

*Research Article*

# Positive Solutions for Fourth-Order Singular $p$ -Laplacian Differential Equations with Integral Boundary Conditions

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By employing upper and lower solutions method together with maximal principle, we establish a necessary and sufficient condition for the existence of pseudo- $C^3[0, 1]$  as well as  $C^2[0, 1]$  positive solutions for fourth-order singular  $p$ -Laplacian differential equations with integral boundary conditions. Our nonlinearity  $f$  may be singular at  $t = 0$ ,  $t = 1$ , and  $u = 0$ . The dual results for the other integral boundary condition are also given.

## 1. Introduction

In this paper, we consider the existence of positive solutions for the following nonlinear fourth-order singular  $p$ -Laplacian differential equations with integral boundary conditions:

$$\begin{aligned}(\varphi_p(x''(t)))'' &= f(t, x(t), x'(t)), \quad 0 < t < 1, \\ x(0) &= \int_0^1 g(s)x(s)ds, \quad x(1) = 0, \\ \varphi_p(x''(0)) &= \varphi_p(x''(1)) = \int_0^1 h(s)\varphi_p(x''(s))ds,\end{aligned}\tag{1.1}$$

where  $\varphi_p(t) = |t|^{p-2} \cdot t$ ,  $p \geq 2$ ,  $\varphi_q = \varphi_p^{-1}$ ,  $1/p + 1/q = 1$ ,  $f \in C(J \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}^+)$ ,  $J = (0, 1)$ ,  $\mathbb{R}_+ = (0, +\infty)$ ,  $\mathbb{R}^+ = [0, +\infty)$ ,  $I = [0, 1]$ , and  $g, h \in L^1[0, 1]$  is nonnegative. Let  $\sigma_1 = \int_0^1 (1-s)g(s)ds$ ,  $\sigma_2 = \int_0^1 h(s)ds$ . Throughout this paper, we always assume that  $0 \leq \int_0^1 g(s)ds < 1$ ,  $0 < \int_0^1 h(s)ds < 1$  and nonlinear term  $f$  satisfies the following hypothesis:

(H)  $f(t, u, v) : J \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}^+$  is continuous, nondecreasing on  $u$  and nonincreasing on  $v$  for each fixed  $t \in J$ , and there exists a real number  $b \in \mathbb{R}^+$  such that, for any  $r \in J$ ,

$$f(t, u, rv) \leq r^{-b} f(t, u, v), \quad \forall (t, u, v) \in J \times \mathbb{R}_+ \times \mathbb{R}_+, \quad (1.2)$$

there exists a function  $\xi : [1, +\infty) \rightarrow \mathbb{R}_+$ ,  $\xi(l) < l$  and  $\xi(l)/l^2$  is integrable on  $(1, +\infty)$  such that

$$f(t, lu, v) \leq \xi(l) f(t, u, v), \quad \forall (t, u, v) \in J \times \mathbb{R}_+ \times \mathbb{R}_+, \quad l \in [1, +\infty). \quad (1.3)$$

*Remark 1.1.* Condition (H) is used to discuss the existence and uniqueness of smooth positive solutions in [1].

(i) Inequality (1.2) implies that

$$f(t, u, cv) \geq c^{-b} f(t, u, v), \quad \text{if } c \geq 1. \quad (1.4)$$

Conversely, (1.4) implies (1.2).

(ii) Inequality (1.3) implies that

$$f(t, cu, v) \geq \left(\xi(c^{-1})\right)^{-1} f(t, u, v), \quad \text{if } 0 < c < 1. \quad (1.5)$$

Conversely, (1.5) implies (1.3).

*Remark 1.2.* Typical functions that satisfy condition (H) are those taking the form  $f(t, u, v) = \sum_{i=1}^n a_i(t)u^{\lambda_i} + \sum_{j=1}^m b_j(t)u^{-\mu_j}$ , where  $a_i, b_j \in C(0, 1)$ ,  $0 < \lambda_i < 1$ ,  $\mu_j > 0$  ( $i = 1, 2, \dots, m$ ;  $j = 1, 2, \dots, m$ ).

*Remark 1.3.* It follows from (1.2) and (1.3) that

$$f(t, u, u) \leq \begin{cases} \xi\left(\frac{u}{v}\right) f(t, v, v), & \text{if } u \geq v \geq 0, \\ \left(\frac{v}{u}\right)^b f(t, v, v), & \text{if } v \geq u \geq 0. \end{cases} \quad (1.6)$$

Boundary value problems with integral boundary conditions arise in variety of different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermoelasticity, and plasma physics can be reduced to nonlocal problems with integral boundary conditions. They include two point, three point, and nonlocal boundary value problems (see [2–5]) as special cases and have attracted much attention of many researchers, such as Gallardo, Karakostas, Tsamatos, Lomtadze, Malaguti, Yang, Zhang, and Feng (see [6–13], e.g.). For more information about the general theory of integral equations and their relation to boundary value problems, the reader is referred to the book by Corduneanu [14] and Agarwal and O'Regan [15].

Recently, Zhang et al. [13] studied the existence and nonexistence of symmetric positive solutions for the following nonlinear fourth-order boundary value problems:

$$\begin{aligned}(\varphi_p(x''(t)))'' &= \omega(t)f(t, x(t)), \quad 0 < t < 1, \\ x(0) &= x(1) = \int_0^1 g(s)x(s)ds, \\ \varphi_p(x''(0)) &= \varphi_p(x''(1)) = \int_0^1 h(s)\varphi_p(x''(s))ds,\end{aligned}\tag{1.7}$$

where  $\varphi_p(t) = |t|^{p-2} \cdot t$ ,  $p > 1$ ,  $\varphi_q = \varphi_p^{-1}$ ,  $1/p + 1/q = 1$ ,  $\omega \in L[0, 1]$  is nonnegative, symmetric on the interval  $[0, 1]$ ,  $f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$  is continuous, and  $g, h \in L^1[0, 1]$  are nonnegative, symmetric on  $[0, 1]$ .

To seek necessary and sufficient conditions for the existence of solutions to the ordinary differential equations is important and interesting, but difficult. Professors Wei [16, 17], Du and Zhao [18], Graef and Kong [19], Zhang and Liu [20], and others have done much excellent work under some suitable conditions in this direction. To the author's knowledge, there are no necessary and sufficient conditions available in the literature for the existence of solutions for integral boundary value problem (1.1). Motivated by above papers, the purpose of this paper is to fill this gap. It is worth pointing out that the nonlinearity  $f(t, u, v)$  permits singularity not only at  $t = 0, 1$  but also at  $v = 0$ . By singularity, we mean that the function  $f$  is allowed to be unbounded at the points  $t = 0, 1$  and  $v = 0$ .

## 2. Preliminaries and Several Lemmas

A function  $x(t) \in C^2[0, 1]$  and  $\varphi_p(x''(t)) \in C^2(0, 1)$  is called a  $C^2[0, 1]$  (positive) solution of BVP (1.1) if it satisfies (1.1) ( $x(t) > 0$  for  $t \in (0, 1)$ ). A  $C^2[0, 1]$  (positive) solution of (1.1) is called a pseudo- $C^3[0, 1]$  (positive) solution if  $\varphi_p(x''(t)) \in C^1[0, 1]$  ( $x(t) > 0$ ,  $-x''(t) > 0$  for  $t \in (0, 1)$ ). Denote that

$$E = \left\{ x : x \in C^2[0, 1], \text{ and } \varphi_p(x''(t)) \in C^2(0, 1) \right\}.\tag{2.1}$$

*Definition 2.1.* A function  $\alpha(t) \in E$  is called a lower solution of BVP (1.1) if  $\alpha(t)$  satisfies

$$\begin{aligned} (\varphi_p(\alpha''(t)))'' &\leq f(t, \alpha(t), \alpha(t)), \quad 0 < t < 1, \\ \alpha(0) - \int_0^1 g(s)\alpha(s)ds &\leq 0, \quad \alpha(1) \leq 0, \\ -\left[ \varphi_p(\alpha''(0)) - \int_0^1 h(s)\varphi_p(\alpha''(s))ds \right] &\leq 0, \\ -\left[ \varphi_p(\alpha''(1)) - \int_0^1 h(s)\varphi_p(\alpha''(s))ds \right] &\leq 0. \end{aligned} \tag{2.2}$$

*Definition 2.2.* A function  $\beta(t) \in E$  is called an upper solution of BVP (1.1) if  $\beta(t)$  satisfies

$$\begin{aligned} (\varphi_p(\beta''(t)))'' &\geq f(t, \beta(t), \beta(t)), \quad 0 < t < 1, \\ \beta(0) - \int_0^1 g(s)\beta(s)ds &\geq 0, \quad \beta(1) \geq 0, \\ -\left[ \varphi_p(\beta''(0)) - \int_0^1 h(s)\varphi_p(\beta''(s))ds \right] &\geq 0, \\ -\left[ \varphi_p(\beta''(1)) - \int_0^1 h(s)\varphi_p(\beta''(s))ds \right] &\geq 0. \end{aligned} \tag{2.3}$$

Suppose that  $0 < a_k < b_k < 1$ , and

$$\begin{aligned} F_k = \left\{ x : x \in C^2[a_k, b_k], \varphi_p(x''(t)) \in C^2(a_k, b_k), x(a_k) - \int_{a_k}^{b_k} g(s)x(s)ds \geq 0, x(b_k) \geq 0, \right. \\ \left. -\left[ \varphi_p(x''(a_k)) - \int_{a_k}^{b_k} h(s)\varphi_p(x''(s))ds \right] \geq 0, -\left[ \varphi_p(x''(b_k)) - \int_{a_k}^{b_k} h(s)\varphi_p(x''(s))ds \right] \geq 0 \right\}. \end{aligned} \tag{2.4}$$

To prove the main results, we need the following maximum principle.

**Lemma 2.3** (Maximum principle). *If  $x \in F_k$ , such that  $[\varphi_p(x''(t))]'' \geq 0$ ,  $t \in (a_k, b_k)$ , then  $x(t) \geq 0$ ,  $-x''(t) \geq 0$ ,  $t \in [a_k, b_k]$ .*

*Proof.* Set

$$-x''(t) = y(t), \quad t \in [a_k, b_k], \quad (2.5)$$

$$[\varphi_p(x''(t))]'' = \sigma(t), \quad t \in (a_k, b_k), \quad (2.6)$$

$$x(a_k) - \int_{a_k}^{b_k} g(s)x(s)ds = r_1, \quad x(b_k) = r_2, \quad (2.7)$$

$$-\left[ \varphi_p(x''(a_k)) - \int_{a_k}^{b_k} h(s)\varphi_p(x''(s))ds \right] = r_3, \quad (2.8)$$

$$-\left[ \varphi_p(x''(b_k)) - \int_{a_k}^{b_k} h(s)\varphi_p(x''(s))ds \right] = r_4, \quad (2.9)$$

then  $r_i \geq 0$ ,  $i = 1, 2, 3, 4$ ,  $\sigma(t) \geq 0$ ,  $t \in (a_k, b_k)$  and

$$-\varphi_p(y)''(t) = \sigma(t), \quad t \in (a_k, b_k),$$

$$\varphi_p(y)(a_k) - \int_{a_k}^{b_k} h(s)\varphi_p(y(s))ds = r_3, \quad (2.10)$$

$$\varphi_p(y)(b_k) - \int_{a_k}^{b_k} h(s)\varphi_p(y(s))ds = r_4.$$

Let

$$\varphi_p(y)(t) = z(t), \quad (2.11)$$

then

$$-z''(t) = \sigma(t), \quad t \in (a_k, b_k), \quad (2.12)$$

$$z(a_k) - \int_{a_k}^{b_k} h(s)z(s)ds = r_3, \quad z(b_k) - \int_{a_k}^{b_k} h(s)z(s)ds = r_4. \quad (2.13)$$

By integration of (2.12), we have

$$z'(t) = z'(a_k) - \int_{a_k}^t \sigma(s)ds. \quad (2.14)$$

Integrating again, we get

$$z(t) = z(a_k) + z'(a_k)(t - a_k) - \int_{a_k}^t (t - s)\sigma(s)ds. \quad (2.15)$$

Let  $t = b_k$  in (2.15), we obtain that

$$z'(a_k) = \frac{r_4 - r_3}{b_k - a_k} + \frac{1}{b_k - a_k} \int_{a_k}^{b_k} (b_k - s)\sigma(s)ds. \quad (2.16)$$

Substituting (2.13) and (2.16) into (2.15), we obtain that

$$z(t) = r_3 + \frac{r_4 - r_3}{b_k - a_k} (t - a_k) + \int_{a_k}^{b_k} G_k(t, s)\sigma(s)ds + \int_{a_k}^{b_k} h(s)z(s)ds, \quad (2.17)$$

where

$$G_k(t, s) = \frac{1}{b_k - a_k} \begin{cases} (b_k - t)(s - a_k), & 0 \leq s \leq t \leq 1, \\ (b_k - s)(t - a_k), & 0 \leq t \leq s \leq 1. \end{cases} \quad (2.18)$$

Notice that

$$\begin{aligned} \int_{a_k}^{b_k} h(s)z(s)ds &= \int_{a_k}^{b_k} h(s) \left[ r_3 + \frac{r_4 - r_3}{b_k - a_k} (s - a_k) + \int_{a_k}^{b_k} G_k(s, \tau)\sigma(\tau)d\tau + \int_{a_k}^{b_k} h(s)z(s)ds \right] ds \\ &= r_3 \int_{a_k}^{b_k} h(s)ds + \frac{r_4 - r_3}{b_k - a_k} \int_{a_k}^{b_k} (s - a_k)h(s)ds + \int_{a_k}^{b_k} h(s) \left[ \int_{a_k}^{b_k} G_k(s, \tau)\sigma(\tau)d\tau \right] ds \\ &\quad + \int_{a_k}^{b_k} h(s)ds \cdot \int_{a_k}^{b_k} h(s)z(s)ds, \end{aligned} \quad (2.19)$$

therefore,

$$\begin{aligned} \int_{a_k}^{b_k} h(s)z(s)ds &= \frac{1}{1 - \int_{a_k}^{b_k} h(s)ds} \left[ r_3 \int_{a_k}^{b_k} h(s)ds + \frac{r_4 - r_3}{b_k - a_k} \int_{a_k}^{b_k} (s - a_k)h(s)ds \right. \\ &\quad \left. + \int_{a_k}^{b_k} h(s) \left( \int_{a_k}^{b_k} G_k(s, \tau)\sigma(\tau)d\tau \right) ds \right]. \end{aligned} \quad (2.20)$$

Substituting (2.20) into (2.17), we have

$$\begin{aligned}
z(t) &= r_3 + \frac{r_4 - r_3}{b_k - a_k}(t - a_k) + \frac{1}{1 - \int_{a_k}^{b_k} h(s) ds} \int_{a_k}^{b_k} h(s) \left[ \int_{a_k}^{b_k} G_k(s, \tau) \sigma(\tau) d\tau \right] ds \\
&\quad + \frac{1}{1 - \int_{a_k}^{b_k} h(s) ds} \left[ r_3 \int_{a_k}^{b_k} h(s) ds + \frac{r_4 - r_3}{b_k - a_k} \int_{a_k}^{b_k} (s - a_k) h(s) ds \right] + \int_{a_k}^{b_k} G_k(t, s) \sigma(s) ds \\
&= \frac{b_k - t}{b_k - a_k} r_3 + \frac{t - a_k}{b_k - a_k} r_4 + \frac{1}{1 - \sigma_{2k}} \left[ \int_{a_k}^{b_k} \frac{b_k - s}{b_k - a_k} h(s) ds r_3 + \int_{a_k}^{b_k} \frac{s - a_k}{b_k - a_k} h(s) ds r_4 \right] \\
&\quad + \int_{a_k}^{b_k} K_k(t, s) \sigma(s) ds \\
&= \left[ \frac{b_k - t}{b_k - a_k} + \frac{1}{1 - \sigma_{2k}} \int_{a_k}^{b_k} \frac{b_k - s}{b_k - a_k} h(s) ds \right] r_3 + \left[ \frac{t - a_k}{b_k - a_k} + \frac{1}{1 - \sigma_{2k}} \int_{a_k}^{b_k} \frac{s - a_k}{b_k - a_k} h(s) ds \right] r_4 \\
&\quad + \int_{a_k}^{b_k} K_k(t, s) \sigma(s) ds,
\end{aligned} \tag{2.21}$$

where

$$K_k(t, s) = G_k(t, s) + \frac{1}{1 - \sigma_{2k}} \int_{a_k}^{b_k} G_k(s, \tau) h(\tau) d\tau, \quad \sigma_{2k} = \int_{a_k}^{b_k} h(s) ds. \tag{2.22}$$

Obviously,  $G_k(t, s) \geq 0$ ,  $K_k(t, s) \geq 0$ ,  $\sigma_{2k} \geq 0$ . From (2.21), it is easily seen that  $z(t) \geq 0$  for  $t \in [a_k, b_k]$ . By (2.11), we know that  $\varphi_p(y)(t) \geq 0$ , that is,  $y(t) \geq 0$ . Thus, we have proved that  $-x''(t) \geq 0$ ,  $t \in [a_k, b_k]$ . Similarly, the solution of (2.5) and (2.7) can be expressed by

$$\begin{aligned}
x(t) &= \left[ \frac{b_k - t}{b_k - a_k} + \frac{b_k - t}{(1 - \sigma_{1k})(b_k - a_k)} \int_{a_k}^{b_k} g(s) \frac{b_k - s}{b_k - a_k} ds \right] r_1 \\
&\quad + \left[ \frac{t - a_k}{b_k - a_k} + \frac{b_k - t}{(1 - \sigma_{1k})(b_k - a_k)} \int_{a_k}^{b_k} g(s) \frac{s - a_k}{b_k - a_k} ds \right] r_2 + \int_{a_k}^{b_k} H_k(t, s) y(s) ds,
\end{aligned} \tag{2.23}$$

where

$$H_k(t, s) = G_k(t, s) + \frac{b_k - t}{1 - \sigma_{1k}} \int_{a_k}^{b_k} G_k(s, \tau) g(\tau) d\tau, \quad \sigma_{1k} = \frac{1}{b_k - a_k} \int_{a_k}^{b_k} g(s)(b_k - s) ds. \tag{2.24}$$

By (2.23), we can get that  $x(t) \geq 0$ ,  $t \in [a_k, b_k]$ . □

**Lemma 2.4.** *Suppose that (H) holds. Let  $x(t)$  be a  $C^2[0, 1]$  positive solution of BVP (1.1). Then there exist two constants  $0 < I_1 < I_2$  such that*

$$I_1(1-t) \leq x(t) \leq I_2(1-t), \quad t \in [0, 1]. \quad (2.25)$$

*Proof.* Assume that  $x(t)$  is a  $C^2[0, 1]$  positive solution of BVP (1.1). Then  $x(t)$  can be stated as

$$x(t) = \int_0^1 G(t, s)(-x''(s))ds + \frac{1-t}{1-\sigma_1} \iint_0^1 G(\tau, s)g(\tau)(-x''(s))d\tau ds, \quad (2.26)$$

where

$$G(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases} \quad (2.27)$$

It is easy to see that

$$x(0) = \frac{1}{1-\sigma_1} \iint_0^1 G(\tau, s)g(\tau)(-x''(s))d\tau ds > 0. \quad (2.28)$$

By (2.26), for  $0 \leq t \leq 1$ , we have that

$$x(t) \geq \frac{1-t}{1-\sigma_1} \iint_0^1 G(\tau, s)g(\tau)(-x''(s))d\tau ds = (1-t)x(0) \geq 0. \quad (2.29)$$

From (2.26) and (2.27), we get that

$$0 \leq x(t) \leq (1-t) \left[ \int_0^1 s(-x''(s))ds + \frac{1}{1-\sigma_1} \iint_0^1 G(\tau, s)g(\tau)(-x''(s))d\tau ds \right]. \quad (2.30)$$

Setting

$$I_1 = x(0), \quad I_2 = \int_0^1 s(-x''(s))ds + \frac{1}{1-\sigma_1} \iint_0^1 G(\tau, s)g(\tau)(-x''(s))d\tau ds, \quad (2.31)$$

then from (2.29) and (2.30), we have (2.25).  $\square$

**Lemma 2.5.** *Suppose that (H) holds. And assume that there exist lower and upper solutions of BVP (1.1), respectively,  $\alpha(t)$  and  $\beta(t)$ , such that  $\alpha(t), \beta(t) \in E$ ,  $0 \leq \alpha(t) \leq \beta(t)$  for  $t \in (0, 1)$ ,  $\alpha(1) = \beta(1) = 0$ . Then BVP (1.1) has at least one  $C^2[0, 1]$  positive solution  $x(t)$  such that  $\alpha(t) \leq x(t) \leq \beta(t)$ ,  $t \in [0, 1]$ . If, in addition, there exists  $F(t) \in L^1[0, 1]$  such that*

$$|f(t, x, x)| \leq F(t), \quad \text{for } \alpha(t) \leq x(t) \leq \beta(t), \quad (2.32)$$



then the solution  $x(t)$  of BVP (1.1) is a pseudo- $C^3[0, 1]$  positive solution.

*Proof.* For each  $k$ , for all  $x(t) \in E_k = \{x : x \in C^2[a_k, b_k], \text{ and } \varphi_p(x''(t)) \in C^2(a_k, b_k)\}$ , we defined an auxiliary function

$$F_k(x)(t) = \begin{cases} f(t, \alpha(t), \alpha(t)), & \text{if } x(t) \leq \alpha(t), \\ f(t, x(t), x(t)), & \text{if } \alpha(t) \leq x(t) \leq \beta(t), \\ f(t, \beta(t), \beta(t)), & \text{if } x(t) \geq \beta(t). \end{cases} \quad (2.33)$$

By condition (H), we have that  $F_k : E_k \rightarrow [0, +\infty)$  is continuous.

Let  $\{a_k\}, \{b_k\}$  be sequences satisfying  $0 < \dots < a_{k+1} < a_k < \dots < a_1 < b_1 < \dots < b_k < b_{k+1} < \dots < 1$ ,  $a_k \rightarrow 0$  and  $b_k \rightarrow 1$  as  $k \rightarrow \infty$ , and let  $\{r_{ki}\}, i = 1, 2, 3$ , be sequences satisfying

$$\begin{aligned} \alpha(a_k) - \int_{a_k}^{b_k} g(s)\alpha(s)ds &\leq r_{k1} \leq \beta(a_k) - \int_{a_k}^{b_k} g(s)\beta(s)ds, \\ \alpha(b_k) &\leq r_{k2} \leq \beta(b_k), \quad r_{k1} \rightarrow 0, \quad r_{k2} \rightarrow 0, \quad \text{as } k \rightarrow \infty, \\ -\left[ \varphi_p(\alpha''(a_k)) - \int_{a_k}^{b_k} h(s)\varphi_p(\alpha''(s))ds \right] &\leq r_{k3} \leq -\left[ \varphi_p(\beta''(a_k)) - \int_{a_k}^{b_k} h(s)\varphi_p(\beta''(s))ds \right], \\ -\left[ \varphi_p(\alpha''(b_k)) - \int_0^1 h(s)\varphi_p(\alpha''(s))ds \right] &\leq r_{k4} \leq -\left[ \varphi_p(\beta''(b_k)) - \int_{a_k}^{b_k} h(s)\varphi_p(\beta''(s))ds \right], \\ r_{k3} &\rightarrow 0, \quad r_{k4} \rightarrow 0, \quad \text{as } k \rightarrow \infty. \end{aligned} \quad (2.34)$$

For each  $k$ , consider the following nonsingular problem:

$$\begin{aligned} (\varphi_p(x''(t)))'' &= F_k(x)(t), \quad t \in [a_k, b_k], \\ x(a_k) - \int_{a_k}^{b_k} g(s)x(s)ds &= r_{k1}, \quad x(b_k) = r_{k2}, \\ -\left[ \varphi_p(x''(a_k)) - \int_{a_k}^{b_k} h(s)\varphi_p(x''(s))ds \right] &= r_{k3}, \\ -\left[ \varphi_p(x''(b_k)) - \int_{a_k}^{b_k} h(s)\varphi_p(x''(s))ds \right] &= r_{k4}. \end{aligned} \quad (2.35)$$

For convenience, we define linear operators as follows:

$$\begin{aligned}
 B_k x(t) &= \left[ \frac{b_k - t}{b_k - a_k} + \frac{1}{1 - \sigma_{2k}} \int_{a_k}^{b_k} \frac{b_k - s}{b_k - a_k} h(s) ds \right] r_3 \\
 &\quad + \left[ \frac{t}{b_k - a_k} + \frac{1}{1 - \sigma_{2k}} \int_{a_k}^{b_k} \frac{s - a_k}{b_k - a_k} h(s) ds \right] r_4 + \int_{a_k}^{b_k} K_k(t, s) x(s) ds, \\
 A_k x(t) &= \left[ \frac{b_k - t}{b_k - a_k} + \frac{b_k - t}{(1 - \sigma_{1k})(b_k - a_k)} \int_{a_k}^{b_k} g(s) \frac{b_k - s}{b_k - a_k} \right] r_1 \\
 &\quad + \left[ \frac{t - a_k}{b_k - a_k} + \frac{b_k - t}{(1 - \sigma_{1k})(b_k - a_k)} \int_{a_k}^{b_k} g(s) \frac{s - a_k}{b_k - a_k} \right] r_2 + \int_{a_k}^{b_k} H_k(t, s) x(s) ds.
 \end{aligned} \tag{2.36}$$

By the proof of Lemma 2.3,  $x(t)$  is a solution of problem (2.35) if and only if it is the fixed point of the following operator equation:

$$x(t) = \left[ A_k \left( \varphi_p^{-1}(B_k F_k) \right) \right] x(t). \tag{2.37}$$

By (2.33), it is easy to verify that  $A_k(\varphi_p^{-1}(B_k F_k)) : E_k \rightarrow E_k$  is continuous and  $F_k(E_k)$  is a bounded set. Moreover, by the continuity of  $G_k(t, s)$ , we can show that  $A_k(\varphi_p^{-1}(B_k))$  is a compact operator and  $A_k(\varphi_p^{-1}(B_k))(E_k)$  is a relatively compact set. So,  $A_k(\varphi_p^{-1}(B_k F_k)) : E_k \rightarrow E_k$  is a completely continuous operator. In addition,  $x \in E_k$  is a solution of (2.35) if and only if  $x$  is a fixed point of operator  $A_k(\varphi_p^{-1}(B_k F_k))x = x$ . Using the Schauder's fixed point theorem, we assert that  $A_k(\varphi_p^{-1}(B_k F_k))$  has at least one fixed point  $x_k \in C^2[a_k, b_k]$ , by  $x_k(t) = A_k(\varphi_p^{-1}(B_k F_k))x_k(t)$ , we can get  $\varphi_p(x_k) \in C^2[a_k, b_k]$ .

We claim that

$$\alpha(t) \leq x_k(t) \leq \beta(t), \quad t \in [a_k, b_k]. \tag{2.38}$$

From this it follows that

$$(\varphi_p(x_k''(t)))'' = f(t, x_k(t), x_k(t)), \quad t \in [a_k, b_k]. \tag{2.39}$$

Indeed, suppose by contradiction that  $x_k(t) \not\leq \beta(t)$  on  $[a_k, b_k]$ . By the definition of  $F_k$ , we have

$$F_k(x_k)(t) = f(t, \beta(t), \beta(t)), \quad t \in [a_k, b_k]. \tag{2.40}$$

Therefore,

$$(\varphi_p(x_k''(t)))'' = f(t, \beta(t), \beta(t)), \quad t \in [a_k, b_k]. \tag{2.41}$$

On the other hand, since  $\beta(t)$  is an upper solution of (1.1), we also have

$$(\varphi_p(\beta''(t)))'' \geq f(t, \beta(t), \beta(t)), \quad t \in [a_k, b_k]. \quad (2.42)$$

Then setting

$$z(t) = \varphi_p(-\beta''(t)) - \varphi_p(-x_k''(t)), \quad t \in [a_k, b_k]. \quad (2.43)$$

By (2.41) and (2.42), we obtain that

$$\begin{aligned} -z''(t) &\geq 0, \quad t \in (a_k, b_k), \quad x \in C^2[a_k, b_k], \\ z(a_k) - \int_{a_k}^{b_k} h(s)z(s)ds &\geq 0, \quad z(b_k) - \int_{a_k}^{b_k} h(s)z(s)ds \geq 0. \end{aligned} \quad (2.44)$$

By Lemma 2.3, we can conclude that

$$z(t) \geq 0, \quad t \in [a_k, b_k]. \quad (2.45)$$

Hence,

$$-[\beta''(t) - x_k''(t)] \geq 0, \quad t \in [a_k, b_k]. \quad (2.46)$$

Set

$$u(t) = \beta(t) - x_k(t), \quad t \in [a_k, b_k]. \quad (2.47)$$

Then

$$\begin{aligned} -u''(t) &\geq 0, \quad t \in (a_k, b_k), \quad x \in C^2[a_k, b_k], \\ u(a_k) - \int_{a_k}^{b_k} g(s)u(s)ds &\geq 0, \quad u(b_k) \geq 0. \end{aligned} \quad (2.48)$$

By Lemma 2.3, we can conclude that

$$u(t) \geq 0, \quad t \in [a_k, b_k], \quad (2.49)$$

which contradicts the assumption that  $x_k(t) \not\leq \beta(t)$ . Therefore,  $x_k(t) \not\leq \beta(t)$  is impossible.

Similarly, we can show that  $\alpha(t) \leq x_k(t)$ . So, we have shown that (2.38) holds.

Using the method of [21] and Theorem 3.2 in [22], we can obtain that there is a  $C^2[0, 1]$  positive solution  $\omega(t)$  of (1.1) such that  $\alpha(t) \leq \omega(t) \leq \beta(t)$ , and a subsequence of  $\{x_k(t)\}$  converging to  $\omega(t)$  on any compact subintervals of  $(0, 1)$ .

In addition, if (2.32) holds, then  $|[\varphi_p(x''(t))]''| \leq F(t)$ . Hence,  $[\varphi_p(x''(t))]''$  is absolutely integrable on  $[0, 1]$ . This implies that  $x(t)$  is a pseudo- $C^3[0, 1]$  positive solution of (1.1).  $\square$

### 3. The Main Results

**Theorem 3.1.** *Suppose that (H) holds, then a necessary and sufficient condition for BVP (1.1) to have a pseudo- $C^3[0, 1]$  positive solution is that the following integral condition holds:*

$$0 < \int_0^1 f(s, (1-s), (1-s)) ds < +\infty. \quad (3.1)$$

*Proof.* The proof is divided into two parts, necessity and sufficiency.

*Necessity.* Suppose that  $x(t)$  is a pseudo- $C^3[0, 1]$  positive solution of (1.1). Then both  $\varphi'_p(x''(0))$  and  $\varphi'_p(x''(1))$  exist. By Lemma 2.4, there exist two constants  $0 < I_1 < I_2$  such that

$$I_1(1-t) < x(t) < I_2(1-t), \quad t \in [0, 1]. \quad (3.2)$$

Without loss of generality, we may assume that  $0 < I_1 < 1 < I_2$ . This together with condition (H) implies that

$$\begin{aligned} \int_0^1 f(s, (1-s), (1-s)) ds &\leq \int_0^1 f\left(s, \frac{1}{I_1}x(s), \frac{1}{I_2}x(s)\right) ds \leq \xi \left(\frac{1}{I_1}\right) I_2^b \int_0^1 f(s, x(s), x(s)) ds \\ &= \xi \left(\frac{1}{I_1}\right) I_2^b \cdot (\varphi'_p(x''(1)) - \varphi'_p(x''(0))) < +\infty. \end{aligned} \quad (3.3)$$

On the other hand, since  $x(t)$  is a pseudo- $C^3[0, 1]$  positive solution of (1.1), we have

$$f(t, x(t), x(t)) \neq 0, \quad t \in (0, 1). \quad (3.4)$$

Otherwise, let  $z(t) = \varphi_p(x''(t))$ . By the proof of Lemma 2.3, we have that  $z(t) \equiv 0$ ,  $t \in (0, 1)$ , that is,  $x''(t) \equiv 0$  which contradicts that  $x(t)$  is a pseudo- $C^3[0, 1]$  positive solution. Therefore, there exists a positive  $t_0 \in (0, 1)$  such that  $f(t_0, x(t_0), x(t_0)) > 0$ . Obviously,  $x(t_0) > 0$ . By (1.6) we have

$$0 < f(t_0, x(t_0), x(t_0)) \leq \begin{cases} \xi \left(\frac{x(t_0)}{1-t_0}\right) f(t_0, 1-t_0, 1-t_0), & \text{if } x(t_0) \geq 1-t_0, \\ \left(\frac{1-t_0}{x(t_0)}\right)^b f(t_0, 1-t_0, 1-t_0), & \text{if } x(t_0) \leq 1-t_0. \end{cases} \quad (3.5)$$

Consequently,  $f(t_0, 1-t_0, 1-t_0) > 0$ , which implies that

$$\int_0^1 f(s, 1-s, 1-s) ds > 0. \quad (3.6)$$

It follows from (3.3) and (3.6) that

$$0 < \int_0^1 f(s, 1-s, 1-s) ds < +\infty, \quad (3.7)$$

which is the desired inequality.

*Sufficiency.* First, we prove the existence of a pair of upper and lower solutions. Since  $\xi(l)/l^2$  is integrable on  $[1, +\infty)$ , we have

$$\lim_{l \rightarrow +\infty} \inf \frac{\xi(l)}{l} = 0. \quad (3.8)$$

Otherwise, if  $\lim_{l \rightarrow +\infty} \inf(\xi(l)/l) = m_0 > 0$ , then there exists a real number  $N > 0$  such that  $\xi(l)/l^2 > m_0/2l$  when  $l > N$ , which contradicts the condition that  $\xi(l)/l^2$  is integrable on  $[1, +\infty)$ . In view of condition (H) and (3.8), we obtain that

$$f(t, ru, v) \geq h(r)f(t, u, v), \quad r \in (0, 1), \quad (3.9)$$

$$\limsup_{r \rightarrow 0^+} \frac{r}{h(r)} = \limsup_{p \rightarrow +\infty} \frac{p^{-1}}{h(p^{-1})} = \liminf_{p \rightarrow +\infty} \frac{\xi(p)}{p} = 0, \quad (3.10)$$

where  $h(r) = (\xi(r^{-1}))^{-1}$ .

Suppose that (3.1) holds. Firstly, we define the linear operators  $A$  and  $B$  as follows:

$$Bx(t) = \int_0^1 G(t, s)x(s)ds + \frac{1}{1-\sigma_2} \iint_0^1 G(s, \tau)h(\tau)x(s)d\tau ds, \quad (3.11)$$

$$Ax(t) = \int_0^1 G(t, s)x(s)ds + \frac{1-t}{1-\sigma_1} \iint_0^1 G(s, \tau)g(\tau)x(s)d\tau ds, \quad (3.12)$$

where  $G(t, s)$  is given by (2.27). Let

$$b_1(t) = A\varphi_p^{-1}Bf(t, 1-t, 1-t), \quad t \in [0, 1]. \quad (3.13)$$

It is easy to know from (3.11) and (3.12) that  $\varphi_p(-b''(t)) \in C^1[0, 1]$ . By Lemma 2.4, we know that there exists a positive number  $k < 1$  such that

$$k_1(1-t) \leq b_1(t) \leq \frac{1}{k_1}(1-t), \quad t \in [0, 1]. \quad (3.14)$$

Take  $0 < l_1 < k_1$  sufficiently small, then by (3.10), we get that  $l_1 k_1 / h(l_1 k_1) \leq k_1$ , that is,

$$h(l_1 k_1) - l_1 \geq 0, \quad \xi\left(\frac{1}{l_1 k_1}\right) - \frac{1}{l_1} \leq 0. \quad (3.15)$$

Let

$$\alpha(t) = l_1 b_1(t), \quad \beta(t) = \frac{1}{l_1} b_1(t), \quad t \in [0, 1]. \quad (3.16)$$

Thus, from (3.14) and (3.16), we have

$$l_1 k_1 (1-t) \leq \alpha(t) \leq (1-t) \leq \beta(t) \leq \frac{1}{l_1 k_1} (1-t), \quad t \in [0, 1]. \quad (3.17)$$

Considering  $p \geq 2$ , it follows from (3.15), (3.17), and condition (H) that

$$\begin{aligned} f(t, \alpha(t), \alpha(t)) &\geq f(t, l_1 k_1 (1-t), (1-t)) \geq h(l_1 k_1) f(t, (1-t), (1-t)) \\ &\geq l_1 f(t, (1-t), (1-t)) \geq l_1^{p-1} f(t, (1-t), (1-t)) \\ &= [\varphi_p(\alpha''(t))]'' , \quad t \in (0, 1), \\ f(t, \beta(t), \beta(t)) &\leq f\left(t, \frac{1}{l_1 k_1} (1-t), (1-t)\right) \leq \xi \left(\frac{1}{l_1 k_1}\right) f(t, (1-t), (1-t)) \\ &\leq \left(\frac{1}{l_1}\right) f(t, (1-t), (1-t)) \\ &\leq \left(\frac{1}{l_1}\right)^{p-1} f(t, (1-t), (1-t)) \\ &= [\varphi_p(\beta''(t))]'' , \quad t \in (0, 1). \end{aligned} \quad (3.18)$$

From (3.13) and (3.16), it follows that

$$\begin{aligned} \alpha(0) &= \int_0^1 g(t) \alpha(t) dt, \quad \beta(0) = \int_0^1 g(t) \beta(t) dt, \quad \alpha(1) = 0, \quad \beta(1) = 0, \\ \varphi_p(\alpha''(0)) &= \varphi_p(\alpha''(1)) = \int_0^1 h(s) \varphi_p(\alpha''(s)) ds, \\ \varphi_p(\beta''(0)) &= \varphi_p(\beta''(1)) = \int_0^1 h(s) \varphi_p(\beta''(s)) ds. \end{aligned} \quad (3.19)$$

Thus, we have shown that  $\alpha(t)$  and  $\beta(t)$  are lower and upper solutions of BVP (1.1), respectively.

Additionally, when  $\alpha(t) \leq x(t) \leq \beta(t)$ ,  $t \in [0, 1]$ , by (3.17) and condition (H), we have

$$\begin{aligned} 0 &\leq f(t, x(t), x(t)) \leq f\left(t, \frac{1}{l_1 k_1} (1-t), l_1 k_1 (1-t)\right) \\ &\leq \xi \left(\frac{1}{l_1 k_1}\right) f(t, (1-t), l_1 k_1 (1-t)) \leq \xi \left(\frac{1}{l_1 k_1}\right) (l_1 k_1)^{-b} f(t, (1-t), (1-t)) = F(t). \end{aligned} \quad (3.20)$$

From (3.1), we have  $\int_0^1 F(t)dt < +\infty$ . So, it follows from Lemma 2.5 that BVP (1.1) admits a pseudo- $C^3[0, 1]$  positive solution such that  $\alpha(t) \leq x(t) \leq \beta(t)$ .  $\square$

*Remark 3.2.* Lin et al. [23, 24] considered the existence and uniqueness of solutions for some fourth-order and  $(k, n - k)$  conjugate boundary value problems when  $f(t, u, v) = q(t)(g(u) + h(v))$ , where

$$\begin{aligned} g : [0, +\infty) &\longrightarrow [0, +\infty) \text{ is continuous and nondecreasing,} \\ h : (0, +\infty) &\longrightarrow (0, +\infty) \text{ is continuous and nonincreasing,} \end{aligned} \quad (3.21)$$

under the following condition:

(P<sub>1</sub>) for  $t \in (0, 1)$  and  $u, v > 0$ , there exists  $\alpha \in (0, 1)$  such that

$$\begin{aligned} g(tu) &\geq t^\alpha g(u), \\ h(t^{-1}v) &\geq t^\alpha h(v). \end{aligned} \quad (3.22)$$

Lei et al. [25] and Liu and Yu [26] investigated the existence and uniqueness of positive solutions to singular boundary value problems under the following condition:

(P<sub>2</sub>)  $f(t, \lambda u, (1/\lambda)v) \geq \lambda^\alpha f(t, u, v)$ , for all  $u, v > 0$ ,  $\lambda \in (0, 1)$ , where  $\alpha \in [0, 1)$  and  $f(t, u, v)$  is nondecreasing on  $u$  and nonincreasing on  $v$ .

Obviously, (3.21)-(3.22) imply condition (P<sub>2</sub>) and condition (P<sub>2</sub>) implies condition (H). So, condition (H) is weaker than conditions (P<sub>1</sub>) and (P<sub>2</sub>). Thus, functions considered in this paper are wider than those in [23–26].

In the following, when  $f(t, u, u)$  admits the form  $f(t, u)$ , that is, nonlinear term  $f$  is not mixed monotone on  $u$ , but monotone with respect  $u$ , BVP (1.1) becomes

$$\begin{aligned} (\varphi_p(x''(t)))'' &= f(t, x(t)), \quad 0 < t < 1, \\ x(0) &= \int_0^1 g(s)x(s)ds, \quad x(1) = 0, \\ \varphi_p(x''(0)) &= \varphi_p(x''(1)) = \int_0^1 h(s)\varphi_p(x''(s))ds. \end{aligned} \quad (3.23)$$

If  $f(t, u)$  satisfies one of the following:

(H\*)  $f(t, u) : J \times \mathbb{R}_+ \rightarrow \mathbb{R}^+$  is continuous, nondecreasing on  $u$ , for each fixed  $t \in (0, 1)$ , there exists a function  $\xi : [1, +\infty) \rightarrow \mathbb{R}_+$ ,  $\xi(l) < l$  and  $\xi(l)/l^2$  is integrable on  $(1, +\infty)$  such that

$$f(t, lu) \leq \xi(l)f(t, u), \quad \forall (t, u) \in J \times \mathbb{R}_+, \quad l \in [1, +\infty). \quad (3.24)$$

**Theorem 3.3.** *Suppose that  $(H^*)$  holds, then a necessary and sufficient condition for BVP (3.23) to have a pseudo- $C^3[0, 1]$  positive solution is that the following integral condition holds*

$$0 < \int_0^1 f(s, 1-s) ds < +\infty. \quad (3.25)$$

*Proof.* The proof is similar to that of Theorem 3.1; we omit the details.  $\square$

**Theorem 3.4.** *Suppose that  $(H^*)$  holds, then a necessary and sufficient condition for problem (3.23) to have a  $C^2[0, 1]$  positive solution is that the following integral condition holds*

$$0 < \int_0^1 s(1-s)f(s, (1-s))ds < +\infty. \quad (3.26)$$

*Proof.* The proof is divided into two parts, necessity and sufficiency.

*Necessity.* Assume that  $x(t)$  is a  $C^2[0, 1]$  positive solution of BVP (3.23). By Lemma 2.4, there exist two constants  $I_1$  and  $I_2$ ,  $0 < I_1 < I_2$ , such that

$$I_1(1-t) \leq x(t) \leq I_2(1-t), \quad t \in [0, 1]. \quad (3.27)$$

Let  $c_1$  be a constant such that  $c_1 I_2 \leq 1, 1/c_1 \geq 1$ . By condition (H), we have

$$\begin{aligned} f(t, x(t)) &= f\left(t, \frac{1}{c_1} \frac{c_1 x(t)}{1-t} (1-t)\right) \geq f\left(t, \frac{c_1 x(t)}{1-t} (1-t)\right) \\ &\geq \xi\left(\frac{1-t}{c_1 x(t)}\right)^{-1} f(t, (1-t)) \geq \frac{c_1 x(t)}{1-t} f(t, (1-t)) \\ &\geq c_1 I_1 f(t, (1-t)), \quad t \in (0, 1). \end{aligned} \quad (3.28)$$

By virtue of (3.28), we obtain that

$$f(t, (1-t)) \leq (c_1 I_1)^{-1} f(t, x(t)) = (c_1 I_1)^{-1} [\varphi_p(x''(t))]'' , \quad t \in (0, 1). \quad (3.29)$$

By boundary value condition, we know that there exists a  $t_0 \in (0, 1)$  such that

$$[\varphi_p(x'')]'(t_0) = 0. \quad (3.30)$$

For  $t \in (t_0, 1)$ , by integration of (3.29), we get

$$\int_{t_0}^t f(s, (1-s)) ds \leq (c_1 I_1)^{-1} [\varphi_p(x'')]'(t), \quad t \in (t_0, 1). \quad (3.31)$$



Integrating (3.31), we have

$$\begin{aligned} \int_{t_0}^1 \int_{t_0}^t f(s, (1-s)) ds dt &\leq (c_1 I_1)^{-1} \int_{t_0}^1 [\varphi_p(x'')]'(t) dt \\ &= (c_1 I_1)^{-1} [\varphi_p(x'')(1) - \varphi_p(x'')(t_0)] < +\infty. \end{aligned} \quad (3.32)$$

Exchanging the order of integration, we obtain that

$$\int_{t_0}^1 \int_{t_0}^t f(s, (1-s)) ds dt = \int_{t_0}^1 (1-s) f(s, (1-s)) ds < +\infty. \quad (3.33)$$

Similarly, by integration of (3.29), we get

$$\int_0^{t_0} s f(s, (1-s)) ds < +\infty. \quad (3.34)$$

Equations (3.33) and (3.34) imply that

$$\int_0^1 s(1-s) f(s, (1-s)) ds < +\infty. \quad (3.35)$$

Since  $x(t)$  is a  $C^2[0, 1]$  positive solution of BVP (1.1), there exists a positive  $t_0 \in (0, 1)$  such that  $f(t_0, x(t_0)) > 0$ . Obviously,  $x(t_0) > 0$ . On the other hand, choose  $c_2 < \min\{1, I_1, 1/I_2\}$ , then  $c_2 I_2 < 1$ . By condition (H), we have

$$\begin{aligned} 0 < f(t_0, x(t_0)) &= f\left(t_0, \frac{1}{c_2} \frac{c_2 x(t_0)}{1-t_0} (1-t_0)\right) \\ &\leq \xi\left(\frac{1}{c_2}\right) f\left(t_0, \frac{c_2 x(t_0)}{1-t_0} (1-t_0)\right) \leq \xi\left(\frac{1}{c_2}\right) f(t_0, (1-t_0)). \end{aligned} \quad (3.36)$$

Consequently,  $f(t_0, t_0) > 0$ , which implies that

$$\int_0^1 s(1-s) f(s, (1-s)) ds > 0. \quad (3.37)$$

It follows from (3.35) and (3.37) that

$$0 < \int_0^1 s(1-s) f(s, (1-s)) ds < +\infty, \quad (3.38)$$

which is the desired inequality.

*Sufficiency.* Suppose that (3.26) holds. Let

$$b_2(t) = A\varphi_p^{-1}Bf(t, 1-t), \quad t \in [0, 1]. \quad (3.39)$$

It is easy to know, from (3.11) and (3.26), that

$$Bf(t, 1-t) \leq \frac{1}{1-\sigma_2} \int_0^1 G(s, s)f(s, 1-s)ds < +\infty. \quad (3.40)$$

Thus, (3.12), (3.39), and (3.40) imply that  $0 \leq b_2(t) < +\infty$ . By Lemma 2.4, we know that there exists a positive number  $k_2 < 1$  such that

$$k_2(1-t) \leq b_2(t) \leq \frac{1}{k_2}(1-t), \quad t \in [0, 1]. \quad (3.41)$$

Take  $0 < l_2 < k_2$  sufficiently small, then by (3.10), we get that  $l_2k_2/h(l_2k_2) \leq k_2$ , that is,

$$h(l_2k_2) - l_2 \geq 0, \quad \xi\left(\frac{1}{l_2k_2}\right) - \frac{1}{l_2} \leq 0. \quad (3.42)$$

Let

$$\alpha(t) = l_2b_2(t), \quad \beta(t) = \frac{1}{l_2}b_2(t), \quad t \in [0, 1]. \quad (3.43)$$

Thus, from (3.41) and (3.43), we have

$$l_2k_2(1-t) \leq \alpha(t) \leq (1-t) \leq \beta(t) \leq \frac{1}{l_2k_2}(1-t), \quad t \in [0, 1]. \quad (3.44)$$

Notice that  $p \geq 2$ , it follows from (3.42)–(3.44) and condition (H) that

$$\begin{aligned} f(t, \alpha(t)) &\geq f(t, l_2k_2(1-t)) \geq h(l_2k_2)f(t, (1-t)) \\ &\geq l_2f(t, (1-t)) \geq l_2^{p-1}f(t, (1-t)) \\ &= [\varphi_p(\alpha''(t))]'' , \quad t \in (0, 1), \\ f(t, \beta(t)) &\leq f\left(t, \frac{1}{l_2k_2}(1-t)\right) \leq \xi\left(\frac{1}{l_2k_2}\right)f(t, (1-t)) \\ &\leq \left(\frac{1}{l_2}\right)f(t, (1-t)) \leq \left(\frac{1}{l_2}\right)^{p-1}f(t, (1-t)) \\ &= [\varphi_p(\beta''(t))]'' , \quad t \in (0, 1). \end{aligned} \quad (3.45)$$

From (3.39) and (3.43), it follows that

$$\begin{aligned}\alpha(0) &= \int_0^1 g(t)\alpha(t)dt, & \beta(0) &= \int_0^1 g(t)\beta(t)dt, & \alpha(1) &= 0, & \beta(1) &= 0, \\ \varphi_p(\alpha''(0)) &= \varphi_p(\alpha''(1)) &= \int_0^1 h(s)\varphi_p(\alpha''(s))ds, & & & & & (3.46) \\ \varphi_p(\beta''(0)) &= \varphi_p(\beta''(1)) &= \int_0^1 h(s)\varphi_p(\beta''(s))ds.\end{aligned}$$

Thus, we have shown that  $\alpha(t)$  and  $\beta(t)$  are lower and upper solutions of BVP (1.1), respectively.

From the first conclusion of Lemma 2.5, we conclude that problem (1.1) has at least one  $C^2[0, 1]$  positive solution  $x(t)$ .  $\square$

#### 4. Dual Results

Consider the fourth-order singular  $p$ -Laplacian differential equations with integral conditions:

$$\begin{aligned}(\varphi_p(x''(t)))'' &= f(t, x(t), x(t)), & 0 < t < 1, \\ x(0) &= \int_0^1 g(s)x(s)ds, & x(1) &= 0,\end{aligned}\tag{4.1}$$

$$\begin{aligned}\varphi_p(x''(0)) &= \int_0^1 h(s)\varphi_p(x''(s))ds, & x''(1) &= 0, \\ (\varphi_p(x''(t)))'' &= f(t, x(t)), & 0 < t < 1, \\ x(0) &= \int_0^1 g(s)x(s)ds, & x(1) &= 0,\end{aligned}\tag{4.2}$$

$$\varphi_p(x''(0)) = \int_0^1 h(s)\varphi_p(x''(s))ds, \quad x''(1) = 0.$$

Firstly, we define the linear operator  $B_1$  as follows:

$$B_1x(t) = \int_0^1 G(t, s)x(s)ds + \frac{1-t}{1 - \int_0^1 (1-s)h(s)ds} \iint_0^1 G(s, \tau)h(\tau)x(s)d\tau ds, \tag{4.3}$$

where  $G(t, s)$  is given by (2.27).

By analogous methods, we have the following results.

Assume that  $x(t)$  is a  $C^2[0, 1]$  positive solution of problem (4.1). Then  $x(t)$  can be expressed by

$$x(t) = A\varphi_p^{-1}B_1f(t, x(t), x(t)). \quad (4.4)$$

**Theorem 4.1.** *Suppose that (H) holds, then a necessary and sufficient condition for (4.1) to have a pseudo- $C^3[0, 1]$  positive solution is that the following integral condition holds:*

$$0 < \int_0^1 f(s, (1-s), (1-s))ds < +\infty. \quad (4.5)$$

**Theorem 4.2.** *Suppose that (H\*) holds, then a necessary and sufficient condition for problem (4.2) to have a pseudo- $C^3[0, 1]$  positive solution is that the following integral condition holds:*

$$0 < \int_0^1 f(s, 1-s)ds < +\infty. \quad (4.6)$$

**Theorem 4.3.** *Suppose that (H\*) holds, then a necessary and sufficient condition for problem (4.2) to have a  $C^2[0, 1]$  positive solution is that the following integral condition holds:*

$$0 < \int_0^1 s(1-s)f(s, (1-s))ds < +\infty. \quad (4.7)$$

Consider the fourth-order singular  $p$ -Laplacian differential equations with integral conditions:

$$\begin{aligned} (\varphi_p(x''(t)))'' &= f(t, x(t), x(t)), \quad 0 < t < 1, \\ x(0) &= 0, \quad x(1) = \int_0^1 g(s)x(s)ds, \end{aligned} \quad (4.8)$$

$$\begin{aligned} \varphi_p(x''(0)) &= \int_0^1 h(s)\varphi_p(x''(s))ds, \quad x''(1) = 0, \\ (\varphi_p(x''(t)))'' &= f(t, x(t)), \quad 0 < t < 1, \\ x(0) &= 0, \quad x(1) = \int_0^1 g(s)x(s)ds, \end{aligned} \quad (4.9)$$

$$\varphi_p(x''(0)) = \int_0^1 h(s)\varphi_p(x''(s))ds, \quad x''(1) = 0.$$

Define the linear operator  $A_1$  as follows:

$$A_1x(t) = \int_0^1 G(t, s)x(s)ds + \frac{t}{1 - \int_0^1 sg(s)ds} \iint_0^1 G(s, \tau)g(\tau)x(s)d\tau ds. \quad (4.10)$$

If  $x(t)$  is a  $C^2[0, 1]$  positive solution of problem (4.8). Then  $x(t)$  can be expressed by

$$x(t) = A_1 \varphi_p^{-1} B_1 f(t, x(t), x(t)). \quad (4.11)$$

**Theorem 4.4.** *Suppose that (H) holds, then a necessary and sufficient condition for problem (4.8) to have a pseudo- $C^3[0, 1]$  positive solution is that the following integral condition holds:*

$$0 < \int_0^1 f(s, s, s) ds < +\infty. \quad (4.12)$$

**Theorem 4.5.** *Suppose that (H\*) holds, then a necessary and sufficient condition for problem (4.9) to have a pseudo- $C^3[0, 1]$  positive solution is that the following integral condition holds:*

$$0 < \int_0^1 f(s, s) ds < +\infty. \quad (4.13)$$

**Theorem 4.6.** *Suppose that (H\*) holds, then a necessary and sufficient condition for problem (4.9) to have a  $C^2[0, 1]$  positive solution is that the following integral condition holds:*

$$0 < \int_0^1 s(1-s) f(s, s) ds < +\infty. \quad (4.14)$$

Consider the fourth-order singular  $p$ -Laplacian differential equations with integral conditions:

$$(\varphi_p(x''(t)))'' = f(t, x(t), x(t)), \quad 0 < t < 1,$$

$$x(0) = 0, \quad x(1) = \int_0^1 g(s)x(s)ds, \quad (4.15)$$

$$\varphi_p(x''(0)) = 0, \quad \varphi_p(x''(1)) = \int_0^1 h(s)\varphi_p(x''(s))ds,$$

$$(\varphi_p(x''(t)))'' = f(t, x(t)), \quad 0 < t < 1,$$

$$x(0) = 0, \quad x(1) = \int_0^1 g(s)x(s)ds, \quad (4.16)$$

$$\varphi_p(x''(0)) = 0, \quad \varphi_p(x''(1)) = \int_0^1 h(s)\varphi_p(x''(s))ds.$$

Define the linear operator  $B_2$  as follows:

$$B_2 x(t) = \int_0^1 G(t, s)x(s)ds + \frac{t}{1 - \int_0^1 sh(s)ds} \int_0^1 G(s, \tau)h(\tau)x(s)d\tau ds. \quad (4.17)$$

If  $x(t)$  is a  $C^2[0, 1]$  positive solution of problem (4.15). Then  $x(t)$  can be expressed by

$$x(t) = A_1 \varphi_p^{-1} B_2 f(t, x(t), x(t)). \quad (4.18)$$

**Theorem 4.7.** *Suppose that (H) holds, then a necessary and sufficient condition for problem (4.15) to have a pseudo- $C^3[0, 1]$  positive solution is that the following integral condition holds:*

$$0 < \int_0^1 f(s, s, s) ds < +\infty. \quad (4.19)$$

**Theorem 4.8.** *Suppose that (H\*) holds, then a necessary and sufficient condition for problem (4.16) to have a pseudo- $C^3[0, 1]$  positive solution is that the following integral condition holds:*

$$0 < \int_0^1 f(s, s) ds < +\infty. \quad (4.20)$$

**Theorem 4.9.** *Suppose that (H\*) holds, then a necessary and sufficient condition for problem (4.16) to have a  $C^2[0, 1]$  positive solution is that the following integral condition holds:*

$$0 < \int_0^1 s(1-s) f(s, s) ds < +\infty. \quad (4.21)$$

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