar{B}_{N_0}) \cap P, P\big) = i(\mathcal{Q}, B_{R_2} \cap P, P) - i(\mathcal{Q}, B_{N_0} \cap P, P) = -1,$$
$$i\big(\mathcal{Q}, (B_{N_0} \setminus \bar{B}_{r_1}) \cap P, P\big) = i(\mathcal{Q}, B_{N_0} \cap P, P) - i(\mathcal{Q}, B_{r_1} \cap P, P) = 1.$$

Then $\mathcal{Q}$ has at least one fixed point $(u^1, v^1) \in (B_{R_2} \setminus \bar{B}_{N_0}) \cap P$ and has at least one fixed point $(u^2, v^2) \in (B_{N_0} \setminus \bar{B}_{r_1}) \cap P$. Therefore, problem (S)–(BC) has two distinct positive solutions (u^1, v^1), (u^2, v^2). □

Remark 3.1 In $(H3)$, if $a(v) = v^p$ with $p \leq 1$ and $b(u) = u^q$ with $q > 0$, the condition from $(H3)$(c) is satisfied if $pq < 1$. In $(H5)$, if $c(v) = v^p$ with $p \leq 1$, and $d(u) = u^q$ with $q > 0$, the condition from $(H5)$(c) is satisfied if $pq > 1$.

Examples

(1) We consider $f(n,u,v) = \frac{n}{n+1}(1 + e^{-(u+v)})$ and $g(n,u,v) = (1 + e^{-n})u^{\theta}$ for $(n,u,v) \in \{1, \ldots, N-1\} \times \mathbb{R}_+ \times \mathbb{R}_+$. For $a(v) = v^p$ with $p \leq 1$, and $b(u) = u^q$ for $q > 0$ and $pq < 1$, then assumptions $(H3)$ and $(H4)$ are satisfied if $q > \theta$ and $\alpha_2 > \theta$. For example, if $\theta = \frac{5}{4}$, $p = \frac{1}{3}$, $q = \frac{4}{3}$, $\alpha_1 = \frac{1}{3}$, and $\alpha_2 = 3$, we can apply Theorem 3.1, and we deduce that problem (S)–(BC) has at least one positive solution.

(2) We consider $f(n,u,v) = (1 + e^{-u})v^{\theta_1}$ and $g(n,u,v) = (1 + e^{-v})u^{\theta_2}$ for $(n,u,v) \in \{1, \ldots, N-1\} \times \mathbb{R}_+ \times \mathbb{R}_+$. For $c(v) = v^p$ with $p \leq 1$, and $d(u) = u^q$ for $q > 0$ and $pq > 1$, then assumptions $(H5)$ and $(H6)$ are satisfied if $p < \theta_1$, $q < \theta_2$, $\beta_1 < \theta_1$, and $\beta_2 < \theta_2$. For example, if $\theta_1 = 4$, $\theta_2 = 2$, $p = \frac{3}{5}$, $q = \frac{9}{5}$, $\beta_1 = 3$, and $\beta_2 = \frac{1}{3}$, we can apply Theorem 3.2, and we conclude that problem (S)–(BC) has at least one positive solution.

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