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Research Article

Positive Solutions for a System of Semipositone Fractional Difference Boundary Value Problems

Cheng Chen,[1] Jiafa Xu,[1] Donal O'Regan,[2] and Zhengqing Fu[3]

[1] *School of Mathematical Sciences, Chongqing Normal University, Chongqing 40133, China*
[2] *School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland*
[3] *College of Mathematics and System Sciences, Shandong University of Science and Technology, Qingdao 266590, China*

Correspondence should be addressed to Jiafa Xu; xujiafa292@sina.com

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Using the fixed point index, we establish two existence theorems for positive solutions to a system of semipositone fractional difference boundary value problems. We adopt nonnegative concave functions and nonnegative matrices to characterize the coupling behavior of our nonlinear terms.

1. Introduction

In this paper we study the existence of positive solutions for the system of fractional difference boundary value problems involving semipositone nonlinearities:

$$\begin{aligned} -\Delta_{\nu-3}^{\nu} x(t) &= f(t+\nu-1, x(t+\nu-1), y(t+\nu-1)), \\ &\qquad t \in [0, b+2]_{\mathbb{N}_0}, \\ -\Delta_{\nu-3}^{\nu} y(t) &= g(t+\nu-1, x(t+\nu-1), y(t+\nu-1)), \\ &\qquad t \in [0, b+2]_{\mathbb{N}_0}, \\ x(\nu-3) &= \left[\Delta_{\nu-3}^{\alpha} x(t)\right]\Big|_{t=\nu-\alpha-2} \\ &= \left[\Delta_{\nu-3}^{\beta} x(t)\right]\Big|_{t=\nu+b+2-\beta} = 0, \\ y(\nu-3) &= \left[\Delta_{\nu-3}^{\alpha} y(t)\right]\Big|_{t=\nu-\alpha-2} \\ &= \left[\Delta_{\nu-3}^{\beta} y(t)\right]\Big|_{t=\nu+b+2-\beta} = 0, \end{aligned} \tag{1}$$

where $2 < \nu \leq 3$, $1 < \beta < 2$, $\nu - \beta > 1$, $0 < \alpha < 1$, $b > 3$ ($b \in \mathbb{N}$), and $\Delta_{\nu-3}^{\nu}$ is a discrete fractional operator. For the nonlinear terms f, g, we assume the following.

(H0) $f, g : [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}} \times \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ are two continuous functions; moreover, there exists a positive constant $M > 0$ such that

$$\begin{aligned} &f, g(t, x, y) \geq -M, \\ &\quad \text{for all } (t, x, y) \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}} \times \mathbb{R}^+ \times \mathbb{R}^+. \end{aligned} \tag{2}$$

Note that, in this paper, we use $[a, b]_{\mathbb{N}_a}$ to stand for $\{a, a+1, a+2, \ldots, b\}$ with $b-a \in \mathbb{N}_1$, where $\mathbb{N}_a := \{a, a+1, a+2, \cdots\}$.

Fractional calculus has been applied in physics, chemistry, aerodynamics, biophysics, and blood flow phenomena. For example, $CD4^+T$ cells' infections can be depicted by a fractional order model

$$\begin{aligned} D^{\alpha_1}(T) &= s - KVT - dT + bI, \\ D^{\alpha_2}(I) &= KVT - (b+\delta) I, \\ D^{\alpha_3}(V) &= N\delta I - cV, \end{aligned} \tag{3}$$

where D^{α_i} $(i = 1, 2, 3)$ are fractional derivatives (see [1, 2]); we also refer the reader to [1–45] and the references therein. In [3], the authors considered the existence of positive solutions for the semipositone discrete fractional system

$$\begin{aligned} &-\Delta^{\nu_1} y_1(t) \\ &\quad = \lambda_1 f_1(t+\nu_1-1, y_1(t+\nu_1-1), y_2(t+\nu_2-1)), \\ &\qquad t \in [1, b+1]_{\mathbb{N}}, \end{aligned}$$

$$
\begin{aligned}
&-\Delta^{\nu_2} y_2(t) \\
&\quad = \lambda_2 f_2\left(t+\nu_2-1, y_1\left(t+\nu_1-1\right), y_2\left(t+\nu_2-1\right)\right), \\
&\qquad t \in [1, b+1]_{\mathbb{N}}, \\
&y_1\left(\nu_1-2\right) = y_1\left(\nu_1+b+1\right) = 0, \\
&y_2\left(\nu_2-2\right) = y_2\left(\nu_2+b+1\right) = 0,
\end{aligned} \tag{4}
$$

where $\nu_1, \nu_2 \in (1, 2]$. Using the Guo-Krasnosel'skiĭ fixed point theorem, the authors showed that the problem has positive solutions for sufficiently small values of $\lambda_1, \lambda_2 > 0$. The growth conditions on f_i $(i = 1, 2)$ are superlinear; i.e.,

$$
\begin{aligned}
\lim_{y_1+y_2 \longrightarrow +\infty} \frac{f_i(t, x, y)}{y_1+y_2} &= +\infty, \\
\lim_{y_1+y_2 \longrightarrow 0^+} \frac{f_i(t, x, y)}{y_1+y_2} &= 0,
\end{aligned} \tag{5}
$$

uniformly for $t \in [\nu_i, \nu_i + b]_{\mathbb{N}_{\nu_i - 2}}$. Using conditions of (5) type the existence of solutions for various fractional boundary value problems was considered in [1, 4–6, 9, 11–13].

In this paper, we use the fixed point index to obtain two existence theorems for positive solutions to (1) with semipositone nonlinearities. We adopt some appropriate nonnegative concave functions and nonnegative matrices to characterize the coupling behavior of our nonlinear terms. Moreover, the growth conditions on $+\infty$ of our nonlinearities f, g are an improvement of (5); see conditions (H1) and (H3) in Section 3.

2. Preliminaries

We first recall some background materials from discrete fractional calculus; for more details we refer the reader to [10].

Definition 1. We define $t^{\underline{\nu}} := \Gamma(t+1)/\Gamma(t+1-\nu)$ for any $t, \nu \in \mathbb{R}$ for which the right-hand side is well-defined. We use the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} = 0$.

Definition 2. For $\nu > 0$, the ν–th fractional sum of a function f is

$$
\Delta_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s), \quad \text{for } t \in \mathbb{N}_{a+\nu}. \tag{6}
$$

We also define the ν–th fractional difference for $\nu > 0$ by

$$
\Delta_a^{\nu} f(t) = \Delta^N \Delta_a^{\nu-N} f(t), \quad \text{for } t \in \mathbb{N}_{a+N-\nu}, \tag{7}
$$

where $N \in \mathbb{N}$ with $0 \leq N-1 < \nu \leq N$.

Let $h : [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}} \longrightarrow \mathbb{R}$ be a continuous function. Then we consider the fractional difference boundary value problems

$$
\begin{aligned}
-\Delta_{\nu-3}^{\nu} y(t) &= h(t+\nu-1), \quad t \in [0, b+2]_{\mathbb{N}_0}, \\
y(\nu-3) &= \left[\Delta_{\nu-3}^{\alpha} y(t)\right]\Big|_{t=\nu-\alpha-2} \\
&= \left[\Delta_{\nu-3}^{\beta} y(t)\right]\Big|_{t=\nu+b+2-\beta} = 0,
\end{aligned} \tag{8}
$$

where ν, α, β, b are as in (1). The following two lemmas are in [9], so we omit their proofs.

Lemma 3 (see [9], Lemma 4). *Problem (8) has a unique solution*

$$
y(t) = \sum_{s=0}^{b+2} G(t, s) h(s+\nu-1), \qquad t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}, \tag{9}
$$

where

$$
G(t, s) = \frac{1}{\Gamma(\nu)} \begin{cases} \dfrac{t^{\underline{\nu-1}} (\nu+b-\beta-s+1)^{\underline{\nu-\beta-1}}}{(\nu+b-\beta+2)^{\underline{\nu-\beta-1}}} - (t-s-1)^{\underline{\nu-1}}, & 0 \leq s < t-\nu+1 \leq b+2, \\ \dfrac{t^{\underline{\nu-1}} (\nu+b-\beta-s+1)^{\underline{\nu-\beta-1}}}{(\nu+b-\beta+2)^{\underline{\nu-\beta-1}}}, & 0 \leq t-\nu+1 \leq s \leq b+2. \end{cases} \tag{10}
$$

Lemma 4 (see [9], Lemma 5). *Green's function (10) has the following properties.*

(i) $G(t, s) > 0$, $(t, s) \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}} \times [0, b+2]_{\mathbb{N}_0}$,

(ii) $q^*(t) G(b+\nu+1, s) \leq G(t, s) \leq G(b+\nu+1, s)$, $(t, s) \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}} \times [0, b+2]_{\mathbb{N}_0}$, *where* $q^*(t) = t^{\underline{\nu-1}}/(b+\nu+1)^{\underline{\nu-1}}$.

Let $\varphi(s+\nu-1) = G(b+\nu+1, s)$ *for* $s \in [0, b+2]_{\mathbb{N}_0}$. *Then* $\varphi(t) = G(b+\nu+1, t-\nu+1)$ *for* $t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}$. *From Lemma* **??**, *the following inequalities are satisfied:*

$$
\begin{aligned}
\sum_{t=\nu-1}^{b+\nu+1} q^*(t) \varphi(t) \cdot \varphi(s+\nu-1) &\leq \sum_{t=\nu-1}^{b+\nu+1} G(t, s) \varphi(t) \\
&\leq \sum_{t=\nu-1}^{b+\nu+1} \varphi(t) \cdot \varphi(s+\nu-1), \quad \text{for } s \in [0, b+2]_{\mathbb{N}_0}.
\end{aligned} \tag{11}
$$

For convenience, we let

$$\kappa_1 = \sum_{t=\nu-1}^{b+\nu+1} q^*(t)\varphi(t) \quad \text{and}$$
$$\kappa_2 = \sum_{t=\nu-1}^{b+\nu+1} \varphi(t). \tag{12}$$

Let E be the collection of all maps from $[\nu-3, b+\nu+1]_{\mathbb{N}_{\nu-3}}$ to $\mathbb{R}$ equipped with the max norm, $\|\cdot\|$. Then E is a Banach space. Define a set $P \subset E$ by $P = \{y \in E : y(t) \geq 0, t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}\}$. Then P is a cone in E. Note that $E \times E$ is a Banach space with the norm $\|(x,y)\| := \max\{\|x\|, \|y\|\}$, and $P \times P$ is a cone in $E \times E$.

From Lemma 3, for all $t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}$, we have that (1) is equivalent to

$$\begin{aligned} x(t) &= \sum_{s=0}^{b+2} G(t,s) \\ &\quad \cdot f(s+\nu-1, x(s+\nu-1), y(s+\nu-1)), \\ y(t) &= \sum_{s=0}^{b+2} G(t,s) \\ &\quad \cdot g(s+\nu-1, x(s+\nu-1), y(s+\nu-1)), \end{aligned} \tag{13}$$

where G is defined in (10).

Lemma 5 (see [46]). *Let E be a real Banach space and P a cone on E. Suppose that $\Omega \subset E$ is a bounded open set and that $A : \overline{\Omega} \cap P \longrightarrow P$ is a continuous compact operator. If there exists $\omega_0 \in P \setminus \{0\}$ such that*

$$\omega - A\omega \neq \lambda\omega_0, \quad \forall \lambda \geq 0, \ \omega \in \partial\Omega \cap P, \tag{14}$$

then $i(A, \Omega \cap P, P) = 0$, where i denotes the fixed point index on P.

Lemma 6 (see [46]). *Let E be a real Banach space and P a cone on E. Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$ and that $A : \overline{\Omega} \cap P \longrightarrow P$ is a continuous compact operator. If*

$$\omega - \lambda A\omega \neq 0, \quad \forall \lambda \in [0,1], \ \omega \in \partial\Omega \cap P, \tag{15}$$

then $i(A, \Omega \cap P, P) = 1$.

3. Main Results

Let ω be a solution of

$$\begin{aligned} -\Delta_{\nu-3}^{\nu} y(t) &= 1, \quad t \in [0, b+2]_{\mathbb{N}_0}, \\ y(\nu-3) &= \left[\Delta_{\nu-3}^{\alpha} y(t)\right]\Big|_{t=\nu-\alpha-2} \\ &= \left[\Delta_{\nu-3}^{\beta} y(t)\right]\Big|_{t=\nu+b+2-\beta} = 0, \end{aligned} \tag{16}$$

where ν, α, β, b are as in (1). Define $z = M\omega$, and then, from Lemmas 3 and **??**, we have

$$\begin{aligned} z(t) &= M\omega(t) = M\sum_{s=0}^{b+2} G(t,s) \leq M\sum_{s=0}^{b+2} \varphi(s+\nu-1) \\ &= M\sum_{s=\nu-1}^{b+\nu+1} \varphi(s) = M\kappa_2. \end{aligned} \tag{17}$$

We note that (1) has a positive solution $(x,y) \in (P \times P) \setminus \{\mathbf{0}\}$ if and only if $(\tilde{x}, \tilde{y}) = (x+z, y+z)$ is a solution of the fractional difference boundary value problems

$$\begin{aligned} -\Delta_{\nu-3}^{\nu} x(t) &= \tilde{f}(t+\nu-1, x(t+\nu-1) \\ &\quad - z(t+\nu-1), y(t+\nu-1) - z(t+\nu-1)), \\ &\qquad t \in [0, b+2]_{\mathbb{N}_0}, \\ -\Delta_{\nu-3}^{\nu} y(t) &= \tilde{g}(t+\nu-1, x(t+\nu-1) \\ &\quad - z(t+\nu-1), y(t+\nu-1) - z(t+\nu-1)), \\ &\qquad t \in [0, b+2]_{\mathbb{N}_0}, \\ x(\nu-3) &= \left[\Delta_{\nu-3}^{\alpha} x(t)\right]\Big|_{t=\nu-\alpha-2} \\ &= \left[\Delta_{\nu-3}^{\beta} x(t)\right]\Big|_{t=\nu+b+2-\beta} = 0, \\ y(\nu-3) &= \left[\Delta_{\nu-3}^{\alpha} y(t)\right]\Big|_{t=\nu-\alpha-2} \\ &= \left[\Delta_{\nu-3}^{\beta} y(t)\right]\Big|_{t=\nu+b+2-\beta} = 0, \end{aligned} \tag{18}$$

and $(\tilde{x}, \tilde{y})(t) \geq (z,z)(t)$ for $t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}$, where ν, α, β, b are as in (1) and

$$\tilde{f}(t,x,y) = \begin{cases} f(t,x,y) + M, & t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}, \ x, y \geq 0, \\ f(t,x,0) + M, & t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}, \ x > 0, \ y < 0, \\ f(t,0,y) + M, & t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}, \ x < 0, \ y > 0, \\ f(t,0,0) + M, & t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}, \ x, y < 0, \end{cases} \tag{19}$$

and

$$\tilde{g}(t,x,y) = \begin{cases} g(t,x,y) + M, & t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}, \ x, y \geq 0, \\ g(t,x,0) + M, & t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}, \ x > 0, y < 0, \\ g(t,0,y) + M, & t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}, \ x < 0, y > 0, \\ g(t,0,0) + M, & t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}, \ x, y < 0. \end{cases} \tag{20}$$

Note that for $(x_1, y_1)(t) \geq (x_2, y_2)(t)$, we mean $x_1(t) \geq x_2(t)$, $y_1(t) \geq y_2(t)$ for all $t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}$.

For $(x,y)\in P\times P$, and $t\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}$, we define the operators

$$\begin{aligned}B_1(x,y)(t)&=\sum_{s=0}^{b+2}G(t,s)\tilde{f}(s+\nu-1,x(s+\nu-1)\\&\quad-z(s+\nu-1),y(s+\nu-1)-z(s+\nu-1)),\\B_2(x,y)(t)&=\sum_{s=0}^{b+2}G(t,s)\tilde{g}(s+\nu-1,x(s+\nu-1)\\&\quad-z(s+\nu-1),y(s+\nu-1)-z(s+\nu-1)),\end{aligned}\tag{21}$$

and

$$B(x,y)(t)=(B_1,B_2)(x,y)(t).\tag{22}$$

Then (H0) and using the Arzelà-Ascoli theorem in a standard way establish that $B:P\times P\longrightarrow P\times P$ is a completely continuous operator. It is clear that $(x,y)\in(P\times P)\setminus\{\mathbf{0}\}$ is a positive solution for (18) if and only if $(x,y)\in(P\times P)\setminus\{\mathbf{0}\}$ is a fixed point of B.

Let $P_0=\{y\in P:y(t)\geq q^*(t)\|y\|,\forall t\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}\}$. Then from Lemma **??** we have

$$B_i(P\times P)\subset P_0,\quad i=1,2.\tag{23}$$

If we seek a fixed point $(\tilde{x},\tilde{y})$ of B then $\tilde{x},\tilde{y}\in P_0$ and

$$\begin{aligned}w(t)-z(t)\geq q^*(t)\|w\|-M\kappa_2\geq q_0\|w\|-M\kappa_2,\\\text{for } t\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}},\end{aligned}\tag{24}$$

where $w=\tilde{x},\tilde{y}$, and $q_0=\min_{t\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}}q^*(t)>0$, so as a result if $\|\tilde{x}\|,\|\tilde{y}\|\geq q_0^{-1}M\kappa_2$ then $(\tilde{x},\tilde{y})(t)\geq(z,z)(t)$ for $t\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}$ (i.e., $(\tilde{x}-z,\tilde{y}-z)(t)$ is a positive solution for (1)).

For convenience, we use $c_1,c_2,\ldots$ to stand for different positive constants. Let $B_\varrho:=\{x\in E:\|x\|<\varrho\}$ for $\varrho>0$. Now, we list our assumptions on $\tilde{f},\tilde{g}$ (the first two are needed for Theorem 10 and the last two are needed for Theorem 11):

(H1) There exist $p,q\in C(\mathbb{R}^+,\mathbb{R}^+)$ such that
(i) p is concave and strictly increasing on $\mathbb{R}^+$,
(ii) there exists $c_1>0$ such that

$$\begin{aligned}\tilde{f}(t,x,y)&\geq x+p(y)-c_1,\\\tilde{g}(t,x,y)&\geq q(x)-c_1,\\&\forall(t,x,y)\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}\times\mathbb{R}^+\times\mathbb{R}^+,\end{aligned}\tag{25}$$

(iii) there is a $\gamma_1>0$ such that $\kappa_1(1+\gamma_1\kappa_1)>1$ and

$$\begin{aligned}&p(\kappa_2q(x(t+\nu-1)-z(t+\nu-1)))\\&\quad\geq\kappa_2\gamma_1(x(t+\nu-1)-z(t+\nu-1))-c_1\end{aligned}\tag{26}$$

for $x\in\mathbb{R}^+$ and $t\in[0,b+2]_{\mathbb{N}_0}$.

(H2) For any $(t,x,y)\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}\times[0,q_0^{-1}M\kappa_2]\times[0,q_0^{-1}M\kappa_2]$, assume

$$\tilde{f},\tilde{g}(t,x,y)<q_0^{-1}M.\tag{27}$$

(H3) There exist $e_i\geq0$ $(i=1,2,3,4)$ with $e_1^2+e_2^2\neq0,e_3^2+e_4^2\neq0$ such that
(i) $\kappa=(1-e_1\kappa_2)(1-e_4\kappa_2)-e_2e_3\kappa_2^2>0$, $e_1,e_4<\kappa_2^{-1}$,
(ii) there exist $c_2>0$ such that

$$\begin{aligned}\begin{pmatrix}\tilde{f}(t,x,y)\\\tilde{g}(t,x,y)\end{pmatrix}\leq\begin{pmatrix}e_1x+e_2y+c_2\\e_3x+e_4y+c_2\end{pmatrix},\\\forall(t,x,y)\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}\times\mathbb{R}^+\times\mathbb{R}^+.\end{aligned}\tag{28}$$

(H4) For any $(t,x,y)\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}\times[0,q_0^{-1}M\kappa_2]\times[0,q_0^{-1}M\kappa_2]$, assume

$$\tilde{f},\tilde{g}(t,x,y)>q_0^{-2}M.\tag{29}$$

Example 7. Let $p(y)=y^{19/20}$ and $q(x)=x^2$ for $x,y\in\mathbb{R}^+$. Then, for any $\omega>0$, we have

$$\liminf_{x\longrightarrow+\infty}\frac{p(\omega q(x))}{x}=\liminf_{x\longrightarrow+\infty}\frac{\omega^{19/20}x^{19/10}}{x}=+\infty.\tag{30}$$

Let $f(t,x,y)=(1/2\kappa_2)x+(1/2\kappa_2e^{\beta_1+\cos(tx)})y-M$ and $g(t,x,y)=(1/e^{\beta_2+\sin(tx)})(q_0^{-1}M)^{1-\beta_3}\kappa_2^{-\beta_3}x^{\beta_3}-M$, where $\beta_1,\beta_2>1$, $\beta_3>2$, for $(t,x,y)\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}\times\mathbb{R}^+\times\mathbb{R}^+$. Then, for any $(t,x,y)\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}\times[0,q_0^{-1}M\kappa_2]\times[0,q_0^{-1}M\kappa_2]$, we have

$$\begin{aligned}f(t,x,y)+M&=\frac{1}{2\kappa_2}x+\frac{1}{2\kappa_2e^{\beta_1+\cos(tx)}}y\\&<\frac{1}{2\kappa_2}q_0^{-1}M\kappa_2+\frac{1}{2\kappa_2}q_0^{-1}M\kappa_2\\&=q_0^{-1}M,\end{aligned}\tag{31}$$

and

$$\begin{aligned}g(t,x,y)+M&=\frac{1}{e^{\beta_2+\sin(tx)}}(q_0^{-1}M)^{1-\beta_3}\kappa_2^{-\beta_3}x^{\beta_3}\\&<(q_0^{-1}M)^{1-\beta_3}\kappa_2^{-\beta_3}(q_0^{-1}M\kappa_2)^{\beta_3}\\&=q_0^{-1}M.\end{aligned}\tag{32}$$

Also,

$$\begin{aligned}&\liminf_{x\longrightarrow+\infty,y\longrightarrow+\infty}\frac{f(t,x,y)+M}{x+p(y)}\\&=\liminf_{x\longrightarrow+\infty,y\longrightarrow+\infty}\frac{(1/2\kappa_2)x+(1/2\kappa_2e^{\beta_1+\cos(tx)})y}{x+y^{19/20}}\\&=+\infty,\quad\text{for } t\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}},\end{aligned}\tag{33}$$

and

$$\begin{aligned}
&\liminf_{x\to+\infty}\frac{g(t,x,y)+M}{q(x)} \\
&\quad=\liminf_{x\to+\infty}\frac{\left(1/e^{\beta_2+\sin(tx)}\right)\left(q_0^{-1}M\right)^{1-\beta_3}\kappa_2^{-\beta_3}x^{\beta_3}}{x^2} \\
&\quad=+\infty, \\
&\quad\text{uniformly on } (t,y)\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}\times\mathbb{R}^+.
\end{aligned} \tag{34}$$

Thus, (H1)-(H2) are satisfied.

Example 8. Let $f(t,x,y)=(q_0^{-2}Me^{q_0^{-1}M\kappa_2}+e^{|\sin(txy)|})e^{-(x+y)/2}-M$ and $g(t,x,y)=(q_0^{-2}Me^{q_0^{-1}M\kappa_2}+e^{|\cos(txy)|})e^{-(x+y)/2}-M$, for $(t,x,y)\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}\times\mathbb{R}^+\times\mathbb{R}^+$. Then, for any $(t,x,y)\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}\times[0,q_0^{-1}M\kappa_2]\times[0,q_0^{-1}M\kappa_2]$, we have

$$\begin{aligned}
&f(t,x,y)+M,\,g(t,x,y)+M \\
&\quad>q_0^{-2}Me^{q_0^{-1}M\kappa_2}e^{-q_0^{-1}M\kappa_2}=q_0^{-2}M.
\end{aligned} \tag{35}$$

Also,

$$\begin{aligned}
&\limsup_{x\to+\infty,y\to+\infty}\frac{f(t,x,y)+M}{e_1x+e_2y} \\
&\quad=\limsup_{x\to+\infty,y\to+\infty}\frac{\left(q_0^{-2}Me^{q_0^{-1}M\kappa_2}+e^{|\sin(txy)|}\right)e^{-(x+y)/2}}{e_1x+e_2y} \\
&\quad=0,
\end{aligned} \tag{36}$$

and

$$\begin{aligned}
&\limsup_{x\to+\infty,y\to+\infty}\frac{g(t,x,y)+M}{e_3x+e_4y} \\
&\quad=\limsup_{x\to+\infty,y\to+\infty}\frac{\left(q_0^{-2}Me^{q_0^{-1}M\kappa_2}+e^{|\cos(txy)|}\right)e^{-(x+y)/2}}{e_3x+e_4y} \\
&\quad=0,
\end{aligned} \tag{37}$$

for $t\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}$. Thus, (H3)-(H4) hold.

Remark 9. (i) In (H1), the growth condition for nonlinear term f depends on two variables x,y; however, in [7], this corresponding condition only involves one variable.

(ii) When nonlinear terms f,g grow sublinearly at $+\infty$, nonnegative matrices are used to depict the coupling behavior of our nonlinearities. This is different from condition (H4) in [7].

Theorem 10. *Suppose that (H0)-(H2) hold. Then (1) has at least one positive solution.*

Proof. We first claim that there exists a sufficiently large positive number $R>q_0^{-1}M\kappa_2$ such that

$$\begin{aligned}
&(x,y)\neq B(x,y)+\lambda(x_0,y_0), \\
&\qquad\forall(x,y)\in\partial(B_R\times B_R)\cap(P\times P),\ \lambda\geq0,
\end{aligned} \tag{38}$$

where $x_0,y_0\in P_0$ are two given functions. Suppose not. Then there exist $(x,y)\in\partial(B_R\times B_R)\cap(P\times P)$ and $\lambda\geq0$ such that $(x,y)=B(x,y)+\lambda(x_0,y_0)$, and so

$$\begin{aligned}
&x(t)=B_1(x,y)(t)+\lambda x_0(t), \\
&y(t)=B_2(x,y)(t)+\lambda y_0(t), \\
&\qquad\text{for } t\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}.
\end{aligned} \tag{39}$$

This implies $x(t)\geq B_1(x,y)(t)$, and $y(t)\geq B_2(x,y)(t)$ for $t\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}$. From (H1) we have

$$\begin{aligned}
&x(t)\geq B_1(x,y)(t)\geq\sum_{s=0}^{b+2}G(t,s)\left[x(s+\nu-1)\right. \\
&\quad-z(s+\nu-1)+p\left(y(s+\nu-1)-z(s+\nu-1)\right) \\
&\quad\left.-c_1\right]\geq\sum_{s=0}^{b+2}G(t,s)\,x(s+\nu-1)+\sum_{s=0}^{b+2}G(t,s) \\
&\quad\cdot p\left(y(s+\nu-1)-z(s+\nu-1)\right)-c_3\geq\sum_{s=0}^{b+2}G(t,s) \\
&\quad\cdot x(s+\nu-1)+\sum_{s=0}^{b+2}G(t,s)\left[p\left(y(s+\nu-1)\right)\right. \\
&\quad\left.-p\left(z(s+\nu-1)\right)\right]-c_3\geq\sum_{s=0}^{b+2}G(t,s)\,x(s+\nu-1) \\
&\quad+\sum_{s=0}^{b+2}G(t,s)\,p\left(y(s+\nu-1)\right)-c_4, \\
&\qquad\text{for } t\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}},
\end{aligned} \tag{40}$$

and

$$\begin{aligned}
&y(t)\geq B_2(x,y)(t) \\
&\quad\geq\sum_{s=0}^{b+2}G(t,s)\left[q\left(x(s+\nu-1)-z(s+\nu-1)\right)-c_1\right] \\
&\quad\geq\sum_{s=0}^{b+2}G(t,s)\,q\left(x(s+\nu-1)-z(s+\nu-1)\right)-c_3, \\
&\qquad\text{for } t\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}.
\end{aligned} \tag{41}$$

As a result, for $t \in [0, b+2]_{\mathbb{N}_0}$, we have

$$\begin{aligned} p(y(t+\nu-1)) + p(c_3) &\geq p(y(t+\nu-1)+c_3) \\ &\geq p\left[\sum_{s=0}^{b+2} G(t+\nu-1,s) \cdot q(x(s+\nu-1)-z(s+\nu-1))\right] \\ &= p\left[\sum_{s=0}^{b+2} G(t+\nu-1,s) \cdot \kappa_2^{-1}\kappa_2 q(x(s+\nu-1)-z(s+\nu-1))\right] \geq \sum_{s=0}^{b+2} G(t+\nu-1,s)\kappa_2^{-1} p(\kappa_2 q(x(s+\nu-1)-z(s+\nu-1))) \\ &\geq \sum_{s=0}^{b+2} G(t+\nu-1,s) \cdot \kappa_2^{-1}\left[\kappa_2\gamma_1(x(s+\nu-1)-z(s+\nu-1))-c_1\right] \\ &\geq \gamma_1 \sum_{s=0}^{b+2} G(t+\nu-1,s)(x(s+\nu-1)-z(s+\nu-1)) - c_5 \geq \gamma_1 \sum_{s=0}^{b+2} G(t+\nu-1,s)x(s+\nu-1)-c_6. \end{aligned} \tag{42}$$

Thus

$$p(y(t+\nu-1)) \geq \gamma_1 \sum_{s=0}^{b+2} G(t+\nu-1,s)x(s+\nu-1) - c_7, \tag{43}$$

and, therefore,

$$\begin{aligned} x(t) &\geq \sum_{s=0}^{b+2} G(t,s)x(s+\nu-1) + \sum_{s=0}^{b+2} G(t,s)p(y(s+\nu-1)) - c_4 \\ &\geq \sum_{s=0}^{b+2} G(t,s)x(s+\nu-1) + \sum_{s=0}^{b+2} G(t,s) \cdot \left[\gamma_1 \sum_{\tau=0}^{b+2} G(s+\nu-1,\tau)x(\tau+\nu-1)-c_7\right] - c_4 \\ &\geq \sum_{s=0}^{b+2} G(t,s)x(s+\nu-1) + \gamma_1 \sum_{s=0}^{b+2} G(t,s) \cdot \sum_{\tau=0}^{b+2} G(s+\nu-1,\tau)x(\tau+\nu-1) - c_8, \\ &\qquad \text{for } t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}. \end{aligned} \tag{44}$$

Multiply both sides of the above inequality by $\varphi(t)$ and sum from $\nu-1$ to $b+\nu+1$ and together with (11) we obtain

$$\begin{aligned} \sum_{t=\nu-1}^{b+\nu+1} x(t)\varphi(t) &= \sum_{t=0}^{b+2} x(t+\nu-1)\varphi(t+\nu-1) \\ &\geq \sum_{t=0}^{b+2} \varphi(t+\nu-1)\left[\sum_{s=0}^{b+2} G(t+\nu-1,s)x(s+\nu-1) + \gamma_1 \sum_{s=0}^{b+2} G(t+\nu-1,s) \cdot \sum_{\tau=0}^{b+2} G(s+\nu-1,\tau)x(\tau+\nu-1) - c_8\right] \\ &\geq (\kappa_1+\gamma_1\kappa_1^2)\sum_{t=0}^{b+2} x(t+\nu-1)\varphi(t+\nu-1) - c_9 = (\kappa_1+\gamma_1\kappa_1^2)\sum_{t=\nu-1}^{b+\nu+1} x(t)\varphi(t) - c_9. \end{aligned} \tag{45}$$

From (23), (39), and $x_0 \in P_0$ we have $x \in P_0$. This implies

$$\begin{aligned} \kappa_1\|x\| &= \|x\|\sum_{t=\nu-1}^{b+\nu+1} q^*(t)\varphi(t) \leq \sum_{t=\nu-1}^{b+\nu+1} x(t)\varphi(t) \\ &\leq \frac{c_9}{\kappa_1(1+\gamma_1\kappa_1)-1}, \end{aligned} \tag{46}$$

and

$$\|x\| \leq \frac{\kappa_1^{-1}c_9}{\kappa_1(1+\gamma_1\kappa_1)-1}. \tag{47}$$

Note that, from (23), (39), and $y_0 \in P_0$, we find $y \in P_0$. Moreover, we may assume $y(t) \not\equiv 0$, for $t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}$. Then $\|y\| > 0$ and $p(\|y\|) > 0$. Thus, from the concavity of p, we have

$$\begin{aligned}\kappa_1\|y\| &= \|y\|\sum_{t=\nu-1}^{b+\nu+1} q^*(t)\varphi(t) \le \sum_{t=\nu-1}^{b+\nu+1} y(t)\varphi(t)\\ &= \sum_{t=0}^{b+2} y(t+\nu-1)\varphi(t+\nu-1)\\ &= \frac{\|y\|}{p(\|y\|)}\sum_{t=0}^{b+2}\frac{y(t+\nu-1)}{\|y\|}p(\|y\|)\varphi(t+\nu-1)\\ &\le \frac{\|y\|}{p(\|y\|)}\sum_{t=0}^{b+2} p(y(t+\nu-1))\varphi(t+\nu-1).\end{aligned} \tag{48}$$

This implies that

$$p(\|y\|) \le \kappa_1^{-1}\sum_{t=0}^{b+2} p(y(t+\nu-1))\varphi(t+\nu-1). \tag{49}$$

From (40) and Lemma **??** we obtain

$$\begin{aligned}x(t)+c_4 &\ge \sum_{s=0}^{b+2} G(t,s)\,p(y(s+\nu-1))\\ &\ge \sum_{s=0}^{b+2} q^*(t)\varphi(s+\nu-1)\,p(y(s+\nu-1))\\ &\ge q_0\sum_{t=0}^{b+2} p(y(t+\nu-1))\varphi(t+\nu-1).\end{aligned} \tag{50}$$

Combining the above two inequalities, we get

$$\begin{aligned}p(\|y\|) &\le (\kappa_1 q_0)^{-1}(x(t)+c_4)\\ &\le (\kappa_1 q_0)^{-1}\left[\frac{\kappa_1^{-1}c_9}{\kappa_1(1+\gamma_1\kappa_1)-1}+c_4\right].\end{aligned} \tag{51}$$

From (H1), $\lim_{z\longrightarrow+\infty}p(z)=+\infty$, and thus there exists $\mathscr{M}_1>0$ such that $\|y\|\le\mathscr{M}_1$.

Hence, we have $\|x\|\le\kappa_1^{-1}c_9/(\kappa_1(1+\gamma_1\kappa_1)-1)$ and $\|y\|\le\mathscr{M}_1$. As a result, choosing $R>\max\{q_0^{-1}M\kappa_2,\kappa_1^{-1}c_9/(\kappa_1(1+\gamma_1\kappa_1)-1),\mathscr{M}_1\}$ we have a contradiction (recall in general $\partial(A\times B)=(\partial A\times\overline{B})\cup(\overline{A}\times\partial B)$). Thus (38) is true. Consequently Lemma 5 (with R chosen above) implies

$$i(B,(B_R\times B_R)\cap(P\times P),P\times P)=0. \tag{52}$$

Now we show that

$$\begin{aligned}&(x,y)\ne\lambda B(x,y),\\ &\forall(x,y)\in\partial\left(B_{q_0^{-1}M\kappa_2}\times B_{q_0^{-1}M\kappa_2}\right)\cap(P\times P),\ \lambda\in[0,1].\end{aligned} \tag{53}$$

Suppose not. Then there exist $(x,y)\in\partial(B_{q_0^{-1}M\kappa_2}\times B_{q_0^{-1}M\kappa_2})\cap(P\times P)$, $\lambda_0\in[0,1]$ such that $(x,y)=\lambda_0B(x,y)$. This implies that

$$\begin{aligned}&x(t)\le B_1(x,y)(t),\\ &y(t)\le B_2(x,y)(t),\\ &\qquad\text{for } t\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}.\end{aligned} \tag{54}$$

Hence, $\|x\|\le\|B_1(x,y)\|$ and $\|y\|\le\|B_2(x,y)\|$. However, from (H2) we have

$$\begin{aligned}B_1(x,y)(t) &= \sum_{s=0}^{b+2} G(t,s)\tilde{f}(s+\nu-1,x(s+\nu-1)\\ &\quad -z(s+\nu-1),y(s+\nu-1)-z(s+\nu-1))\\ &< \sum_{s=0}^{b+2}\varphi(s+\nu-1)q_0^{-1}M = q_0^{-1}M\kappa_2,\end{aligned} \tag{55}$$

for all $t\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}$. This implies $\|B_1(x,y)\|<q_0^{-1}M\kappa_2$. Similarly, $\|B_2(x,y)\|<q_0^{-1}M\kappa_2$. Thus, note that $(x,y)\in\partial(B_{q_0^{-1}M\kappa_2}\times B_{q_0^{-1}M\kappa_2})\cap(P\times P)$, and we have

$$\begin{aligned}\|(x,y)\| &= \max\{\|x\|,\|y\|\}\\ &\le \max\{\|B_1(x,y)\|,\|B_2(x,y)\|\} < q_0^{-1}M\kappa_2\\ &= \|(x,y)\|.\end{aligned} \tag{56}$$

Clearly, this is a contradiction. Thus (53) is true. It follows from Lemma 6 that

$$i\left(B,\left(B_{q_0^{-1}M\kappa_2}\times B_{q_0^{-1}M\kappa_2}\right)\cap(P\times P),P\times P\right)=1. \tag{57}$$

From (52) and (57) we have

$$\begin{aligned}&i\left(B,\left((B_R\times B_R)\setminus\left(\overline{B}_{q_0^{-1}M\kappa_2}\times\overline{B}_{q_0^{-1}M\kappa_2}\right)\right)\cap(P\times P),P\right.\\ &\quad\left.\times P\right)=0-1=-1.\end{aligned} \tag{58}$$

Therefore the operator B has at least one fixed point (x,y) in $((B_R\times B_R)\setminus(\overline{B}_{q_0^{-1}M\kappa_2}\times\overline{B}_{q_0^{-1}M\kappa_2}))\cap(P\times P)$ with $\|x\|,\|y\|\ge q_0^{-1}M\kappa_2$, and then $(x-z,y-z)(t)$ is a positive solution for (1). This completes the proof. □

Theorem 11. *Suppose that (H0), (H3), and (H4) hold. Then (1) has at least one positive solution.*

Proof. We show there exists a positive constant $R>q_0^{-1}M\kappa_2$ such that

$$\begin{aligned}&(x,y)\ne\lambda B(x,y),\\ &\qquad\forall(x,y)\in\partial(B_R\times B_R)\cap(P\times P),\ \lambda\in[0,1].\end{aligned} \tag{59}$$

Suppose not. Then there exist $(x,y)\in\partial(B_R\times B_R)\cap(P\times P)$, $\lambda_0\in[0,1]$ such that $(x,y)=\lambda_0B(x,y)$. This implies that

$$\begin{aligned}&x(t)\le B_1(x,y)(t),\\ &y(t)\le B_2(x,y)(t),\\ &\qquad\text{for } t\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}.\end{aligned} \tag{60}$$

From (H3) we have

$$\begin{aligned} x(t) &\le \sum_{s=0}^{b+2} G(t,s)\left[e_1\left(x(s+\nu-1)-z(s+\nu-1)\right)\right.\\ &\quad \left.+ e_2\left(y(s+\nu-1)-z(s+\nu-1)\right)+c_2\right]\\ &\le \sum_{s=0}^{b+2} G(t,s)\left[e_1 x(s+\nu-1)+e_2 y(s+\nu-1)\right]\\ &\quad + c_{10},\\ &\qquad \forall (t,x,y)\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}\times\mathbb{R}^+\times\mathbb{R}^+. \end{aligned} \tag{61}$$

Similarly, we have

$$\begin{aligned} y(t) &\le \sum_{s=0}^{b+2} G(t,s)\left[e_3 x(s+\nu-1)+e_4 y(s+\nu-1)\right]\\ &\quad + c_{10},\\ &\qquad \forall (t,x,y)\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}\times\mathbb{R}^+\times\mathbb{R}^+. \end{aligned} \tag{62}$$

Consequently, for all $t\in[0,b+2]_{\mathbb{N}_0}$, multiply both sides of the above two inequalities by $\varphi(t)$ and sum from $\nu-1$ to $b+\nu+1$ and together with (11) we obtain

$$\begin{aligned} \sum_{t=\nu-1}^{b+\nu+1} x(t)\varphi(t) &= \sum_{t=0}^{b+2} x(t+\nu-1)\varphi(t+\nu-1)\\ &\le \sum_{t=0}^{b+2}\varphi(t+\nu-1)\\ &\quad\cdot\left[\sum_{s=0}^{b+2} G(t,s)\left[e_1 x(s+\nu-1)+e_2 y(s+\nu-1)\right]\right.\\ &\quad\left.+c_{10}\right] \le e_1\kappa_2\sum_{t=\nu-1}^{b+\nu+1} x(t)\varphi(t)\\ &\quad + e_2\kappa_2\sum_{t=\nu-1}^{b+\nu+1} y(t)\varphi(t)+c_{10}\sum_{t=\nu-1}^{b+\nu+1}\varphi(t). \end{aligned} \tag{63}$$

Similarly, we have

$$\begin{aligned} \sum_{t=\nu-1}^{b+\nu+1} y(t)\varphi(t) &= \sum_{t=0}^{b+2} y(t+\nu-1)\varphi(t+\nu-1)\\ &\le \sum_{t=0}^{b+2}\varphi(t+\nu-1)\\ &\quad\cdot\left[\sum_{s=0}^{b+2} G(t,s)\left[e_3 x(s+\nu-1)+e_4 y(s+\nu-1)\right]\right.\\ &\quad\left.+c_{10}\right] \le e_3\kappa_2\sum_{t=\nu-1}^{b+\nu+1} x(t)\varphi(t)\\ &\quad + e_4\kappa_2\sum_{t=\nu-1}^{b+\nu+1} y(t)\varphi(t)+c_{10}\sum_{t=\nu-1}^{b+\nu+1}\varphi(t). \end{aligned} \tag{64}$$

Consequently, we obtain

$$\begin{aligned} &\begin{pmatrix} 1-e_1\kappa_2 & -e_2\kappa_2\\ -e_3\kappa_2 & 1-e_4\kappa_2\end{pmatrix}\begin{pmatrix}\displaystyle\sum_{t=\nu-1}^{b+\nu+1} x(t)\varphi(t)\\ \displaystyle\sum_{t=\nu-1}^{b+\nu+1} y(t)\varphi(t)\end{pmatrix}\\ &\quad\le\begin{pmatrix}\kappa_2 c_{10}\\ \kappa_2 c_{10}\end{pmatrix}. \end{aligned} \tag{65}$$

From (H3)(i) we have

$$\begin{aligned} &\begin{pmatrix}\displaystyle\sum_{t=\nu-1}^{b+\nu+1} x(t)\varphi(t)\\ \displaystyle\sum_{t=\nu-1}^{b+\nu+1} y(t)\varphi(t)\end{pmatrix}\\ &\quad\le\frac{1}{\kappa}\begin{pmatrix} 1-e_4\kappa_2 & e_2\kappa_2\\ e_3\kappa_2 & 1-e_1\kappa_2\end{pmatrix}\begin{pmatrix}\kappa_2 c_{10}\\ \kappa_2 c_{10}\end{pmatrix}\\ &\quad=\begin{pmatrix}\dfrac{\kappa_2 c_{10}\left(1+e_2\kappa_2-e_4\kappa_2\right)}{\kappa}\\ \dfrac{\kappa_2 c_{10}\left(1+e_3\kappa_2-e_1\kappa_2\right)}{\kappa}\end{pmatrix}. \end{aligned} \tag{66}$$

Note that $x,y\in P_0$ from the fact that $B_i(P\times P)\subset P_0$ $(i=1,2)$. This implies

$$\begin{aligned} \|x\|\sum_{t=\nu-1}^{b+\nu+1} q^*(t)\varphi(t) &\le \sum_{t=\nu-1}^{b+\nu+1} x(t)\varphi(t)\\ &\le\frac{\kappa_2 c_{10}\left(1+e_2\kappa_2-e_4\kappa_2\right)}{\kappa},\\ \|y\|\sum_{t=\nu-1}^{b+\nu+1} q^*(t)\varphi(t) &\le \sum_{t=\nu-1}^{b+\nu+1} y(t)\varphi(t)\\ &\le\frac{\kappa_2 c_{10}\left(1+e_3\kappa_2-e_1\kappa_2\right)}{\kappa}. \end{aligned} \tag{67}$$

Hence

$$\begin{aligned} \|x\| &\le \frac{\kappa_2 c_{10}\left(1+e_2\kappa_2-e_4\kappa_2\right)}{\kappa_1\kappa},\\ \|y\| &\le \frac{\kappa_2 c_{10}\left(1+e_3\kappa_2-e_1\kappa_2\right)}{\kappa_1\kappa}. \end{aligned} \tag{68}$$

Thus if we choose $R>\max\{q_0^{-1}M\kappa_2,\kappa_2c_{10}(1+e_2\kappa_2-e_4\kappa_2)/\kappa_1\kappa$, and $\kappa_2c_{10}(1+e_3\kappa_2-e_1\kappa_2)/\kappa_1\kappa\}$ we have a contradiction. Thus (59) is true. Lemma 6 (with R chosen above) implies

$$i\left(B,\left(B_R\times B_R\right)\cap(P\times P),P\times P\right)=1. \tag{69}$$

We next prove that

$$\begin{aligned} &(x,y)\ne B(x,y)+\lambda\left(x_0,y_0\right),\\ &\quad\forall (x,y)\in\partial\left(B_{q_0^{-1}M\kappa_2}\times B_{q_0^{-1}M\kappa_2}\right)\cap(P\times P),\ \lambda\ge 0, \end{aligned} \tag{70}$$

where $x_0, y_0 \in P$ are two fixed functions. Indeed, if not, there exist $(x, y) \in \partial(B_{q_0^{-1}M\kappa_2} \times B_{q_0^{-1}M\kappa_2}) \cap (P \times P)$, $\lambda_0 \geq 0$ such that $(x, y) = B(x, y) + \lambda_0(x_0, y_0)$. This implies that

$$\begin{gathered} x(t) \geq B_1(x, y)(t), \\ y(t) \geq B_2(x, y)(t), \\ \text{for } t \in [\nu - 1, b + \nu + 1]_{\mathbb{N}_{\nu-1}}. \end{gathered} \tag{71}$$

Hence, $\|x\| \geq \|B_1(x, y)\|$ and $\|y\| \geq \|B_2(x, y)\|$. However, from (H4) we have

$$\begin{aligned} B_1(x, y)(t) &= \sum_{s=0}^{b+2} G(t, s) \widetilde{f}(s + \nu - 1, x(s + \nu - 1) \\ &\quad - z(s + \nu - 1), y(s + \nu - 1) - z(s + \nu - 1)) \\ &> \sum_{s=0}^{b+2} q^*(t) \varphi(s + \nu - 1) q_0^{-2} M \geq q_0^{-1} M \kappa_2, \end{aligned} \tag{72}$$

for all $t \in [\nu - 1, b + \nu + 1]_{\mathbb{N}_{\nu-1}}$. This implies $\|B_1(x, y)\| > q_0^{-1} M\kappa_2$. Similarly, $\|B_2(x, y)\| > q_0^{-1} M\kappa_2$. Thus, note that $(x, y) \in \partial(B_{q_0^{-1}M\kappa_2} \times B_{q_0^{-1}M\kappa_2}) \cap (P \times P)$, and we have

$$\begin{aligned} \|(x, y)\| &= \max\{\|x\|, \|y\|\} \\ &\geq \max\{\|B_1(x, y)\|, \|B_2(x, y)\|\} > q_0^{-1} M\kappa_2 \\ &= \|(x, y)\|. \end{aligned} \tag{73}$$

This is a contradiction. So (70) is true. It follows from Lemma 5 that

$$i\left(B, \left(B_{q_0^{-1}M\kappa_2} \times B_{q_0^{-1}M\kappa_2}\right) \cap (P \times P), P \times P\right) = 0. \tag{74}$$

From (69) and (74) we have

$$\begin{aligned} &i\left(B, \left((B_R \times B_R) \setminus \left(\overline{B}_{q_0^{-1}M\kappa_2} \times \overline{B}_{q_0^{-1}M\kappa_2}\right)\right) \cap (P \times P), P \right. \\ &\quad \left. \times P\right) = 1 - 0 = 1. \end{aligned} \tag{75}$$

Therefore the operator B has at least one fixed point (x, y) in $((B_R \times B_R) \setminus (\overline{B}_{q_0^{-1}M\kappa_2} \times \overline{B}_{q_0^{-1}M\kappa_2})) \cap (P \times P)$ with $\|x\|, \|y\| \geq q_0^{-1} M\kappa_2$, and then $(x - z, y - z)(t)$ is a positive solution for (1). This completes the proof. □

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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