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# Positive solutions for Hadamard differential systems with fractional integral conditions on an unbounded domain

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**Abstract:** In this paper, we investigate the existence of positive solutions for Hadamard type fractional differential system with coupled nonlocal fractional integral boundary conditions on an infinite domain. Our analysis relies on Guo-Krasnoselskii's and Leggett-Williams fixed point theorems. The obtained results are well illustrated with the aid of examples.

**Keywords:** Hadamard fractional differential systems, Nonlocal boundary conditions, Hadamard fractional integral conditions, Positive solutions, Fixed point theorems

**MSC:** 26A33, 34A08, 34B18, 34B40

## 1 Introduction

Fractional calculus and fractional differential equations have been studied extensively during the last decades. Fractional derivatives provide a more excellent tool for the description of memory and hereditary properties of various materials and processes than integer derivatives. Engineers and scientists have developed new models that involve fractional differential equations. These models have been applied successfully in, e.g., physics, biomathematics, blood flow phenomena, ecology, environmental issues, viscoelasticity, aerodynamics, electrodynamics of complex medium, electrical circuits, electron-analytical chemistry, control theory, etc. For a systematic development of the topic, we refer to the books [1]-[7]. As an important issue for the theory of fractional differential equations, the existence of solutions to kinds of boundary value problems has attracted many scholars attention, and lots of excellent results have been obtained by means of fixed point theorems, upper and lower solutions technique, and so forth. A variety of results on initial and boundary value problems of fractional differential equations and inclusions can easily be found in the literature on the topic. For some recent results, we can refer to [8]-[18] and references cited therein.

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Boundary value problems on infinite intervals appear often in applied mathematics and physics. Due to the fact that an infinite interval is noncompact, the discussion about boundary value problem on the infinite intervals is more complicated. Results on the existence of solutions of boundary value problems on infinite intervals for differential, difference and integral equations may be found in the monographs [19, 20]. For boundary value problems of fractional order on infinite intervals we refer to [21]-[25].

Many researchers have shown their interest in the study of systems of fractional differential equations. The motivation for those works stems from both the intensive development of the theory of fractional calculus itself and the applications. See for example [26]-[30] where systems for fractional differential equations were studied by using Banach contraction mapping principle and Schaefer's fixed point theorem.

Recently in [31] we investigated the existence of positive solutions for fractional differential equations of Hadamard type, with integral boundary condition on infinite intervals

$$D^{\alpha}u(t) + a(t)f(u(t)) = 0, \quad 1 < \alpha \le 2, \ t \in (1,\infty), \tag{1}$$

$$u(1) = 0, \qquad D^{\alpha-1}u(\infty) = \sum_{i=1}^{m} \lambda_i I^{\beta_i} u(\eta), \tag{2}$$

where $D^{\alpha}$ denotes the Hadamard fractional derivative of order $\alpha$, $\eta \in (1,\infty)$ and $I^{\beta_i}$ is the Hadamard fractional integral of order $\beta_i > 0$, $i = 1, 2, \ldots, m$ and $\lambda_i \ge 0$, $i = 1, 2, \ldots, m$ are given constants. For some recent results on positive solutions of fractional differential equations we refer to [32]-[37] and references cited therein.

In [38] the existence of positive solutions were studied for the following fractional system of differential equations subject to the nonlocal Riemann-Liouville fractional integral boundary conditions of the form

$$\begin{cases} D^{p}x(t) + f(t, x(t), y(t)) = 0, \quad 1 < p \le 2, \ t \in (0,1), \\ D^{q}y(t) + g(t, x(t), y(t)) = 0, \quad 1 < q \le 2, \ t \in (0,1), \\ x(0) = 0, \quad x(1) = \sum\limits_{i=1}^{m} \alpha_i I^{\gamma_i} y(\eta), \\ y(0) = 0, \quad y(1) = \sum\limits_{j=1}^{n} \beta_j I^{\mu_j} x(\xi), \end{cases} \tag{3}$$

where $D^{\phi}$ are Riemann-Liouville fractional derivatives of orders $\phi \in \{p, q\}$, $f, g \in C([0,1] \times \mathbb{R}_+^2, \mathbb{R}_+)$, $I^{\Phi}$ are Riemann-Liouville fractional integrals of order $\Phi \in \{\gamma_i, \mu_j\}$, $\alpha_i, \beta_j > 0$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ and the fixed constants $0 < \eta < \xi < 1$.

In this paper we investigate the existence of positive solutions for the following fractional system of Hadamard differential equations subject to the fractional integral boundary conditions on an unbounded domain

$$\begin{cases} D^{p}x(t) + a(t)f(x(t), y(t)) = 0, \quad 1 < p \le 2, \ t \in (1,\infty), \\ D^{q}y(t) + b(t)g(x(t), y(t)) = 0, \quad 1 < q \le 2, \ t \in (1,\infty), \\ x(1) = 0, \qquad D^{p-1}x(\infty) = \sum\limits_{i=1}^{m} \lambda_i I^{\alpha_i} y(\eta), \\ y(1) = 0, \qquad D^{q-1}y(\infty) = \sum\limits_{j=1}^{n} \sigma_j I^{\beta_j} x(\xi), \end{cases} \tag{4}$$

where $D^{\phi}$ are Hadamard fractional derivatives of orders $\phi \in \{p, q\}$ with lower limit 1, $f, g \in C([1,\infty) \times \mathbb{R}_+^2, \mathbb{R}_+)$, $I^{\Phi}$ are Hadamard fractional integrals of order $\Phi \in \{\alpha_i, \beta_j\}$ with lower limit 1, $\lambda_i, \sigma_j > 0$, $i = 1, \ldots, m$, $j = 1, \ldots, n$.

Applying first the well-known Guo-Krasnoselskii's fixed point theorem we obtain the existence of at least one positive solution. Next we prove the existence of at least three distinct nonnegative solutions by using Leggett-Williams fixed point theorem.

The rest of the paper is organized as follows: In section 2, we present some preliminaries and lemmas that will be used to prove our main results. We also obtain the corresponding Green's function and some of its properties. The main results are formulated and proved in Section 3. Especially in Subsection 3.1 we prove the existence of at least one positive solution while, in Subsection 3.2, we prove the existence of at least three distinct nonnegative solutions. Examples illustrating our results are presented in Section 4.

## 2 Background materials and preliminaries

In this section, we present some notations and definitions of Hadamard fractional calculus (see [4]) and present preliminary results needed in our proofs later.

**Definition 2.1** ([4]). *The Hadamard fractional integral of order $\Phi$ with lower limit 1 for a function $g : [1,\infty) \to \mathbb{R}$ is defined as*

$$I^{\Phi} g(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\log\frac{t}{s}\right)^{\Phi-1} \frac{g(s)}{s} ds, \quad \Phi > 0, \tag{5}$$

*provided the integral exists, where $\log(\cdot) = \log_e(\cdot)$.*

**Definition 2.2** ([4]). *The Hadamard derivative of fractional order $\phi$ with lower limit 1 for a function $g : [1,\infty) \to \mathbb{R}$ is defined as*

$$D^{\phi} g(t) = \frac{1}{\Gamma(n-\phi)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log\frac{t}{s}\right)^{n-\phi-1} \frac{g(s)}{s} ds, \quad n-1 < \phi < n, \tag{6}$$

*where $n = [\phi]+1$, $[\phi]$ denotes the integer part of the real number $\phi$.*

**Lemma 2.3** ([4, Property 2.24]). *If $a, \alpha, \beta > 0$ then*

$$\left(D_a^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{x}{a}\right)^{\beta-\alpha-1}. \tag{7}$$

*and*

$$\left(I^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\log\frac{x}{a}\right)^{\beta+\alpha-1}. \tag{8}$$

**Lemma 2.4** ([4]). *Let $q > 0$ and $x \in C[1,\infty) \cap L^1[1,\infty)$. Then the Hadamard fractional differential equation $D^q x(t) = 0$ has the solutions*

$$x(t) = \sum_{i=1}^{n} c_i (\log t)^{q-i},$$

*and the following formula holds:*

$$I^q D^q x(t) = x(t) + \sum_{i=1}^{n} c_i (\log t)^{q-i},$$

*where $c_i \in \mathbb{R}$, $i = 1,2,\ldots,n$, and $n-1 < q < n$.*

**Lemma 2.5.** *Suppose that the functions $u, v \in C([1,\infty),\mathbb{R}^+)$ and $1 < p, q \le 2$. Then the following system*

$$\begin{cases} D^p x(t) + u(t) = 0, & t \in (1,\infty), \\ D^q y(t) + v(t) = 0, & t \in (1,\infty), \\ x(1) = 0, \quad D^{p-1}x(\infty) = \sum\limits_{i=1}^{m} \lambda_i I^{\alpha_i} y(\eta), \\ y(1) = 0, \quad D^{q-1}y(\infty) = \sum\limits_{j=1}^{n} \sigma_j I^{\beta_j} x(\xi), \end{cases} \tag{9}$$

*can be written in the equivalent integral equations of the form*

$$x(t) = -\frac{1}{\Gamma(p)} \int_1^t \left(\log\frac{t}{s}\right)^{p-1} u(s)\frac{ds}{s} + (\log t)^{p-1}\left[\frac{\Gamma(q)}{\Omega}\int_1^{\infty} u(s)\frac{ds}{s}\right.$$

$$-\frac{\Gamma(q)}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(q+\alpha_i)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{q+\alpha_i-1}v(s)\frac{ds}{s}+\frac{\Lambda_1}{\Omega}\int_1^{\infty}v(s)\frac{ds}{s}$$
$$-\frac{\Lambda_1}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j}{\Gamma(p+\beta_j)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{p+\beta_j-1}u(s)\frac{ds}{s}\Bigg],\tag{10}$$

*and*

$$y(t)=-\frac{1}{\Gamma(q)}\int_1^{t}\left(\log\frac{t}{s}\right)^{q-1}v(s)\frac{ds}{s}+(\log t)^{q-1}\Bigg[\frac{\Gamma(p)}{\Omega}\int_1^{\infty}v(s)\frac{ds}{s}$$
$$-\frac{\Gamma(p)}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j}{\Gamma(p+\beta_j)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{p+\beta_j-1}u(s)\frac{ds}{s}+\frac{\Lambda_2}{\Omega}\int_1^{\infty}u(s)\frac{ds}{s}$$
$$-\frac{\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(q+\alpha_i)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{q+\alpha_i-1}v(s)\frac{ds}{s}\Bigg],\tag{11}$$

*where*

$$\Omega:=\Gamma(p)\Gamma(q)-\Lambda_1\Lambda_2>0,$$

*with*

$$\Lambda_1:=\sum_{i=1}^{m}\frac{\lambda_i\Gamma(q)}{\Gamma(q+\alpha_i)}(\log\eta)^{q+\alpha_i-1}\quad\textit{and}\quad\Lambda_2:=\sum_{j=1}^{n}\frac{\sigma_j\Gamma(p)}{\Gamma(p+\beta_j)}(\log\xi)^{p+\beta_j-1}.$$

*Proof.* By applying the Hadamard fractional integral of orders $p$ and $q$ to both sides of the first two equations of system (9), respectively, we obtain

$$x(t)=-\frac{1}{\Gamma(p)}\int_1^{t}\left(\log\frac{t}{s}\right)^{p-1}u(s)\frac{ds}{s}+c_1(\log t)^{p-1}+c_2(\log t)^{p-2},$$
$$y(t)=-\frac{1}{\Gamma(q)}\int_1^{t}\left(\log\frac{t}{s}\right)^{q-1}v(s)\frac{ds}{s}+k_1(\log t)^{q-1}+k_2(\log t)^{q-2},$$

where $c_1,c_2,k_1,k_2\in\mathbb{R}$.

The conditions of (9) that $x(1)=0$ and $y(1)=0$ imply $c_2=0$ and $k_2=0$. Hence

$$x(t)=-\frac{1}{\Gamma(p)}\int_1^{t}\left(\log\frac{t}{s}\right)^{p-1}u(s)\frac{ds}{s}+c_1(\log t)^{p-1},\tag{12}$$

and

$$y(t)=-\frac{1}{\Gamma(q)}\int_1^{t}\left(\log\frac{t}{s}\right)^{q-1}v(s)\frac{ds}{s}+k_1(\log t)^{q-1}.\tag{13}$$

From Lemma 2.3, we obtain

$$D^{p-1}x(t)=-\int_1^{t}u(s)\frac{ds}{s}+c_1\Gamma(p)\qquad\text{and}\qquad D^{q-1}y(t)=-\int_1^{t}v(s)\frac{ds}{s}+k_1\Gamma(q).$$

By applying the Hadamard fractional integral of orders $\beta_j$ and $\alpha_i$ to (12) and (13), and also substitution $t=\xi$ and $t=\eta$, respectively, we get

$$I^{\beta_j}x(\xi)=-\frac{1}{\Gamma(p+\beta_j)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{p+\beta_j-1}u(s)\frac{ds}{s}+c_1\frac{\Gamma(p)}{\Gamma(p+\beta_j)}(\log\xi)^{p+\beta_j-1},$$

and

$$I^{\alpha_i} y(\eta) = -\frac{1}{\Gamma(q+\alpha_i)} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{q+\alpha_i-1} v(s)\frac{ds}{s} + k_1 \frac{\Gamma(q)}{\Gamma(q+\alpha_i)} (\log \eta)^{q+\alpha_i-1}.$$

Using the conditions of (9) and solving the system of linear equations we find the constants $c_1$ and $k_1$ as

$$c_1 = \frac{\Gamma(q)}{\Omega} \int_1^{\infty} u(s)\frac{ds}{s} - \frac{\Gamma(q)}{\Omega} \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(q+\alpha_i)} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{q+\alpha_i-1} v(s)\frac{ds}{s}$$
$$+ \frac{\Lambda_1}{\Omega} \int_1^{\infty} v(s)\frac{ds}{s} - \frac{\Lambda_1}{\Omega} \sum_{j=1}^{n} \frac{\sigma_j}{\Gamma(p+\beta_j)} \int_1^{\xi} \left(\log \frac{\xi}{s}\right)^{p+\beta_j-1} u(s)\frac{ds}{s},$$

and

$$k_1 = \frac{\Gamma(p)}{\Omega} \int_1^{\infty} v(s)\frac{ds}{s} - \frac{\Gamma(p)}{\Omega} \sum_{j=1}^{n} \frac{\sigma_j}{\Gamma(p+\beta_j)} \int_1^{\xi} \left(\log \frac{\xi}{s}\right)^{p+\beta_j-1} u(s)\frac{ds}{s}$$
$$+ \frac{\Lambda_2}{\Omega} \int_1^{\infty} u(s)\frac{ds}{s} - \frac{\Lambda_2}{\Omega} \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(q+\alpha_i)} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{q+\alpha_i-1} v(s)\frac{ds}{s}.$$

Substituting the values of $c_1$ and $k_1$ in (12) and (13), we deduce the integral equations (10) and (11), respectively. The converse follows by direct computation. This completes the proof. □

**Lemma 2.6** (Green's function). *The integral equations* (10) *and* (11)*, in Lemma 2.5, can be expressed in the form of Green's functions as*

$$x(t) = \int_1^{\infty} G_1(t,s)u(s)\frac{ds}{s} + \int_1^{\infty} G_2(t,s)v(s)\frac{ds}{s}, \tag{14}$$

$$y(t) = \int_1^{\infty} G_3(t,s)v(s)\frac{ds}{s} + \int_1^{\infty} G_4(t,s)u(s)\frac{ds}{s}, \tag{15}$$

*where the Green's functions $G_i(t,s)$, $i = 1,2,3,4$ are given by*

$$G_1(t,s) = g_p(t,s) + \frac{\Lambda_1}{\Omega} \sum_{j=1}^{n} \frac{\sigma_j (\log t)^{p-1}}{\Gamma(p+\beta_j)} g^p_{\beta_j}(\xi,s),$$
$$G_2(t,s) = \frac{\Gamma(q)}{\Omega} \sum_{i=1}^{m} \frac{\lambda_i (\log t)^{p-1}}{\Gamma(q+\alpha_i)} g^q_{\alpha_i}(\eta,s),$$
$$G_3(t,s) = g_q(t,s) + \frac{\Lambda_2}{\Omega} \sum_{i=1}^{m} \frac{\lambda_i (\log t)^{q-1}}{\Gamma(q+\alpha_i)} g^q_{\alpha_i}(\eta,s),$$
$$G_4(t,s) = \frac{\Gamma(p)}{\Omega} \sum_{j=1}^{n} \frac{\sigma_j (\log t)^{q-1}}{\Gamma(p+\beta_j)} g^p_{\beta_j}(\xi,s),$$

*where*

$$g_\phi(t,s) = \begin{cases} \dfrac{(\log t)^{\phi-1} - (\log(t/s))^{\phi-1}}{\Gamma(\phi)}, & 1 \le s \le t < \infty, \\ \dfrac{(\log t)^{\phi-1}}{\Gamma(\phi)}, & 1 \le t \le s < \infty, \end{cases} \tag{16}$$

*and*

$$g^\phi_\psi(\rho,s) = \begin{cases} (\log \rho)^{\phi+\psi-1} - (\log(\rho/s))^{\phi+\psi-1}, & 1 \le s \le \rho < \infty, \\ (\log \rho)^{\phi+\psi-1}, & 1 \le \rho \le s < \infty, \end{cases} \tag{17}$$

*with* $\phi \in \{p, q\}$, $\psi \in \{\alpha_j, \beta_i\}$, $\rho \in \{\xi, \eta\}$.

*Proof.* By Lemma 2.5, we have

$$
\begin{aligned}
x(t) = &-\frac{1}{\Gamma(p)}\int_1^t \left(\log\frac{t}{s}\right)^{p-1} u(s)\frac{ds}{s} + (\log t)^{p-1}\left[\frac{\Gamma(q)}{\Omega}\int_1^\infty u(s)\frac{ds}{s}\right.\\
&-\frac{\Gamma(q)}{\Omega}\sum_{i=1}^m \frac{\lambda_i}{\Gamma(q+\alpha_i)}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{q+\alpha_i-1} v(s)\frac{ds}{s} + \frac{\Lambda_1}{\Omega}\int_1^\infty v(s)\frac{ds}{s}\\
&\left.-\frac{\Lambda_1}{\Omega}\sum_{j=1}^n \frac{\sigma_j}{\Gamma(p+\beta_j)}\int_1^\xi \left(\log\frac{\xi}{s}\right)^{p+\beta_j-1} u(s)\frac{ds}{s}\right] + \frac{1}{\Gamma(p)}\int_1^\infty (\log t)^{p-1} u(s)\frac{ds}{s}\\
&-\frac{(\Gamma(p)\Gamma(q)-\Lambda_1\Lambda_2)}{\Omega\Gamma(p)}\int_1^\infty (\log t)^{p-1} u(s)\frac{ds}{s}\\
= &\int_1^\infty g_p(t,s)u(s)\frac{ds}{s} + \frac{\Lambda_1}{\Omega}\int_1^\infty (\log t)^{p-1} v(s)\frac{ds}{s}\\
&-\frac{\Gamma(q)}{\Omega}\sum_{i=1}^m \frac{\lambda_i\,(\log t)^{p-1}}{\Gamma(q+\alpha_i)}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{q+\alpha_i-1} v(s)\frac{ds}{s}\\
&-\frac{\Lambda_1}{\Omega}\sum_{j=1}^n \frac{\sigma_j\,(\log t)^{p-1}}{\Gamma(p+\beta_j)}\int_1^\xi \left(\log\frac{\xi}{s}\right)^{p+\beta_j-1} u(s)\frac{ds}{s} + \frac{\Lambda_1\Lambda_2}{\Omega\Gamma(p)}\int_1^\infty (\log t)^{p-1} u(s)\frac{ds}{s}\\
= &\int_1^\infty g_p(t,s)u(s)\frac{ds}{s} + \frac{\Gamma(q)}{\Omega}\sum_{i=1}^m \frac{\lambda_i\,(\log t)^{p-1}}{\Gamma(q+\alpha_i)}\int_1^\infty (\log\eta)^{q+\alpha_i-1} v(s)\frac{ds}{s}\\
&-\frac{\Gamma(q)}{\Omega}\sum_{i=1}^m \frac{\lambda_i\,(\log t)^{p-1}}{\Gamma(q+\alpha_i)}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{q+\alpha_i-1} v(s)\frac{ds}{s}\\
&-\frac{\Lambda_1}{\Omega}\sum_{j=1}^n \frac{\sigma_j\,(\log t)^{p-1}}{\Gamma(p+\beta_j)}\int_1^\xi \left(\log\frac{\xi}{s}\right)^{p+\beta_j-1} u(s)\frac{ds}{s}\\
&+\frac{\Lambda_1}{\Omega}\sum_{j=1}^n \frac{\sigma_j\,(\log t)^{p-1}}{\Gamma(p+\beta_j)}\int_1^\infty (\log\xi)^{p+\beta_j-1} u(s)\frac{ds}{s}\\
= &\int_1^\infty g_p(t,s)u(s)\frac{ds}{s} + \frac{\Lambda_1}{\Omega}\sum_{j=1}^n \frac{\sigma_j\,(\log t)^{p-1}}{\Gamma(p+\beta_j)}\int_1^\infty g^p_{\beta_j}(\xi,s)u(s)\frac{ds}{s}\\
&+\frac{\Gamma(q)}{\Omega}\sum_{i=1}^m \frac{\lambda_i\,(\log t)^{p-1}}{\Gamma(q+\alpha_i)}\int_1^\infty g^q_{\alpha_i}(\eta,s)v(s)\frac{ds}{s}\\
= &\int_1^\infty G_1(t,s)u(s)\frac{ds}{s} + \int_1^\infty G_2(t,s)v(s)\frac{ds}{s},
\end{aligned}
$$

which yields that (14) is satisfied. In a similar way, we have

$$
y(t) = -\frac{1}{\Gamma(q)}\int_1^t \left(\log\frac{t}{s}\right)^{q-1} v(s)\frac{ds}{s} + (\log t)^{q-1}\left[\frac{\Gamma(p)}{\Omega}\int_1^\infty v(s)\frac{ds}{s}\right.
$$

$$
\begin{aligned}
&-\frac{\Gamma(p)}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j}{\Gamma(p+\beta_j)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{p+\beta_j-1}u(s)\frac{ds}{s}+\frac{\Lambda_2}{\Omega}\int_1^{\infty}u(s)\frac{ds}{s}\\
&-\frac{\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(q+\alpha_i)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{q+\alpha_i-1}v(s)\frac{ds}{s}\bigg]+\frac{1}{\Gamma(q)}\int_1^{\infty}(\log t)^{q-1}\,v(s)\frac{ds}{s}\\
&-\frac{(\Gamma(p)\Gamma(q)-\Lambda_1\Lambda_2)}{\Omega\Gamma(q)}\int_1^{\infty}(\log t)^{q-1}\,v(s)\frac{ds}{s}\\
=&\int_1^{\infty}g_q(t,s)v(s)\frac{ds}{s}+\frac{\Lambda_2}{\Omega}\int_1^{\infty}(\log t)^{q-1}\,u(s)\frac{ds}{s}\\
&-\frac{\Gamma(p)}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j\,(\log t)^{q-1}}{\Gamma(p+\beta_j)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{p+\beta_j-1}u(s)\frac{ds}{s}\\
&-\frac{\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i\,(\log t)^{q-1}}{\Gamma(q+\alpha_i)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{q+\alpha_i-1}v(s)\frac{ds}{s}+\frac{\Lambda_1\Lambda_2}{\Omega\Gamma(q)}\int_1^{\infty}(\log t)^{q-1}\,v(s)\frac{ds}{s}\\
=&\int_1^{\infty}g_q(t,s)v(s)\frac{ds}{s}+\frac{\Gamma(p)}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j\,(\log t)^{q-1}}{\Gamma(p+\beta_j)}\int_1^{\infty}(\log\xi)^{p+\beta_j-1}\,u(s)\frac{ds}{s}\\
&-\frac{\Gamma(p)}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j\,(\log t)^{q-1}}{\Gamma(p+\beta_j)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{p+\beta_j-1}u(s)\frac{ds}{s}\\
&-\frac{\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i\,(\log t)^{q-1}}{\Gamma(q+\alpha_i)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{q+\alpha_i-1}v(s)\frac{ds}{s}\\
&+\frac{\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i\,(\log t)^{q-1}}{\Gamma(q+\alpha_i)}\int_1^{\infty}(\log\eta)^{q+\alpha_i-1}\,v(s)\frac{ds}{s}\\
=&\int_1^{\infty}g_q(t,s)v(s)\frac{ds}{s}+\frac{\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i\,(\log t)^{q-1}}{\Gamma(q+\alpha_i)}\int_1^{\infty}g^{q}_{\alpha_i}(\eta,s)v(s)\frac{ds}{s}\\
&+\frac{\Gamma(p)}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j\,(\log t)^{q-1}}{\Gamma(p+\beta_j)}\int_1^{\xi}g^{p}_{\beta_j}(\xi,s)u(s)\frac{ds}{s}\\
=&\int_1^{\infty}G_3(t,s)v(s)\frac{ds}{s}+\int_1^{\infty}G_4(t,s)u(s)\frac{ds}{s},
\end{aligned}
$$

which proves that (15) holds. This completes the proof. □

Before establishing some properties of the Green's functions, we set

$$
M_1=\frac{1}{\Gamma(p)}+\frac{\Lambda_1}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j\,(\log\xi)^{p+\beta_j-1}}{\Gamma(p+\beta_j)},\quad M_2=\frac{\Gamma(q)}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i\,(\log\eta)^{q+\alpha_i-1}}{\Gamma(q+\alpha_i)},
$$

$$
M_3=\frac{1}{\Gamma(q)}+\frac{\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i\,(\log\eta)^{q+\alpha_i-1}}{\Gamma(q+\alpha_i)},\quad M_4=\frac{\Gamma(p)}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j\,(\log\xi)^{p+\beta_j-1}}{\Gamma(p+\beta_j)}.
$$

**Lemma 2.7.** *The Green's functions $G_i(t,s)$, $i=1,2,3,4$, satisfy the following properties:*
$(P_1)G_i(t,s)$, $i=1,2,3,4$ *are continuous for* $(t,s)\in[1,\infty)\times[1,\infty)$*;*

$(P_2) G_i(t,s) \geq 0$, $i = 1,2,3,4$ *for all* $(t,s) \in [1,\infty) \times [1,\infty)$;

$(P_3)$ $\dfrac{G_1(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}} \leq M_1, \quad \dfrac{G_2(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}} \leq M_2,$

$\dfrac{G_3(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}} \leq M_3, \quad \dfrac{G_4(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}} \leq M_4;$

$(P_4)$ $\inf\limits_{\xi \leq t \leq \xi k_1} \dfrac{G_1(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}} \geq \dfrac{\Lambda_1}{\Omega} \sum\limits_{j=1}^{n} \dfrac{\sigma_j (\log \xi)^{p-1} g^p_{\beta_j}(\xi,s)}{\Gamma(p+\beta_j)(1+(\log \xi)^{p-1}+(\log \xi)^{q-1})},$

$\inf\limits_{\eta \leq t \leq \eta k_2} \dfrac{G_2(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}} \geq \dfrac{\Gamma(q)}{\Omega} \sum\limits_{i=1}^{m} \dfrac{\lambda_i (\log \eta)^{p-1} g^q_{\alpha_i}(\eta,s)}{\Gamma(q+\alpha_i)(1+(\log \eta)^{p-1}+(\log \eta)^{q-1})},$

$\inf\limits_{\eta \leq t \leq \eta k_2} \dfrac{G_3(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}} \geq \dfrac{\Lambda_2}{\Omega} \sum\limits_{i=1}^{m} \dfrac{\lambda_i (\log \eta)^{q-1} g^q_{\alpha_i}(\eta,s)}{\Gamma(q+\alpha_i)(1+(\log \eta)^{p-1}+(\log \eta)^{q-1})},$

$\inf\limits_{\xi \leq t \leq \xi k_1} \dfrac{G_4(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}} \geq \dfrac{\Gamma(p)}{\Omega} \sum\limits_{j=1}^{n} \dfrac{\sigma_j (\log \xi)^{q-1} g^p_{\beta_j}(\xi,s)}{\Gamma(p+\beta_j)(1+(\log \xi)^{p-1}+(\log \xi)^{q-1})},$

*for any* $\kappa_1, \kappa_2 > 1$.

*Proof.* It is easy to prove that $(P_1)$ and $(P_2)$ hold.

To prove $(P_3)$, for $(s,t) \in [1,\infty) \times [1,\infty)$, we have

$$\begin{aligned}
&\frac{G_1(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}} \\
&= \frac{g_p(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}} + \frac{\Lambda_1}{\Omega} \sum_{j=1}^{n} \frac{\sigma_j (\log t)^{p-1} g^p_{\beta_j}(\xi,s)}{\Gamma(p+\beta_j)(1+(\log t)^{p-1}+(\log t)^{q-1})} \\
&\leq \frac{(\log t)^{p-1}}{\Gamma(p)(1+(\log t)^{p-1}+(\log t)^{q-1})} + \frac{\Lambda_1}{\Omega} \sum_{j=1}^{n} \frac{\sigma_j (\log t)^{p-1} g^p_{\beta_j}(\xi,s)}{\Gamma(p+\beta_j)(1+(\log t)^{p-1}+(\log t)^{q-1})} \\
&\leq \frac{1}{\Gamma(p)} + \frac{\Lambda_1}{\Omega} \sum_{j=1}^{n} \frac{\sigma_j g^p_{\beta_j}(\xi,s)}{\Gamma(p+\beta_j)}.
\end{aligned}$$

In similar manner, we can prove that

$$\begin{aligned}
\frac{G_2(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}} &\leq \frac{\Gamma(q)}{\Omega} \sum_{i=1}^{m} \frac{\lambda_i g^q_{\alpha_i}(\eta,s)}{\Gamma(q+\alpha_i)}, \\
\frac{G_3(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}} &\leq \frac{1}{\Gamma(q)} + \frac{\Lambda_2}{\Omega} \sum_{i=1}^{m} \frac{\lambda_i g^q_{\alpha_i}(\eta,s)}{\Gamma(q+\alpha_i)}, \\
\frac{G_4(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}} &\leq \frac{\Gamma(p)}{\Omega} \sum_{j=1}^{n} \frac{\sigma_j g^p_{\beta_j}(\xi,s)}{\Gamma(p+\beta_j)}.
\end{aligned}$$

From (17), it follows that $(P_3)$ holds.

To prove $(P_4)$, from the positivity of $g_p(t,s)$ and $g^p_{\beta_j}(\xi,s)$, $j = 1,\ldots,n$, we have for any $\kappa_1 > 1$,

$$\begin{aligned}
&\min_{\xi \leq t \leq \xi \kappa_1} \frac{G_1(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}} \\
&= \min_{\xi \leq t \leq \xi \kappa_1} \left[ \frac{g_p(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}} + \frac{\Lambda_1}{\Omega} \sum_{j=1}^{n} \frac{\sigma_j (\log t)^{p-1} g^p_{\beta_j}(\xi,s)}{\Gamma(p+\beta_j)(1+(\log t)^{p-1}+(\log t)^{q-1})} \right] \\
&\geq \min_{\xi \leq t \leq \xi \kappa_1} \left[ \frac{\Lambda_1}{\Omega} \sum_{j=1}^{n} \frac{\sigma_j (\log t)^{p-1} g^p_{\beta_j}(\xi,s)}{\Gamma(p+\beta_j)(1+(\log t)^{p-1}+(\log t)^{q-1})} \right] \\
&\geq \frac{\Lambda_1}{\Omega} \sum_{j=1}^{n} \frac{\sigma_j (\log \xi)^{p-1} g^p_{\beta_j}(\xi,s)}{\Gamma(p+\beta_j)(1+(\log \xi)^{p-1}+(\log \xi)^{q-1})}.
\end{aligned}$$

In the same way, for any $\kappa_2 > 1$, we have

$$\min_{\eta\le t\le\eta\kappa_2} \frac{G_2(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}} \ge \frac{\Gamma(q)}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i\,(\log\eta)^{p-1}\,g^q_{\alpha_i}(\eta,s)}{\Gamma(q+\alpha_i)(1+(\log\eta)^{p-1}+(\log\eta)^{q-1})},$$

$$\min_{\eta\le t\le\eta\kappa_2} \frac{G_3(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}} \ge \frac{\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i\,(\log\eta)^{q-1}\,g^q_{\alpha_i}(\eta,s)}{\Gamma(q+\alpha_i)(1+(\log\eta)^{p-1}+(\log\eta)^{q-1})},$$

$$\min_{\xi\le t\le\xi\kappa_1} \frac{G_4(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}} \ge \frac{\Gamma(p)}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j\,(\log\xi)^{q-1}\,g^p_{\beta_j}(\xi,s)}{\Gamma(p+\beta_j)(1+(\log\xi)^{p-1}+(\log\xi)^{q-1})}.$$

Therefore, $(P_4)$ is proved. □

## 3 Main results

Define the set

$$E=\left\{(x,y)\in C([1,\infty),\mathbb{R})\times C([1,\infty),\mathbb{R}) : \sup_{t\in[1,\infty)}\frac{|x(t)|+|y(t)|}{1+(\log t)^{p-1}+(\log t)^{q-1}}<\infty\right\},$$

and the norm

$$\|(x,y)\| = \sup_{t\in[1,\infty)}\frac{|x(t)|+|y(t)|}{1+(\log t)^{p-1}+(\log t)^{q-1}}.$$

It is clear that $(E,\|\cdot\|)$ is a Banach space.

**Lemma 3.1.** *Let $U\subset E$ be a bounded set. If the following conditions hold:*

*(i) for any $(u,v)(t)\in U$, $\dfrac{u(t)+v(t)}{1+(\log t)^{p-1}+(\log t)^{q-1}}$ is equicontinuous on any compact interval of* $[1,\infty)$*;*

*(ii) for any $\varepsilon>0$, there exists a constant $T=T(\varepsilon)>0$ such that*

$$\left|\frac{u(t_1)+v(t_1)}{1+(\log t_1)^{p-1}+(\log t_1)^{q-1}}-\frac{u(t_2)+v(t_2)}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}}\right|<\varepsilon \tag{18}$$

*for any $t_1,t_2\ge T$ and $(u,v)\in U$,*

*then $U$ is relatively compact in $E$.*

The proof is similar to that of Lemma 2.8 in [31], and is omitted.

Now, we define the positive cone $P\subset E$ by

$$P=\{(x,y)\in E : x(t)\ge 0,\ y(t)\ge 0 \text{ on } [1,\infty)\},$$

and the operator $\mathcal{T}: P\to E$ by $\mathcal{T}(x,y)(t)=(A(x,y)(t),B(x,y)(t))$ for all $t\in[1,\infty)$, where the operators $A: P\to E$ and $B: P\to E$ are defined by

$$\begin{cases} A(x,y)(t) := \displaystyle\int_1^\infty G_1(t,s)a(s)f(x(s),y(s))\frac{ds}{s}+\int_1^\infty G_2(t,s)b(s)g(x(s),y(s))\frac{ds}{s},\\ B(x,y)(t) := \displaystyle\int_1^\infty G_3(t,s)b(s)g(x(s),y(s))\frac{ds}{s}+\int_1^\infty G_4(t,s)a(s)f(x(s),y(s))\frac{ds}{s}. \end{cases} \tag{19}$$

Throughout this paper we assume that the following conditions hold:

$(H_1)$ The functions $f,g\in C([0,\infty)\times[0,\infty),[0,\infty))$, $f(x,y),g(x,y)\ne 0$ on any subinterval of $(0,\infty)\times(0,\infty)$ and $f,g$ are bounded on $[0,\infty)\times[0,\infty)$;

$(H_2) a, b : [1, \infty) \to [1, \infty)$ are not identical zero on any closed subinterval of $[1, \infty)$ and

$$0 < \int_1^\infty a(s)\frac{ds}{s} < \infty, \qquad 0 < \int_1^\infty b(s)\frac{ds}{s} < \infty.$$

Next, we are going to prove that the operator $\mathcal{T}$ is completely continuous.

**Lemma 3.2.** *Let* $(H_1)$ *and* $(H_2)$ *hold. Then* $\mathcal{T} : P \to P$ *is completely continuous.*

*Proof.* Firstly, we will show that $\mathcal{T}$ is uniformly bounded on $P$. Let $\Omega^* = \{(x, y) \in P : \|(x, y)\| \le r\} \subseteq P$ be a bounded set. From $(H_1)$, there exist two positive constants $L_1$ and $L_2$ such that

$$|f(x(t), y(t))| \le L_1, \qquad |g(x(t), y(t))| \le L_2, \ \ \forall (x, y) \in \Omega^*.$$

Then, from $(H_2)$, for any $(x, y) \in \Omega^*$, we have

$$\begin{aligned}
\|\mathcal{T}(x, y)\| &= \sup_{t\in[1,\infty)} \frac{1}{1 + (\log t)^{p-1} + (\log t)^{q-1}}\Bigg[\int_1^\infty G_1(t, s)a(s)f(x(s), y(s))\frac{ds}{s} \\
&\quad + \int_1^\infty G_2(t, s)b(s)g(x(s), y(s))\frac{ds}{s} + \int_1^\infty G_3(t, s)b(s)g(x(s), y(s))\frac{ds}{s} \\
&\quad + \int_1^\infty G_4(t, s)a(s)f(x(s), y(s))\frac{ds}{s}\Bigg] \\
&\le M_1 \int_1^\infty a(s)f(x(s), y(s))\frac{ds}{s} + M_2 \int_1^\infty b(s)g(x(s), y(s))\frac{ds}{s} \\
&\quad + M_3 \int_1^\infty b(s)g(x(s), y(s))\frac{ds}{s} + M_4 \int_1^\infty a(s)f(x(s), y(s))\frac{ds}{s} \\
&\le (M_1 + M_4)L_1 \int_1^\infty a(s)\frac{ds}{s} + (M_2 + M_3)L_2 \int_1^\infty b(s)\frac{ds}{s} < \infty.
\end{aligned}$$

This means that the operator $\mathcal{T}$ is uniformly bounded.

Secondly, we will show that $\mathcal{T}$ is equicontinuous on any compact interval of $[1, \infty)$. For any $S > 1$, $t_1, t_2 \in [1, S]$, and $(x, y) \in \Omega^*$, without loss of generality, we assume that $t_1 < t_2$. Then, we have

$$\begin{aligned}
&\left|\frac{A(x, y)(t_2)}{1 + (\log t_2)^{p-1} + (\log t_2)^{q-1}} - \frac{A(x, y)(t_1)}{1 + (\log t_1)^{p-1} + (\log t_1)^{q-1}}\right| \\
&= \Bigg|\int_1^\infty \frac{G_1(t_2, s)}{1 + (\log t_2)^{p-1} + (\log t_2)^{q-1}} a(s)f(x(s), y(s))\frac{ds}{s} \\
&\quad + \int_1^\infty \frac{G_2(t_2, s)}{1 + (\log t_2)^{p-1} + (\log t_2)^{q-1}} b(s)g(x(s), y(s))\frac{ds}{s} \\
&\quad - \int_1^\infty \frac{G_1(t_1, s)}{1 + (\log t_1)^{p-1} + (\log t_1)^{q-1}} a(s)f(x(s), y(s))\frac{ds}{s} \\
&\quad - \int_1^\infty \frac{G_2(t_1, s)}{1 + (\log t_1)^{p-1} + (\log t_1)^{q-1}} b(s)g(x(s), y(s))\frac{ds}{s}\Bigg|
\end{aligned}$$

$$\leq \left| \int_1^\infty \left( \frac{G_1(t_2,s)}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}} - \frac{G_1(t_1,s)}{1+(\log t_1)^{p-1}+(\log t_1)^{q-1}} \right) a(s) f(x(s),y(s)) \frac{ds}{s} \right|$$
$$+ \left| \int_1^\infty \left( \frac{G_2(t_2,s)}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}} - \frac{G_2(t_1,s)}{1+(\log t_1)^{p-1}+(\log t_1)^{q-1}} \right) b(s) g(x(s),y(s)) \frac{ds}{s} \right|.$$

Since

$$\begin{aligned} &\left| \frac{G_1(t_2,s)}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}} - \frac{G_1(t_1,s)}{1+(\log t_1)^{p-1}+(\log t_1)^{q-1}} \right| \\ &\leq \frac{|g_p(t_2,s)-g_p(t_1,s)|}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}} \\ &\quad + \frac{|(\log t_2)^{p-1}-(\log t_1)^{p-1}+(\log t_2)^{q-1}-(\log t_1)^{q-1}| g_p(t_1,s)}{(1+(\log t_2)^{p-1}+(\log t_2)^{q-1})(1+(\log t_1)^{p-1}+(\log t_1)^{q-1})} \\ &\quad + \frac{|(\log t_2)^{p-1}-(\log t_1)^{p-1}+(\log t_2)^{p-1}(\log t_1)^{q-1}-(\log t_1)^{p-1}(\log t_2)^{q-1}|}{(1+(\log t_2)^{p-1}+(\log t_2)^{q-1})(1+(\log t_1)^{p-1}+(\log t_1)^{q-1})} \frac{\Lambda_1}{\Omega} \sum_{j=1}^n \frac{\sigma_j g_{\beta_j}^p(\xi,s)}{\Gamma(p+\beta_j)}, \end{aligned}$$

and

$$\begin{aligned} &\int_1^\infty \frac{|g_p(t_2,s)-g_p(t_1,s)|}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}} a(s) f(x(s),y(s)) \frac{ds}{s} \\ &\leq \int_1^{t_1} \frac{|(\log t_2)^{p-1}-(\log t_1)^{p-1}| + |(\log(t_2/s))^{p-1}-(\log(t_1/s))^{p-1}|}{\Gamma(p)(1+(\log t_2)^{p-1}+(\log t_2)^{q-1})} a(s) f(x(s),y(s)) \frac{ds}{s} \\ &\quad + \int_{t_1}^{t_2} \frac{|(\log t_2)^{p-1}-(\log t_1)^{p-1}| + |(\log(t_2/s))^{p-1}|}{\Gamma(p)(1+(\log t_2)^{p-1}+(\log t_2)^{q-1})} a(s) f(x(s),y(s)) \frac{ds}{s} \\ &\quad + \int_{t_2}^{\infty} \frac{|(\log t_2)^{p-1}-(\log t_1)^{p-1}|}{\Gamma(p)(1+(\log t_2)^{p-1}+(\log t_2)^{q-1})} a(s) f(x(s),y(s)) \frac{ds}{s}, \end{aligned}$$

we have that

$$\left| \int_1^\infty \left( \frac{G_1(t_2,s)}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}} - \frac{G_1(t_1,s)}{1+(\log t_1)^{p-1}+(\log t_1)^{q-1}} \right) a(s) f(x(s),y(s)) \frac{ds}{s} \right| \to 0,$$

uniformly as $t_1 \to t_2$. In addition, we can find that

$$\begin{aligned} &\left| \frac{G_2(t_2,s)}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}} - \frac{G_2(t_1,s)}{1+(\log t_1)^{p-1}+(\log t_1)^{q-1}} \right| \\ &\leq \frac{(|(\log t_2)^{p-1}-(\log t_1)^{p-1}| + |(\log t_2)^{p-1}(\log t_1)^{q-1}-(\log t_1)^{p-1}(\log t_2)^{q-1}|)}{(1+(\log t_2)^{p-1}+(\log t_2)^{q-1})(1+(\log t_1)^{p-1}+(\log t_1)^{q-1})} \\ &\quad \times \frac{\Gamma(q)}{\Omega} \sum_{i=1}^m \frac{\lambda_i g_{\alpha_i}^q(\eta,s)}{\Gamma(q+\alpha_i)}, \end{aligned}$$

which leads to

$$\left| \int_1^\infty \left( \frac{G_2(t_2,s)}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}} - \frac{G_2(t_1,s)}{1+(\log t_1)^{p-1}+(\log t_1)^{q-1}} \right) b(s) g(x(s),y(s)) \frac{ds}{s} \right| \to 0,$$

uniformly as $t_1 \to t_2$. Hence,

$$\left| \frac{A(x,y)(t_2)}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}} - \frac{A(x,y)(t_1)}{1+(\log t_1)^{p-1}+(\log t_1)^{q-1}} \right| \to 0, \text{ uniformly as } t_1 \to t_2.$$

In the similar way, we can prove that

$$\left|\frac{B(x,y)(t_2)}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}}-\frac{B(x,y)(t_1)}{1+(\log t_1)^{p-1}+(\log t_1)^{q-1}}\right|\to 0,$$

as $t_1 \to t_2$ independently of $(x,y)\in\Omega^*$. Therefore $\mathcal{T}\Omega^*$ is equicontinuous on $[1,\infty)$.

Thirdly, we will show that the operator $\mathcal{T}$ is equiconvergent at $\infty$. From the first step, for any $(x,y)\in\Omega^*$, we have

$$\int_1^\infty a(s)f(x(s),y(s))\frac{ds}{s}\le L_1\int_1^\infty a(s)\frac{ds}{s}<\infty,$$

and

$$\int_1^\infty b(s)g(x(s),y(s))\frac{ds}{s}\le L_2\int_1^\infty b(s)\frac{ds}{s}<\infty.$$

Then

$$\begin{aligned}
&\lim_{t\to\infty}\left|\frac{A(x,y)(t)}{1+(\log t)^{p-1}+(\log t)^{q-1}}\right|\\
&=\lim_{t\to\infty}\left|\int_1^\infty\frac{G_1(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}}a(s)f(x(s),y(s))\frac{ds}{s}\right.\\
&\quad\left.+\int_1^\infty\frac{G_2(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}}b(s)g(x(s),y(s))\frac{ds}{s}\right|\\
&\le\lim_{t\to\infty}\int_1^\infty\left(\frac{(\log t)^{p-1}}{\Gamma(p)(1+(\log t)^{p-1}+(\log t)^{q-1})}\right.\\
&\quad\left.+\frac{\Lambda_1}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j\,(\log t)^{p-1}\,g^{p}_{\beta_j}(\xi,s)}{\Gamma(p+\beta_j)(1+(\log t)^{p-1}+(\log t)^{q-1})}\right)a(s)f(x(s),y(s))\frac{ds}{s}\\
&\quad+\lim_{t\to\infty}\int_1^\infty\left(\frac{\Gamma(q)}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i\,(\log t)^{p-1}\,g^{q}_{\alpha_i}(\eta,s)}{\Gamma(q+\alpha_i)(1+(\log t)^{p-1}+(\log t)^{q-1})}\right)b(s)g(x(s),y(s))\frac{ds}{s}\\
&\le M_1L_1\int_1^\infty a(s)\frac{ds}{s}+M_2L_2\int_1^\infty b(s)\frac{ds}{s}<\infty,
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{t\to\infty}\left|\frac{B(x,y)(t)}{1+(\log t)^{p-1}+(\log t)^{q-1}}\right|\\
&=\lim_{t\to\infty}\left|\int_1^\infty\frac{G_3(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}}b(s)g(x(s),y(s))\frac{ds}{s}\right.\\
&\quad\left.+\int_1^\infty\frac{G_4(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}}a(s)f(x(s),y(s))\frac{ds}{s}\right|\\
&\le\lim_{t\to\infty}\int_1^\infty\left(\frac{(\log t)^{q-1}}{\Gamma(q)(1+(\log t)^{p-1}+(\log t)^{q-1})}\right.\\
&\quad\left.+\frac{\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i\,(\log t)^{q-1}\,g^{q}_{\alpha_i}(\eta,s)}{\Gamma(q+\alpha_i)(1+(\log t)^{p-1}+(\log t)^{q-1})}\right)b(s)g(x(s),y(s))\frac{ds}{s}
\end{aligned}$$

$$+\lim_{t\to\infty}\int_1^\infty\left(\frac{\Gamma(p)}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j\,(\log t)^{q-1}\,g_{\beta_j}^{p}(\xi,s)}{\Gamma(p+\beta_j)(1+(\log t)^{p-1}+(\log t)^{q-1})}\right)a(s)f(x(s),y(s))\frac{ds}{s}$$

$$\le M_3L_2\int_1^\infty b(s)\frac{ds}{s}+M_4L_1\int_1^\infty a(s)\frac{ds}{s}<\infty,$$

which imply

$$\lim_{t\to\infty}\left|\frac{\mathcal{T}(x,y)(t)}{1+(\log t)^{p-1}+(\log t)^{q-1}}\right|<\infty.$$

Hence $\mathcal{T}\Omega^*$ is equiconvergent at infinity.

Finally, we will prove that the operator $\mathcal{T}$ is continuous. Let $(x_n,y_n)\to\infty$ as $n\to\infty$ in $P$. By applying the Lebesgue dominated convergence and the continuity of $f$ and $g$ guarantee that

$$\int_1^\infty a(s)f(x_n(s),y_n(s))\frac{ds}{s}\to\int_1^\infty a(s)f(x(s),y(s))\frac{ds}{s}\quad\text{as } n\to\infty,$$

and

$$\int_1^\infty b(s)g(x_n(s),y_n(s))\frac{ds}{s}\to\int_1^\infty b(s)g(x(s),y(s))\frac{ds}{s}\quad\text{as } n\to\infty.$$

Therefore, we get

$$\begin{aligned}\|\mathcal{T}(x_n,y_n)-\mathcal{T}(x,y)\| &= \|(A(x_n,y_n)(t)-A(x,y)(t),B(x_n,y_n)(t)-B(x,y)(t))\|\\ &= \sup_{t\in[1,\infty)}\frac{|A(x_n,y_n)(t)-A(x,y)(t)|+|B(x_n,y_n)(t)-B(x,y)(t)|}{1+(\log t)^{p-1}+(\log t)^{q-1}}\\ &\le M_1\int_1^\infty a(s)|f(x_n(s),y_n(s))-f(x(s),y(s))|\frac{ds}{s}\\ &\quad +M_2\int_1^\infty b(s)|g(x_n(s),y_n(s))-g(x(s),y(s))|\frac{ds}{s}\\ &\quad +M_3\int_1^\infty b(s)|g(x_n(s),y_n(s))-g(x(s),y(s))|\frac{ds}{s}\\ &\quad +M_4\int_1^\infty a(s)|f(x_n(s),y_n(s))-f(x(s),y(s))|\frac{ds}{s}\to 0\quad\text{as } n\to\infty.\end{aligned}$$

Thus, the operator $\mathcal{T}$ is continuous.

By applying Lemma 3.1, we deduce that the operator $\mathcal{T}:P\to P$ is completely continuous. This completes the proof. □

To prove our main results, we set the following constants

$$m_1=\frac{\Lambda_1}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j\,(\log\xi)^{2p+\beta_j-2}}{\Gamma(p+\beta_j)(1+(\log\xi)^{p-1}+(\log\xi)^{q-1})},$$

$$m_2=\frac{\Gamma(q)}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i\,(\log\eta)^{p+q+\alpha_i-2}}{\Gamma(q+\alpha_i)(1+(\log\eta)^{p-1}+(\log\eta)^{q-1})},$$

$$m_3=\frac{\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i\,(\log\eta)^{2q+\alpha_i-2}}{\Gamma(q+\alpha_i)(1+(\log\eta)^{p-1}+(\log\eta)^{q-1})},$$

$$m_4 = \frac{\Gamma(p)}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j\,(\log\xi)^{p+q+\beta_j-2}}{\Gamma(p+\beta_j)(1+(\log\xi)^{p-1}+(\log\xi)^{q-1})},$$

$$n_1 = \int_1^{\infty} a(s)\frac{ds}{s},\quad n_2 = \int_1^{\infty} b(s)\frac{ds}{s},\quad n_3 = \int_{\xi}^{k_1\xi} a(s)\frac{ds}{s},$$

$$n_4 = \int_{\eta}^{k_2\eta} b(s)\frac{ds}{s},\quad \Lambda_3 = n_3(m_1+m_4),\quad \Lambda_4 = n_4(m_2+m_3),$$

$$\Lambda_5 = n_1(M_1+M_4),\quad \Lambda_6 = n_2(M_2+M_3).$$

## 3.1 Existence result via Guo-Krasnoselskii's fixed point theorem

In this subsection, the existence theorems of at least one positive solution will be established using the Guo-Krasnoselskii fixed point theorem.

**Theorem 3.3.** (Guo-Krasnoselskii fixed point theorem) *[39] Let $E$ be a Banach space, and let $\mathcal{P}\subset E$ be a cone. Assume that $\Phi_1,\Phi_2$ are bounded open subsets of $E$ with $0\in\Phi_1$, $\overline{\Phi}_1\subset\Phi_2$, and let $Q:\mathcal{P}\cap(\overline{\Phi}_2\setminus\Phi_1)\to\mathcal{P}$ be a completely continuous operator such that:*
*(i) $\|Qx\|\geq\|x\|$, $x\in\mathcal{P}\cap\partial\Phi_1$, and $\|Qx\|\leq\|x\|$, $x\in\mathcal{P}\cap\partial\Phi_2$; or*
*(ii) $\|Qx\|\leq\|x\|$, $x\in\mathcal{P}\cap\partial\Phi_1$, and $\|Qx\|\geq\|x\|$, $x\in\mathcal{P}\cap\partial\Phi_2$.*
*Then $Q$ has a fixed point in $\mathcal{P}\cap(\overline{\Phi}_2\setminus\Phi_1)$.*

**Theorem 3.4.** *Let $f,g:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ be continuous functions. Suppose that there exist positive constants $\rho_1<\rho_2$, and $\theta_1\in(\Lambda_3^{-1},\infty)$, $\theta_2\in(\Lambda_4^{-1},\infty)$, $\theta_3\in(0,\Lambda_5^{-1})$ and $\theta_4\in(0,\Lambda_6^{-1})$. In addition, assume that the following conditions hold:*
*$(H_3)\, f(x,y)\geq\frac{\theta_1\rho_1}{2}$ for $(x,y)\in[0,\rho_1]\times[0,\rho_1]$ and $g(x,y)\geq\frac{\theta_2\rho_1}{2}$ for $(x,y)\in[0,\rho_1]\times[0,\rho_1]$;*
*$(H_4)\, f(x,y)\leq\frac{\theta_3\rho_2}{2}$ for $(x,y)\in[0,\rho_2]\times[0,\rho_2]$ and $g(x,y)\leq\frac{\theta_4\rho_2}{2}$ for $(x,y)\in[0,\rho_2]\times[0,\rho_2]$.*
*Then, the problem (4) has at least one positive solution $(x,y)$ such that*

$$\rho_1<\|(x,y)\|<\rho_2.$$

*Proof.* It follows from Lemma 3.2 that the operator $\mathcal{T}:P\to P$ is completely continuous. Define $\Phi_1=\{(x,y)\in E:\|(x,y)\|<\rho_1\}$. Hence, for any $(x,y)\in P\cap\partial\Phi_1$, we have $0\leq x(t)\leq\rho_1$ and $0\leq y(t)\leq\rho_1$ for all $t\in[1,\infty)$. From assumption $(H_3)$ and Lemma 2.7, we get

$$\begin{aligned}
\|\mathcal{T}(x,y)\| &= \sup_{t\in[1,\infty)}\frac{|A(x,y)(t)|+|B(x,y)(t)|}{1+(\log t)^{p-1}+(\log t)^{q-1}}\\
&= \sup_{t\in[1,\infty)}\frac{1}{1+(\log t)^{p-1}+(\log t)^{q-1}}\Bigg[\int_1^{\infty}G_1(t,s)a(s)f(x(s),y(s))\frac{ds}{s}\\
&\quad+\int_1^{\infty}G_2(t,s)b(s)g(x(s),y(s))\frac{ds}{s}+\int_1^{\infty}G_3(t,s)b(s)g(x(s),y(s))\frac{ds}{s}\\
&\quad+\int_1^{\infty}G_4(t,s)a(s)f(x(s),y(s))\frac{ds}{s}\Bigg]\\
&\geq \int_1^{\infty}\inf_{t\in[\xi,k_1\xi]}\frac{G_1(t,s)a(s)f(x(s),y(s))}{1+(\log t)^{p-1}+(\log t)^{q-1}}\frac{ds}{s}+\int_1^{\infty}\inf_{t\in[\eta,k_2\eta]}\frac{G_2(t,s)b(s)g(x(s),y(s))}{1+(\log t)^{p-1}+(\log t)^{q-1}}\frac{ds}{s}
\end{aligned}$$

$$+\int_1^\infty \inf_{t\in[\eta,k_2\eta]} \frac{G_3(t,s)b(s)g(x(s),y(s))}{1+(\log t)^{p-1}+(\log t)^{q-1}}\frac{ds}{s} + \int_1^\infty \inf_{t\in[\xi,k_1\xi]} \frac{G_4(t,s)a(s)f(x(s),y(s))}{1+(\log t)^{p-1}+(\log t)^{q-1}}\frac{ds}{s}$$

$$\geq \frac{\Lambda_1}{\Omega}\sum_{j=1}^{n} \frac{\sigma_j(\log\xi)^{p-1}}{\Gamma(p+\beta_j)(1+(\log\xi)^{p-1}+(\log\xi)^{q-1})}\int_\xi^{k_1\xi} g^p_{\beta_j}(\xi,s)a(s)f(x(s),y(s))\frac{ds}{s}$$

$$+\frac{\Gamma(q)}{\Omega}\sum_{i=1}^{m} \frac{\lambda_i(\log\eta)^{p-1}}{\Gamma(q+\alpha_i)(1+(\log\eta)^{p-1}+(\log\eta)^{q-1})}\int_\eta^{k_2\eta} g^q_{\alpha_i}(\eta,s)b(s)g(x(s),y(s))\frac{ds}{s}$$

$$+\frac{\Lambda_2}{\Omega}\sum_{i=1}^{m} \frac{\lambda_i(\log\eta)^{q-1}}{\Gamma(q+\alpha_i)(1+(\log\eta)^{p-1}+(\log\eta)^{q-1})}\int_\eta^{k_2\eta} g^q_{\alpha_i}(\eta,s)b(s)g(x(s),y(s))\frac{ds}{s}$$

$$+\frac{\Gamma(p)}{\Omega}\sum_{j=1}^{n} \frac{\sigma_j(\log\xi)^{q-1}}{\Gamma(p+\beta_j)(1+(\log\xi)^{p-1}+(\log\xi)^{q-1})}\int_\xi^{k_1\xi} g^p_{\beta_j}(\xi,s)a(s)f(x(s),y(s))\frac{ds}{s}$$

$$\geq \frac{1}{2}m_1\rho_1\theta_1\int_\xi^{k_1\xi} a(s)\frac{ds}{s} + \frac{1}{2}m_2\rho_1\theta_2\int_\eta^{k_2\eta} b(s)\frac{ds}{s} + \frac{1}{2}m_3\rho_1\theta_1\int_\eta^{k_2\eta} b(s)\frac{ds}{s} + \frac{1}{2}m_4\rho_1\theta_2\int_\xi^{k_1\xi} a(s)\frac{ds}{s}$$

$$= \frac{1}{2}\Lambda_3\theta_1\rho_1 + \frac{1}{2}\Lambda_4\theta_2\rho_1 \geq \rho_1,$$

which implies that $\|\mathcal{T}(x,y)\| \geq \|(x,y)\|$ for $(x,y)\in P\cap\partial\Phi_1$.

Next, we define $\Phi_2 = \{(x,y)\in E : \|(x,y)\| < \rho_2\}$. Hence, for any $(x,y)\in P\cap\partial\Phi_2$, we have $0\leq x(t)\leq\rho_2$ and $0\leq y(t)\leq\rho_2$ for all $t\in[1,\infty)$. Using the condition $(H_4)$, we obtain

$$\|\mathcal{T}(x,y)\| = \sup_{t\in[1,\infty)} \frac{1}{1+(\log t)^{p-1}+(\log t)^{q-1}}\Bigg[\int_1^\infty G_1(t,s)a(s)f(x(s),y(s))\frac{ds}{s}$$

$$+\int_1^\infty G_2(t,s)b(s)g(x(s),y(s))\frac{ds}{s} + \int_1^\infty G_3(t,s)b(s)g(x(s),y(s))\frac{ds}{s}$$

$$+\int_1^\infty G_4(t,s)a(s)f(x(s),y(s))\frac{ds}{s}\Bigg]$$

$$\leq \frac{1}{2}M_1\theta_3\rho_2\int_1^\infty a(s)\frac{ds}{s} + \frac{1}{2}M_2\theta_4\rho_2\int_1^\infty b(s)\frac{ds}{s} + \frac{1}{2}M_3\theta_4\rho_2\int_1^\infty b(s)\frac{ds}{s} + \frac{1}{2}M_4\theta_3\rho_2\int_1^\infty a(s)\frac{ds}{s}$$

$$= \frac{1}{2}\Lambda_5\theta_3\rho_2 + \frac{1}{2}\Lambda_6\theta_4\rho_2 \leq \rho_2,$$

which leads to $\|\mathcal{T}(x,y)\| \leq \|(x,y)\|$ for $(x,y)\in P\cap\partial\Phi_2$.

Therefore, by the first part of Theorem 3.3, we deduce that the operator $\mathcal{T}$ has a fixed point in $P\cap(\overline{\Phi}_2\setminus\Phi_1)$ which is a positive solution of problem (4). Thus the problem (4) has at least one positive solution $(x,y)$ such that

$$\rho_1 < \|(x,y)\| < \rho_2.$$

This completes the proof. □

Similarly to the previous theorem, we can prove the following result.

**Theorem 3.5.** *Let $f,g:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ be continuous functions. Assume that there exist positive constants $\rho_1<\rho_2$, and $\theta_1\in(\Lambda_3^{-1},\infty)$, $\theta_2\in(\Lambda_4^{-1},\infty)$, $\theta_3\in(0,\Lambda_5^{-1})$ and $\theta_4\in(0,\Lambda_6^{-1})$. In addition, assume that the following conditions hold:*

$(H_5)$ $f(x,y) \le \frac{\theta_3\rho_1}{2}$ *for* $(x,y) \in [0,\rho_1]\times[0,\rho_1]$ *and* $g(x,y) \le \frac{\theta_4\rho_1}{2}$ *for* $(x,y) \in [0,\rho_1]\times[0,\rho_1]$*;*
$(H_6)$ $f(x,y) \ge \frac{\theta_1\rho_2}{2}$ *for* $(x,y) \in [0,\rho_2]\times[0,\rho_2]$ *and* $g(x,y) \ge \frac{\theta_2\rho_2}{2}$ *for* $(x,y) \in [0,\rho_2]\times[0,\rho_2]$.
*Then, the problem* (4) *has at least one positive solution* $(x,y)$ *such that*

$$\rho_1 < \|(x,y)\| < \rho_2.$$

## 3.2 Existence result via Leggett-Williams fixed point theorem

In this subsection, the existence of at least three positive solutions will be proved using the Leggett-Williams fixed point theorem.

**Definition 3.6.** *A continuous mapping* $\omega : \mathcal{P} \to [0,\infty)$ *is said to be a nonnegative continuous concave functional on the cone* $\mathcal{P}$ *of a real Banach space* $E$ *provided that*

$$\omega(\lambda x + (1-\lambda)y) \ge \lambda\omega(x) + (1-\lambda)\omega(y)$$

*for all* $x,y \in \mathcal{P}$ *and* $\lambda \in [0,1]$.

Let $a,b,d > 0$ be given constants and define $\mathcal{P}_d = \{(x,y) \in \mathcal{P} : \|(x,y)\| < d\}$, $\overline{\mathcal{P}}_d = \{(x,y) \in \mathcal{P} : \|(x,y)\| \le d\}$ and $\mathcal{P}(\omega,a,b) = \{(x,y) \in \mathcal{P} : \omega((x,y)) \ge a,\ \|(x,y)\| \le b\}$.

**Theorem 3.7.** (Leggett-Williams fixed point theorem) *[40] Let* $\mathcal{P}$ *be a cone in the real Banach space* $E$ *and* $c > 0$ *be a constant. Assume that there exists a concave nonnegative continuous functional* $\omega$ *on* $\mathcal{P}$ *with* $\omega(x) \le \|x\|$ *for all* $x \in \overline{\mathcal{P}}_c$*. Let* $Q : \overline{\mathcal{P}}_c \to \overline{\mathcal{P}}_c$ *be a completely continuous operator. Suppose that there exist constants* $0 < a < b < d \le c$ *such that the following conditions hold:*
*(i)* $\{x \in \mathcal{P}(\omega,b,d) : \omega(x) > b\} \neq \emptyset$ *and* $\omega(Qx) > b$ *for* $x \in \mathcal{P}(\omega,b,d)$*;*
*(ii)* $\|Qx\| < a$ *for* $x \le a$*;*
*(iii)* $\omega(Qx) > b$ *for* $x \in \mathcal{P}(\omega,b,c)$ *with* $\|Qx\| > d$.
*Then* $Q$ *has at least three fixed points* $x_1, x_2$ *and* $x_3$ *in* $\overline{\mathcal{P}}_c$*. In addition,* $\|x_1\| < a$*,* $\omega(x_2) > b$*,* $\|x_3\| > a$ *with* $\omega(x_3) < b$.

**Theorem 3.8.** *Let* $f,g : \mathbb{R}^+\times\mathbb{R}^+ \to \mathbb{R}^+$ *be continuous functions. Suppose that there exist two constants* $k_1, k_2 > 1$ *such that* $[\xi, k_1\xi] \cap [\eta, k_2\eta] \neq \emptyset$*. In addition, assume that there exist positive constants* $a < b < c$ *satisfying*
$(H_7)$ $f(x,y) < \frac{a}{2\Lambda_5}$ *and* $g(x,y) < \frac{a}{2\Lambda_6}$ *for* $(x,y) \in [0,a]\times[0,a]$*;*
$(H_8)$ $f(x,y) > \frac{b}{2\Lambda_3}$ *and* $g(x,y) > \frac{b}{2\Lambda_4}$ *for* $(x,y) \in [b,c]\times[b,c]$*;*
$(H_9)$ $f(x,y) \le \frac{c}{2\Lambda_5}$ *and* $g(x,y) \le \frac{c}{2\Lambda_6}$ *for* $(x,y) \in [0,c]\times[0,c]$.
*Then, the problem* (4) *has at least three positive solutions* $(x_1,y_1)$*,* $(x_2,y_2)$ *and* $(x_3,y_3)$ *such that* $\|(x_1,y_1)\| < a$*,* $\inf_{\tau_1\le t\le\tau_2}(x_2,y_2)(t) > b$ *and* $\|(x_3,y_3)\| > a$ *with* $\inf_{\tau_1\le t\le\tau_2}(x_3,y_3)(t) < b$*, where* $\tau_1 = \max\{\xi,\eta\}$*,* $\tau_2 = \min\{k_1\xi, k_2\eta\}$.

*Proof.* At first, we will show that the operator $\mathcal{T} : \overline{P}_c \to \overline{P}_c$. For any $(x,y) \in \overline{P}_c$, we have $\|(x,y)\| \le c$. Using the condition $(H_9)$ and Lemma 2.7, we obtain

$$\|\mathcal{T}(x,y)\| = \sup_{t\in[1,\infty)} \frac{1}{1+(\log t)^{p-1}+(\log t)^{q-1}}\Bigg[\int_1^\infty G_1(t,s)a(s)f(x(s),y(s))\frac{ds}{s}$$
$$+\int_1^\infty G_2(t,s)b(s)g(x(s),y(s))\frac{ds}{s} + \int_1^\infty G_3(t,s)b(s)g(x(s),y(s))\frac{ds}{s}$$

$$
\begin{aligned}
&+\int_1^\infty G_4(t,s)a(s)f(x(s),y(s))\frac{ds}{s}\Bigg]\\
&\le \int_1^\infty \sup_{t\in[1,\infty)} \frac{G_1(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}}a(s)f(x(s),y(s))\frac{ds}{s}\\
&\quad+\int_1^\infty \sup_{t\in[1,\infty)} \frac{G_2(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}}b(s)g(x(s),y(s))\frac{ds}{s}\\
&\quad+\int_1^\infty \sup_{t\in[1,\infty)} \frac{G_3(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}}b(s)g(x(s),y(s))\frac{ds}{s}\\
&\quad+\int_1^\infty \sup_{t\in[1,\infty)} \frac{G_4(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}}a(s)f(x(s),y(s))\frac{ds}{s}\\
&\le \frac{c}{2\Lambda_5}M_1n_1+\frac{c}{2\Lambda_6}M_2n_2+\frac{c}{2\Lambda_6}M_3n_2+\frac{c}{2\Lambda_5}M_4n_1\\
&= c,
\end{aligned}
$$

which yields $\mathcal{T}:\overline{P}_c\to\overline{P}_c$. Secondly, we let $(x,y)\in\overline{P}_a$. By applying condition $(H_1)$, we have

$$
\begin{aligned}
\|\mathcal{T}(x,y)\| &\le \int_1^\infty \sup_{t\in[1,\infty)} \frac{G_1(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}}a(s)f(x(s),y(s))\frac{ds}{s}\\
&\quad+\int_1^\infty \sup_{t\in[1,\infty)} \frac{G_2(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}}b(s)g(x(s),y(s))\frac{ds}{s}\\
&\quad+\int_1^\infty \sup_{t\in[1,\infty)} \frac{G_3(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}}b(s)g(x(s),y(s))\frac{ds}{s}\\
&\quad+\int_1^\infty \sup_{t\in[1,\infty)} \frac{G_4(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}}a(s)f(x(s),y(s))\frac{ds}{s}\\
&< \frac{a}{2\Lambda_5}M_1n_1+\frac{a}{2\Lambda_6}M_2n_2+\frac{a}{2\Lambda_6}M_3n_2+\frac{a}{2\Lambda_5}M_4n_1\\
&= a.
\end{aligned}
$$

This means that the condition $(ii)$ of Theorem 3.7 is satisfied.

Thirdly, we define $\tau_1=\max\{\xi,\eta\}$, $\tau_2=\min\{k_1\xi,k_2\eta\}$ and also a concave nonnegative continuous functional $\omega$ on $E$ by

$$\omega((x,y))=\inf_{\tau_1\le t\le\tau_2}\frac{|x(t)|+|y(t)|}{1+(\log t)^{p-1}+(\log t)^{q-1}}.$$

By choosing

$$(x,y)(t)=\left(\frac{(b+c)}{2}(1+(\log t)^{p-1}+(\log t)^{q-1}),\frac{(b+c)}{2}(1+(\log t)^{p-1}+(\log t)^{q-1})\right),$$

we deduce that $(x,y)(t)\in\overline{P}(\omega,b,c)$ and $\omega((x,y))>b$. Hence it follows that $\{(x,y)\in\overline{P}(\omega,b,c):\omega((x,y))>b\}\neq\emptyset$. Therefore, if $(x,y)\in\overline{P}(\omega,b,c)$, then we have $b\le x(t)\le c$ and $b\le y(t)\le c$ for $t\in[\tau_1,\tau_2]$. Using condition $(H_8)$ and Lemma 2.7, we obtain

$$
\begin{aligned}
\omega(\mathcal{T}(x,y)(t)) &= \inf_{\tau_1\le t\le\tau_2}\frac{|A(x,y)(t)|+|B(x,y)(t)|}{1+(\log t)^{p-1}+(\log t)^{q-1}}\\
&= \inf_{\tau_1\le t\le\tau_2}\frac{1}{1+(\log t)^{p-1}+(\log t)^{q-1}}\Bigg[\int_1^\infty G_1(t,s)a(s)f(x(s),y(s))\frac{ds}{s}
\end{aligned}
$$

$$
\begin{aligned}
&+\int_1^\infty G_2(t,s)b(s)g(x(s),y(s))\frac{ds}{s}+\int_1^\infty G_3(t,s)b(s)g(x(s),y(s))\frac{ds}{s}\\
&+\int_1^\infty G_4(t,s)a(s)f(x(s),y(s))\frac{ds}{s}\Bigg]\\
&\geq \int_1^\infty \inf_{t\in[\xi,k_1\xi]}\frac{G_1(t,s)a(s)f(x(s),y(s))}{1+(\log t)^{p-1}+(\log t)^{q-1}}\frac{ds}{s}+\int_1^\infty \inf_{t\in[\eta,k_2\eta]}\frac{G_2(t,s)b(s)g(x(s),y(s))}{1+(\log t)^{p-1}+(\log t)^{q-1}}\frac{ds}{s}\\
&+\int_1^\infty \inf_{t\in[\eta,k_2\eta]}\frac{G_3(t,s)b(s)g(x(s),y(s))}{1+(\log t)^{p-1}+(\log t)^{q-1}}\frac{ds}{s}+\int_1^\infty \inf_{t\in[\xi,k_1\xi]}\frac{G_4(t,s)a(s)f(x(s),y(s))}{1+(\log t)^{p-1}+(\log t)^{q-1}}\frac{ds}{s}\\
&> \frac{b}{2\Lambda_3}m_1n_3+\frac{b}{2\Lambda_4}m_2n_4+\frac{b}{2\Lambda_4}m_3n_4+\frac{b}{2\Lambda_3}m_4n_3\\
&= b.
\end{aligned}
$$

Therefore, $\omega(\mathcal{T}(x,y))>b$ for all $(x,y)\in P(\omega,b,c)$. This means that the condition $(i)$ of Theorem 3.7 is fulfilled.

Finally, we assume that $(x,y)(t)\in P(\omega,b,c)$ with $\|\mathcal{T}(x,y)\|>d$, where $b<d\leq c$. Then, we have $b\leq x(t)\leq c$ and $b\leq y(t)\leq c$ for $t\in[\tau_1,\tau_2]$. From condition $(H_8)$ and Lemma 2.7, we get

$$
\begin{aligned}
\omega(\mathcal{T}(x,y)(t)) &= \inf_{\tau_1\leq t\leq\tau_2}\frac{1}{1+(\log t)^{p-1}+(\log t)^{q-1}}\Bigg[\int_1^\infty G_1(t,s)a(s)f(x(s),y(s))\frac{ds}{s}\\
&+\int_1^\infty G_2(t,s)b(s)g(x(s),y(s))\frac{ds}{s}+\int_1^\infty G_3(t,s)b(s)g(x(s),y(s))\frac{ds}{s}\\
&+\int_1^\infty G_4(t,s)a(s)f(x(s),y(s))\frac{ds}{s}\Bigg]\\
&\geq \int_1^\infty \inf_{t\in[\xi,k_1\xi]}\frac{G_1(t,s)a(s)f(x(s),y(s))}{1+(\log t)^{p-1}+(\log t)^{q-1}}\frac{ds}{s}+\int_1^\infty \inf_{t\in[\eta,k_2\eta]}\frac{G_2(t,s)b(s)g(x(s),y(s))}{1+(\log t)^{p-1}+(\log t)^{q-1}}\frac{ds}{s}\\
&+\int_1^\infty \inf_{t\in[\eta,k_2\eta]}\frac{G_3(t,s)b(s)g(x(s),y(s))}{1+(\log t)^{p-1}+(\log t)^{q-1}}\frac{ds}{s}+\int_1^\infty \inf_{t\in[\xi,k_1\xi]}\frac{G_4(t,s)a(s)f(x(s),y(s))}{1+(\log t)^{p-1}+(\log t)^{q-1}}\frac{ds}{s}\\
&> \frac{b}{2\Lambda_3}n_3(m_1+m_4)+\frac{b}{2\Lambda_4}n_4(m_2+m_3)\\
&= b.
\end{aligned}
$$

It follows that the condition $(iii)$ of Theorem 3.7 is satisfied. Hence, by applying Theorem 3.7, we deduce that the problem (4) has at least three positive solutions $(x_1,y_1)$, $(x_2,y_2)$ and $(x_3,y_3)$ such that $\|(x_1,y_1)\|<a$, $\inf_{\tau_1\leq t\leq\tau_2}(x_2,y_2)(t)>b$ and $\|(x_3,y_3)\|>a$ with $\inf_{\tau_1\leq t\leq\tau_2}(x_3,y_3)(t)<b$. This completes the proof. □

## 4 Examples

In this section, we present two examples to illustrate our results.

**Example 4.1.** *Consider the following Hadamard fractional differential system subject to boundary conditions on an unbounded domain*

$$\begin{cases} D^{3/2}x(t) + e^{-t} f(x(t), y(t)) = 0, \quad t \in (1, \infty), \\ D^{5/3}y(t) + t^{-2}g(x(t), y(t)) = 0, \quad t \in (1, \infty), \\ x(1) = 0, \quad D^{1/2}x(\infty) = \dfrac{2}{3}I^{1/2}y\left(\dfrac{7}{4}\right) + \dfrac{\pi}{5}I^{3/2}y\left(\dfrac{7}{4}\right), \\ y(1) = 0, \quad D^{2/3}y(\infty) = \dfrac{\sqrt{2}}{5}I^{1/3}x\left(\dfrac{8}{3}\right) + \dfrac{1}{7}I^{2/3}x\left(\dfrac{8}{3}\right) + \dfrac{2}{\sqrt{e}}I^{4/3}x\left(\dfrac{8}{3}\right), \end{cases} \tag{20}$$

*where*

$$f(x,y) = \begin{cases} \dfrac{x}{1+y}\left(\dfrac{1}{4} - x\right) + \dfrac{y}{1+x}\left(\dfrac{1}{4} - y\right) + 7; \quad 0 \le x, y \le 1/4, \\ 6 + e^{\frac{1}{16} - xy} + \sin^2\left(\left(\dfrac{1}{4} - x\right)\left(\dfrac{1}{4} - y\right)\right); \ 1/4 \le x, y < \infty, \end{cases}$$

*and*

$$g(x,y) = \begin{cases} \dfrac{x^2}{1+x^2}\left(\dfrac{1}{4} - y\right) + \dfrac{y^4}{1+y^4}\left(\dfrac{1}{4} - x\right) + 2; \ 0 \le x, y \le 1/4, \\ 2 + e^{-xy}\sin^4\left(\left(x - \dfrac{1}{4}\right)\left(y - \dfrac{1}{4}\right)\right); \quad 1/4 \le x, y < \infty. \end{cases}$$

Here $p = 3/2$, $q = 5/3$, $a(t) = e^{-t}$, $b(t) = t^{-2}$, $m = 2$, $\eta = 7/4$, $\lambda_1 = 2/3$, $\alpha_1 = 1/2$, $\lambda_2 = \pi/5$, $\alpha_2 = 3/2$, $n = 3$, $\xi = 8/3$, $\sigma_1 = \sqrt{2}/5$, $\beta_1 = 1/3$, $\sigma_2 = 1/7$, $\beta_2 = 2/3$, $\sigma_3 = 2/\sqrt{e}$ and $\beta_3 = 4/3$. We find that $\Lambda_1 = 0.3512388401$ and $\Lambda_2 = 0.9782224108$ which leads to $\Omega = 0.4564474804 > 0$. In addition, we can compute that $M_1 = 1.977763776$, $M_2 = 0.7695054857$, $M_3 = 1.941574800$, $M_4 = 2.143121504$, $m_1 = 0.2825156640$, $m_2 = 0.2371688476$, $m_3 = 0.2332983559$, $m_4 = 0.7105323246$, $n_1 = 0.2193839344$ and $n_2 = 0.5000000000$. By choosing $k_1 = 2$ and $k_2 = 3$, we also obtain $n_3 = 0.01925300492$, $n_4 = 0.1451247166$, $\Lambda_3 = 0.01911915781$, $\Lambda_4 = 0.06827641958$, $\Lambda_5 = 0.9040560259$ and $\Lambda_6 = 1.355540143$. Observe that the functions $f, g, a$ and $b$ satisfy the conditions $(H_1)$-$(H_2)$.

Choosing $\rho_1 = 1/4$, $\rho_2 = 100$, $\theta_1 = 54 \in (\Lambda_3^{-1}, \infty) = (52.30355908, \infty)$, $\theta_2 = 16 \in (\Lambda_4^{-1}, \infty) = (14.64634505, \infty)$, $\theta_3 = 1 \in (0, \Lambda_5^{-1}) = (0, 1.106126137)$ and $\theta_4 = 0.5 \in (0, \Lambda_6^{-1}) = (0, 0.7377133058)$, we obtain

$$f(x,y) \ge 7 \ge \frac{\theta_1\rho_1}{2} \quad \text{and} \quad g(x,y) \ge 2 \ge \frac{\theta_2\rho_1}{2},$$

for $0 \le x, y \le 1/4$. Also we have

$$f(x,y) \le 50 \le \frac{\theta_3\rho_2}{2} \quad \text{and} \quad g(x,y) \le 25 \le \frac{\theta_4\rho_2}{2},$$

for $1/4 \le x, y < \infty$.

Hence the conditions $(H_3)$-$(H_4)$ hold. By Theorem 3.4, we conclude that the problem (20) has at least one positive solution $(x, y)$ such that $1/4 < \|(x, y)\| < 100$.

**Example 4.2.** *Consider the following Hadamard fractional differential system subject to boundary conditions on an unbounded domain*

$$\begin{cases} D^{7/4}x(t) + t^{-3/4} f(x(t), y(t)) = 0, \quad t \in (1, \infty), \\ D^{9/5}y(t) + e^{-2t} g(x(t), y(t)) = 0, \quad t \in (1, \infty), \\ x(1) = 0, \quad D^{3/4}x(\infty) = \dfrac{\sqrt{\pi}}{13}I^{1/4}y\left(\dfrac{9}{5}\right) + \dfrac{7}{12}I^{1/2}y\left(\dfrac{9}{5}\right) + \dfrac{\sqrt{2}}{15}I^{3/4}y\left(\dfrac{9}{5}\right) \\ y(1) = 0, \quad D^{4/5}y(\infty) = \dfrac{3}{16}I^{1/5}x\left(\dfrac{7}{3}\right) + \dfrac{2}{\sqrt{5}}I^{2/5}x\left(\dfrac{7}{3}\right) + \dfrac{1}{3e^2}I^{3/5}x\left(\dfrac{7}{3}\right) \\ \qquad\qquad + \dfrac{3}{8\pi}I^{4/5}x\left(\dfrac{7}{3}\right), \end{cases} \tag{21}$$

*where*

$$f(x,y)=\begin{cases}\dfrac{x}{5(1+x)}\left(\dfrac{3}{5}-x\right)+\dfrac{y}{5(1+y)}\left(\dfrac{3}{5}-y\right)+\dfrac{3}{100}; & 0\le x,y\le 3/5,\\[2mm] \dfrac{3}{100}e^{-|x-y|}+3\left(x-\dfrac{3}{5}\right)+2\left(y-\dfrac{3}{5}\right); & 3/5\le x,y\le 7/5,\\[2mm] \dfrac{403}{100}+e^{-((7/5)-y)}\sin^2\left(\dfrac{7}{5}-x\right); & 7/5\le x,y<\infty,\end{cases}$$

*and*

$$g(x,y)=\begin{cases}xy\left(\dfrac{3}{5}-x\right)\left(\dfrac{3}{5}-y\right)+2; & 0\le x,y\le 3/5,\\[2mm] 2e^{-|x^2-y^2|}+230\left(x-\dfrac{3}{5}\right)+220\left(y-\dfrac{3}{5}\right); & 3/5\le x,y\le 7/5,\\[2mm] 362+e^{-2((7/5)-x)}\sin^4\left(\dfrac{7}{5}-y\right); & 7/5\le x,y<\infty.\end{cases}$$

Here $p=7/4$, $q=9/5$, $a(t)=t^{-3/4}$, $b(t)=e^{-2t}$, $m=3$, $\eta=9/5$, $\lambda_1=\sqrt{\pi}/13$, $\alpha_1=1/4$, $\lambda_2=7/12$, $\alpha_2=1/2$, $\lambda_3=\sqrt{2}/15$, $\alpha_2=3/4$, $n=4$, $\xi=7/3$, $\sigma_1=3/16$, $\beta_1=1/5$, $\sigma_2=2/\sqrt{5}$, $\beta_2=2/5$, $\sigma_3=1/e^2$ and $\beta_3=3/5$, $\sigma_4=3/8\pi$ and $\beta_4=4/5$. We find that $\Lambda_1=0.3324581038$ and $\Lambda_2=0.8725814055$ which leads to $\Omega=0.5659031627>0$. In addition, we can compute that $M_1=1.645835954$, $M_2=0.5874823216$, $M_3=1.624063246$, $M_4=1.541927070$, $m_1=0.1785392532$, $m_2=0.1696242483$, $m_3=0.1547484152$, $m_4=0.4894898091$, $n_1=1.333333333$ and $n_2=0.04890051071$. By choosing $k_1=4$ and $k_2=5$, it is easy to see that $[\xi,k_1\xi]\cap[\eta,k_2\eta]\neq\emptyset$. Then we also obtain $n_3=0.4565504861$, $n_4=0.006160413528$, $\Lambda_3=0.3049889931$, $\Lambda_4=0.001998269744$, $\Lambda_5=4.250350698$ and $\Lambda_6=0.1081457077$. From above information, the conditions $(H_1)$-$(H_2)$ are fulfilled.

Choosing $a=3/5$, $b=7/5$, $c=80$, we get

$$f(x,y)\le 0.06600000000 \quad\text{and}\quad g(x,y)\le 2.008100000000,$$

which yields for $0\le x,y\le 3/5$,

$$f(x,y)<0.07058241103=\frac{a}{2\Lambda_5}\quad\text{and}\quad g(t,x,y)<2.774035201=\frac{a}{2\Lambda_6}.$$

In addition, we obtain

$$f(x,y)\ge 4.030000000 \quad\text{and}\quad g(x,y)\ge 362.000000,$$

which leads to

$$f(x,y)>2.295164796=\frac{b}{2\Lambda_3}\quad\text{and}\quad g(x,y)>350.3030570=\frac{b}{2\Lambda_4},$$

for $7/5\le x,y\le 80$. Also we have for $0\le x,y\le 80$.

$$f(x,y)\le 9.410988138=\frac{c}{2\Lambda_5}\quad\text{and}\quad g(x,y)\le 369.8713601=\frac{c}{2\Lambda_6}.$$

It is easy to see that $\tau_1=\max\{\xi,\eta\}=7/3$, $\tau_2=\min\{k_1\xi,k_2\eta\}=9$.

Therefore, the conditions $(H_7)$-$(H_9)$ of Theorem 3.8 hold. Applying Theorem 3.8, we deduce that the problem (21) has at least three positive solutions $(x_1,y_1)$, $(x_2,y_2)$ and $(x_3,y_3)$ such that $\|(x_1,y_1)\|<3/5$, $\inf_{7/3\le t\le 9}(x_2,y_2)(t)>7/5$ and $\|(x_3,y_3)\|>3/5$ with $\inf_{7/3\le t\le 9}(x_3,y_3)(t)<7/5$.

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