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POSITIVE SOLUTIONS FOR SINGULAR SEMILINEAR ELLIPTIC SYSTEMS

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Abstract. We establish existence results for singular semilinear elliptic systems on bounded domains with homogeneous Dirichlet boundary conditions. The systems considered are the paradigmatic mathematical models of chemical reactions, morphogenesis (singular Gierer-Meinhardt system) and population dynamics. In these systems the operator need not be in divergence form and the systems need not be cooperative. The results have been obtained by the method of sub and supersolutions (appropriately modified) and Schauder's fixed point theorem. Some uniqueness results have been obtained extending a "concavity" argument used for a single equation. We extend some existence results to general elliptic operators and more general nonlinearities and we prove existence for systems that have not been considered in the literature.

1. Introduction

Reaction-diffusion equations (and then systems) have been intensively studied during the last forty years. The main reason for this study is that they provide rather simple, but however interesting, mathematical models when considering very different kinds of phenomena arising in a large variety of fields in applications: chemical reactions, combustion, transmission of nerve impulses (Fitzhugh-Nagumo systems), morphogenesis and, of course,

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population dynamics, where Lotka-Volterra systems have played an important role. A simple example of a parabolic reaction-diffusion equation is

$$\begin{aligned} &\frac{\partial u}{\partial t} - \Delta u = f(x,u) \quad \text{in } \Omega \times (0,T), \\ &\quad u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \quad u(x,0) = u_0(x) \quad \text{on } \bar{\Omega}, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a domain (maybe unbounded), $T > 0$, the Laplacian Δ models the diffusion of the corresponding quantity $u(x,t)$, and $f(x,u)$ accounts for the reaction taking place in the domain Ω. The initial datum u_0 is smooth. Needless to say, the model (1.1) can be generalized in several directions.

Some very general local existence and uniqueness results for classical or weak solutions of (1.1) have been obtained (see the references below). Global existence (i.e, solutions defined for $0 \le t < +\infty$) depends on the nonlinear term $f(x,u)$ and can be obtained using a priori estimates and comparison arguments. The asymptotic behavior (as $t \to +\infty$) of solutions has been studied for many equations. In particular, existence and properties of traveling waves in unbounded domains have attracted a lot of attention. (The reader will find many references and results in the books by Smoller [32] and Pao [28]. See also [30, 14, 16, 38, 37, 39, 40].)

An interesting associated question is the existence of equilibria to the stationary problem corresponding to (1.1), i.e., solutions to the semilinear elliptic problem

$$-\Delta u = f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

More precisely, existence of solutions to (1.2) (often depending on a parameter λ as well) is important. In particular, the stability properties of equilibria are studied using either linearized stability theorems or comparison arguments. Several methods (sub and supersolutions, bifurcation, continuation, etc.) have been used in this context. Very often positive (or non-negative) solutions are the only meaningful ones for the corresponding problem. The surveys by Amann [1] and P.L. Lions [26] include a large overview concerning results and methods for these problems. See also [16] and [30].

However, the mathematical study of most of the interesting problems requires systems with more than one equation. A typical example could be

$$\begin{aligned} \frac{\partial u}{\partial t} - a\Delta u = f(x,u,v) \quad \text{in } \Omega \times (0,T), \\ \frac{\partial v}{\partial t} - b\Delta v = g(x,u,v) \quad \text{in } \Omega \times (0,T), \end{aligned}$$

$$u(x,t) = v(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.3}$$
$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad \text{on } \bar{\Omega},$$

where $\Omega \subset \mathbb{R}^N$ is a domain, $T > 0$, $a, b > 0$ are the diffusion coefficients, f and g represent the interaction between both species u and v, and u_0 and v_0 are the initial data. Again, (1.3) can be generalized in many ways.

The above considerations concerning the interesting problems when dealing with (1.1) can be extended, with suitable modifications, to (1.3) and the associated stationary system

$$\begin{aligned} -a\Delta u &= f(x,u,v) \quad \text{in } \Omega, \\ -b\Delta v &= g(x,u,v) \quad \text{in } \Omega, \\ u &= v = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.4}$$

Existence and multiplicity of positive solutions to (1.4) have been studied using the same methods mentioned above, in particular sub and supersolutions. Many more developments and references can be found in the books by Smoller[32] and Pao [28], and also in [1, 16, 30, 14]. An excellent reference for the Lotka-Volterra system is the recent book by Cantrell and Cosner [2].

Usually, the nonlinear term $f(x,u)$ arising in (1.3) is smooth and satisfies $f(x,0) \geq 0$ (for any x). However nonlinearities such that $f(x,u) \to +\infty$ if $u \to 0$ and $u > 0$ also arise in some applications [10, 9, 8, 11]. After the pioneering paper by Fulks and Maybee [10], rather general existence results for positive solutions were proved in [6] and [33] (see also [25, 31]) and later by several authors. In [18] the authors prove a rather general existence theorem (for classical solutions) in the interval between ordered sub and supersolutions, trying to get an existence theorem as close as possible to the well-known ones in the non-singular case [1]. They also obtain results concerning the differentiability of the Green (solution) operator at the interior of the positive cone in the space $C_0^1(\bar{\Omega})$ and these results lead to a linearized stability theorem and smoothness of branches of positive solutions depending on parameters. These results were applied in [19] to a series of examples, extending and simplifying most of the preceding results in the literature. A local existence (of classical solutions) and uniqueness result for the singular parabolic problem was obtained in [18] working in the framework of sectorial operators [14]. A similar result for weak solutions in weighted Sobolev spaces was given by Takáč [34], where he also obtained some "stabilization" results. See [17] for a recent survey of the field.

The situation concerning systems (both parabolic and elliptic) with singular nonlinearities is quite different from the one described above in the sense that the list of available examples is rather short, and some of them even look a bit "artificial" (see details below). Moreover, some of the results in the literature have been obtained under strong additional restrictions (Ω a ball or even $\Omega =]0,1[$, radial solutions, etc.).

As is well known, the method of sub and supersolutions giving monotone sequences converging to a minimal (respectively maximal) solution can be used for cooperative systems (i.e., systems such that $f(x,u,v)$ (respectively $g(x,u,v)$) is increasing in v (respectively in u) for any x). This is already done in [30], see also [32, 16, 28, 1]. For non-cooperative systems simple counterexamples (see, e.g., pages 66-67 in [16]) show that it is not necessarily true that there is a solution in the interval between ordered sub and supersolutions if sub and supersolutions are defined in the "natural" way. However this difficulty can be overcomed using a modified definition of coupled sub- super-solution (which coincides with the usual one in the cooperative case). This was done independently (and more or less simultaneously) in [15, 23, 35] and has been used also for nonlocal [36] and for hyperbolic problems [27]. See also [16].

In Section 2 we show how to extend the arguments used for one single equation in [18] in order to obtain existence for systems, both in the cooperative and the non-cooperative situations. In the first case we get monotone sequences just as in the case of one equation. In the non-cooperative case we rely on the compactness of the solution operators (as in [15, 16]) and Schauder's fixed point theorem. Also, we extend to cooperative systems a "concavity" argument already used in [19] for proving uniqueness in the case of a single equation. Such an extension allows us to greatly extend most of the preceding uniqueness results, which only apply to very special situations.

In Section 3 we apply the general results to several examples. Some of them were already treated in fairly particular situations. Namely, the results in [4, 5, 20, 21] were obtained using a decoupling argument which only works in quite special situations (essentially, the nonlinear terms in both equations must coincide) and requires considering many different cases. Instead, we can deal with completely independent nonlinear terms and can treat all these cases at the same time. See more details in Remark 4.

The associated parabolic systems have never been considered in the previous literature (to the best of our knowledge). Local existence for a single parabolic equation was obtained in [18] by showing that the problem can be formulated in the framework of fractional operators already defined by Henry [14]. It is possible to develop similar arguments and get local existence for systems. Then, appropriate maximum principles [18, Appendix] could be used to show that solutions whose initial conditions are between ordered sub and supersolutions (for the elliptic system in the sense defined below) remain in this interval for any time $t > 0$. This would provide existence and uniqueness of classical solutions for all the systems arising in Section 3. Linearized stability could be studied extending the results in [18] to general (non-potential, non-selfadjoint) linear systems with singular coefficients.

2. General existence and uniqueness theorems

Let us state and prove the main general existence theorems for singular semilinear elliptic systems. First, we extend to cooperative systems the construction in [18, Theorem 4.1] giving monotone sequences converging to a maximal (a minimal) positive solution in the interval between ordered sub and supersolutions. This is the content of Theorem 2.1. Then we deal with non-cooperative systems and show that using a stronger notion of coupled sub and supersolutions and Schauder's fixed point theorem produces an existence result for solutions in the corresponding interval. We also prove a uniqueness theorem for positive solutions of cooperative systems under the smoothness assumptions in Theorems 2.1 and 2.2, by using a "concavity" argument which is classical in the regular case and was extended to singular problems in [19].

More precisely, we consider the singular semilinear elliptic system

$$\mathcal{L}_1 u = f(x,u,v) \quad \text{in } \Omega, \tag{2.1}$$

$$\mathcal{L}_2 v = g(x,u,v) \quad \text{in } \Omega, \tag{2.2}$$

$$u = v = 0 \quad \text{on } \partial\Omega, \tag{2.3}$$

where

$$\mathcal{L}_l u \equiv -\sum_{i,j=1}^{N} a_{ij}^l(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i^l(x)\frac{\partial u}{\partial x_i} \tag{2.4}$$

for $l = 1, 2$ under the following assumptions:

(H.1) *$\Omega \subset \mathbb{R}^N$ is a bounded domain, with a $C^{3,\gamma}$ boundary, for some $\gamma > 0$, if $N > 1$.*

(H.2) *The second-order part of the operator $\mathcal{L}_l$ is uniformly, strongly elliptic in Ω. Also, for all $i, j, k', l' = 1, \dots, N$, $a_{ij}^l = a_{ji}^l \in C^3(\Omega) \cap C(\bar{\Omega})$, $b_i^l \in C^2(\Omega)$, and there is a constant κ such that $|\partial a_{ij}^l/\partial x_{k'}| + |b_i^l| < \kappa[1 + d(x)^{\alpha^*}]$ and $|\partial^2 a_{ij}^l/\partial x_{k'}\partial x_{l'}| + |\partial b_i^l/\partial x_k'| < \kappa d(x)^{\alpha^*-1}$ for all $x \in \Omega$* and for $l = 1, 2$, where α^* is such that $-1 < \alpha^* < 1$.

(H.3) *There is an integer $m > 0$ such that f, $\partial^j f/\partial u^{j-k}\ \partial v^k$, g, $\partial^j g/\partial u^{j-k}\ \partial v^k$, $\in C(\Omega \times (0,\infty) \times (0,\infty))$ for all $j = 1, \dots, m+1$ and* *all* $k = 0, \ldots, j$, *and moreover,* $\partial^j f \ /\partial u^{j-1-k}\partial v^k \partial x_{k'}$, $\partial^j g/\partial u^{j-1-k}\partial v^k \partial x_{k'}$ $\in C(\Omega \times (0,\infty)\times\ (0,\infty))$ *for all* $k' = 1, \ldots, N$, *all* $j = 1, \ldots, m+1$ *and all* $k = 0, \ldots, j-1$. *If* $u, v : \Omega \to \mathbb{R}$ *are such that* $0 < k_1 d(x) \le u \le k_2 d(x)$, $0 < k_3 d(x) \le v \le k_4 d(x)$, *for all* $x \in \Omega$ *and for* $k_i > 0$, $i = 1, \ldots, 4$, *then* $|f(x, u(x), v(x))| + |g(x, u(x), v(x))| \le k_5(1+d(x)^{\alpha^*})$, *and* $|\ \partial^j f(x, u(x), v(x)) /\partial u^{j-k}\partial v^k| + \ |\partial^j g(x, u(x), v(x))/\ \partial u^{j-k}\partial v^k| \ \le \ k_j d(x)^{\alpha^*-j}$ *for all* $x \in \Omega$, *all* $j = 1, \ldots, m+1$ *and all* $k = 0, \ldots, j$ *and* $\sum_{k'=1}^{N} |\partial^j f(x, u(x), v(x))/ \partial u^{j-k-1}\partial v^k\ \partial x_{k'}| + \sum_{k'=1}^{N} |\partial^j g(x, u(x),\ v(x))\ /\partial u^{j-k-1}\partial v^k\ \partial x_{k'}| \le k_j d(x)^{\alpha^*-j}$ *for all* $x \in \Omega$, *all* $k' = 1, \ldots, N$, *all* $j = 1, \ldots, m+1$ *and all* $k = 0, \ldots, j-1$ *where* k_j (*can depend on* k_i, $i = 1, \ldots, 4$ *but*) *is independent of* u *and* v.

Notice that assumption (H.3) is satisfied by the usual power law nonlinearities $f(x, u, v) = m(x)u^{\alpha_1}\ v^{\alpha_2}$ whenever $\alpha_1 + \alpha_2 > -1$ and $m \in C^1(\bar{\Omega})$ (and similarly for g) and, more generally, when $m \in C^1(\Omega)$ and $|m(x))| \le kd(x)^{\alpha_3}$ for some $k > 0$ and some α_3 such that $|\alpha_1 + \alpha_2 + \alpha_3| < 1$.

Now we collect from [18] the two main results for linear problems which are the essential ingredients in order to apply the method of sub and supersolutions.

The first result concerns existence and uniqueness for a singular linear problem, and a regularity estimate giving the appropriate compactness of the associated Green (solution) operator.

Proposition 1. ([18, Proposition 2.3]) *Let* Ω *and* $\mathcal{L}$ *satisfy assumptions* (H.1)-(H.2) *and* $M(x)$ *satisfy*

(H.4) $M \in C^1(\Omega)$ *and for all* $k = 1, \ldots, N$, $d(x)^{2-\alpha^*}|\partial M(x)/\partial x_k|$ *is bounded in* Ω.

(*As a consequence, the function* $x \to d(x)^2 M(x)$ *is in* $C^{0,\delta}(\bar{\Omega})$ *whenever* $0 < \delta < min\{\gamma, 1+\alpha^*\}$ *and* $d(x)^{1-\alpha^*}M(x)$ *is bounded in* Ω.) *Then if* $v \in C_0^{0,1}(\bar{\Omega})$ *there exists a unique solution* u *to the linear problem*

$$\mathcal{L}u = M(x)v \quad in\ \Omega, \qquad u = 0 \quad on\ \partial\Omega, \tag{2.5}$$

such that $u \in C^2(\Omega) \cap C_0^{1,\delta}(\bar{\Omega})$ *for all* δ *such that* $0 < \delta < \delta_0 = \min\{\gamma, \alpha^*+1\}$. *There is a constant* K, *which (can depend on* δ *but) is independent of* u *such that*

$$\|u\|_{C^{1,\delta}(\bar{\Omega})} \le K\|v\|_{C^{0,1}(\bar{\Omega})}. \tag{2.6}$$

The second result is an extension to singular problems of the strong maximum principle.

Proposition 2. ([18, Theorem B.2, Appendix B]) *Let* $u \in C^2(\Omega) \cap C_0^{1,\delta}(\bar{\Omega})$ *be such that*

$$\mathcal{L}u + M(x)u \geq 0 \quad in\ \Omega, \tag{2.7}$$

where δ, Ω, $\mathcal{L}$, *and* $M(x) \geq 0$ *for all* $x \in \Omega$ *are as in Proposition* 1. *If* $u \geq 0$ *in* $\bar{\Omega}$ *and* $u(x_0) = 0$ *for some* $x_0 \in \bar{\Omega}$, *then the following properties hold:*

i) If $x_0 \in \Omega$, *then* $u \equiv 0$ *in* Ω.

ii) If $x_0 \in \partial\Omega$ *and* $u > 0$ *in* Ω, *then* $\frac{\partial u}{\partial n}(x_0) < 0$.

We say that (u_0, v_0) (respectively (u^0, v^0)), with u_0, u^0, v_0, $v^0 \in C^2(\Omega) \cap C(\bar{\Omega})$, is a subsolution (respectively supersolution) to (2.1)-(2.3) if

$$\mathcal{L}_1 u_0 - f(x, u_0, v_0) \leq 0 \leq \mathcal{L}_1 u^0 - f(x, u^0, v^0) \quad \text{in } \Omega, \tag{2.8}$$

$$\mathcal{L}_2 v_0 - g(x, u_0, v_0) \leq 0 \leq \mathcal{L}_2 v^0 - g(x, u^0, v^0) \quad \text{in } \Omega, \tag{2.9}$$

$$u_0 = u^0 = v_0 = v^0 = 0 \quad \text{on } \partial\Omega. \tag{2.10}$$

If, moreover,

$$0 < k_1 d(x) \leq u_0(x) \leq u^0(x) \qquad \text{for all } x \in \Omega, \tag{2.11}$$

$$0 < k_2 d(x) \leq v_0(x) \leq v^0(x) \qquad \text{for all } x \in \Omega, \tag{2.12}$$

then we say that the subsolution (u_0, v_0) and the supersolution (u^0, v^0) are ordered.

The system (2.1)-(2.3) is called cooperative (or quasi-monotone) if

$$f(x, u, v) \text{is increasing in } u \text{ and } v \text{ for any } x \in \Omega, \tag{2.13}$$

$$g(x, u, v) \text{is increasing in } u \text{ and } v \text{ for any } x \in \Omega. \tag{2.14}$$

Remark 1. Conditions (2.10)-(2.12) are stronger than the usual ones in the non-singular case and were already required in [18] when dealing with one single equation.

Theorem 2.1. *Assume that assumptions* (H.1)-(H.3) *are satisfied and that there exist ordered sub and supersolutions* (u_0, v_0)-(u^0, v^0) *for the cooperative system* (2.1)-(2.3). *Then there exist a minimal and a maximal solution in the subset* $[u_0, u^0] \times [v_0, v^0]$, $(\underline{u}, \underline{v})$ *and* $(\overline{u}, \overline{v})$, *such that* $\underline{u}, \underline{v}, \overline{u}, \overline{v} \in C^2(\Omega) \cap C_0^{1,\delta}(\bar{\Omega})$ *whenever* $0 < \delta < \delta_0 = \min\{\gamma, \alpha^* + 1\}$ *and* $u_0 \leq \underline{u} \leq \overline{u} \leq u^0$, $v_0 \leq \underline{v} \leq \overline{v} \leq v^0$ *in* Ω, *in the sense that if* (u, v) *is a solution such that* $u_0 \leq u \leq u^0$, $v_0 \leq v \leq v^0$, *then* $\underline{u} \leq u \leq \overline{u}$, $\underline{v} \leq v \leq \overline{v}$. *We also have* $(\underline{u}, \underline{v})$ (*respectively* $\overline{u}, \overline{v}$) *as the* $C_0^{1,\delta}(\bar{\Omega})$-*limit from below* (*respectively from above*) *of a monotone sequence of subsolutions* (*respectively supersolutions*) *to* (2.1)-(2.3). *Moreover,* $\partial\underline{u}/\partial n < 0$ $\partial\overline{u}/\partial n < 0$ *on* $\partial\Omega$.

Proof. We consider the sequences $\{u_n\}$, $\{u^n\}$, $\{v_n\}$, $\{v^n\}$, defined inductively by

$$\mathcal{L}_1 u_{n+1} = f(x, u_n, v_n) \quad \text{in } \Omega, \tag{2.15}$$

$$\mathcal{L}_1 u^{n+1} = f(x, u^n, v^n) \quad \text{in } \Omega, \tag{2.16}$$

$$\mathcal{L}_2 v_{n+1} = g(x, u_n, v_n) \quad \text{in } \Omega, \tag{2.17}$$

$$\mathcal{L}_2 v^{n+1} = g(x, u^n, v^n) \quad \text{in } \Omega, \tag{2.18}$$

$$u_{n+1} = u^{n+1} = v_{n+1} = v^{n+1} = 0 \quad \text{on } \partial\Omega, \tag{2.19}$$

for $n = 0, 1, \dots$ with u_0, u^0, v_0, v^0 as above. Indeed, by (H.3) we have

$$f(x, u, v) = M(x)\psi(x, u, v), \tag{2.20}$$

where $M(x) = d(x)^{\alpha^*-2}$ and $\psi(x, u, v) \in C^1(\bar{\Omega})$ for all $u, v > 0$ and satisfies

$$\|\psi(x, u, v)\|_{C^1(\bar{\Omega})} \leq k_1(\|u\|_{C^1(\bar{\Omega})} + \|v\|_{C^1(\bar{\Omega})}), \tag{2.21}$$

$$\|\psi(x, u, v) - \psi(x, \tilde{u}, \tilde{v})\|_{C^1(\bar{\Omega})} \leq k_2(\|u - \tilde{u}\|_{C^1(\bar{\Omega})} + \|v - \tilde{v}\|_{C^1(\bar{\Omega})}) \tag{2.22}$$

whenever $u_0 \leq u$, $\tilde{u} \leq u^0$, $v_0 \leq v$, $\tilde{v} \leq v^0$ in Ω, where the constants k_1, k_2 are independent of u, $\tilde{u}$, v and $\tilde{v}$.

Since $\psi(x, u, v) = d(x)^{2-\alpha^*} f(x, u, v)$ satisfies (2.22) we can apply Proposition 1 to obtain inductively that $\{u_n\}$, $\{u^n\} \in C^2(\Omega) \cap C_0^{1,\delta}(\bar{\Omega})$ for $n = 1, 2, \dots$ and $0 < \delta < \delta_0$; and the maximum principle in Proposition 2 shows that

$$u_0 \leq u_1 \leq \dots \leq u_n \leq \dots \leq u^n \leq \dots \leq u^1 \leq u^0 \quad \text{in } \Omega. \tag{2.23}$$

(A similar result holds for v_n and v^n.)

Hence, the stated result follows if we can prove that u_m and u^m converge in $C_0^{1,\delta}(\bar{\Omega})$ whenever $0 < \delta < \delta_0$. (Then, according to standard local, elliptic estimates, the limits must be in $C^2(\Omega)$.)

We only prove the result for u_n (the result for u^n is similar). Reasoning as in [18] we observe invoking the dominated convergence theorem that u_n converges in $L_q(\Omega)$ for all $q > 1$. Then, by Proposition 1 and (2.21)- (2.22), we have

$$\|u_p - u_n\|_{C^{1,\delta}(\bar{\Omega})} \leq K\|u_{p-1} - u_{n-1}\|_{C^1(\bar{\Omega})}, \tag{2.24}$$

where K is independent of n and p. By using the interpolation inequality

$$\|u\|_{C^1(\bar{\Omega})} \leq \varepsilon\|u\|_{C^{1,\delta}(\bar{\Omega})} + C_{\varepsilon,q}\|u\|_{L_q(\Omega)} \tag{2.25}$$

which holds for any ε and all $q > N+2$ (see [24] page 80), keeping $q > N+2$ fixed, the inequality (2.21)(for ε well chosen) and (2.24) give

$$\begin{aligned} \|u_p - u_n\|_{C^1(\bar{\Omega})} &\leq \frac{1}{4}\|u_{p-1} - u_{n-1}\|_{C^1(\bar{\Omega})} + k_1\|u_p - u_n\|_{L_q(\Omega)} \\ &\leq \frac{1}{4}(\|u_p - u_n\|_{C^1(\bar{\Omega})} + \|u_p - u_{p-1}\|_{C^1(\bar{\Omega})} \\ &\quad + \|u_n - u_{n-1}\|_{C^1(\bar{\Omega})}) + k_1\|u_p - u_n\|_{L_q(\Omega)} \end{aligned} \tag{2.26}$$

with k_1 independent of n and p. Since u_n converges in $L_q(\Omega)$, the first inequality in (2.26) (with $p = n+1$) implies $\|u_{n+1} - u_n\|_{C^1(\bar{\Omega})} \to 0$ as $n \to +\infty$. And, since u_n is a Cauchy sequence in $L_q(\Omega)$, (2.22) and (2.24) imply that u_n is a Cauchy sequence in $C^1(\bar{\Omega})$ and in $C_0^{1,\delta}(\bar{\Omega})$. Thus $\{u_n\}$ converges in $C_0^{1,\delta}(\bar{\Omega})$ to a limit $\underline{u}$. The last statement follows from the strong maximum principle.

Remark 2. Theorem 2.1 is the extension to cooperative singular elliptic systems of the standard existence result in the regular case (see, e.g., [30] and also [28]). In the singular case, most of the existence results were obtained by approximating singular problems by regular ones, applying the method of sub and supersolutions in the standard way, and then going to the limit. An existence theorem for solutions in the interval between ordered sub and supersolutions was proved by the authors [18, Theorem 4.1]. Many more details and references for the singular problem can be found in [18, 17, 19].

However, since most of the systems arising in applications are not cooperative (for example, among systems in population dynamics only symbiosis is cooperative, whereas competition and predator-prey are not) it is quite natural to try to extend the method to such situations. Nevertheless it turns out that it is easy to exhibit simple examples showing that with the above definition of sub and supersolution it is not true that there is always a solution in the interval between ordered sub and supersolutions (see, e.g., pages 66-67 in [16]). This difficulty was circumvented by giving a more stringent definition of coupled sub- super-solution which coincides with the above one for cooperative systems.

Next we extend this idea to the singular case. First, we state the definition of coupled sub-supersolution for (2.1)-(2.3).

We say that (u_0, v_0) (respectively (u^0, v^0)), where u_0, u^0, v_0, $v^0 \in C^2(\Omega) \cap C(\bar{\Omega})$ is a subsolution (respectively supersolution) to (2.1)-(2.3) if

$$\mathcal{L}_1 u_0 - f(x, u_0, v) \leq 0 \leq \mathcal{L}_1 u^0 - f(x, u^0, v) \quad \text{in } \Omega, \text{ for all } v \in [v_0, v^0], \tag{2.27}$$

$$\mathcal{L}_2 v_0 - g(x,u,v_0) \leq 0 \leq \mathcal{L}_2 v^0 - g(x,u,v^0) \quad \text{in } \Omega, \text{ for all } u \in [u_0, u^0], \tag{2.28}$$

$$u_0 = u^0 = v_0 = v^0 = 0 \quad \text{on } \partial\Omega, \tag{2.29}$$

where

$$[w,z] = \{p \in C^1(\bar\Omega) : w(x) \leq p(x) \leq z(x) \quad \text{for all } x \in \Omega\}, \tag{2.30}$$

and if moreover (2.11)-(2.12) are satisfied.

We have the following existence result.

Theorem 2.2. *Assume that conditions* (H.1)-(H.3) *are satisfied and there exists a coupled sub- supersolution* (u_0, v_0)-(u^0, v^0) *for the system* (2.1)-(2.3). *Then there exists at least a solution* (u,v) *of* (2.1)-(2.3) *with* $u, v \in C^2(\Omega) \cap C_0^{1,\delta}(\bar\Omega)$ *for all* $0 < \delta < \delta_0$ *and such that* $u_0 \leq u \leq u^0$ *and* $v_0 \leq v \leq v^0$.

Proof. It was shown in the proof of Theorem 4.1 in [18] that there exists a function $M_1(x)$ (respectively $M_2(x)$) $M_i \in C^1(\Omega)$, $i = 1,2$, satisfying (H.4) and such that, for some constants K_i the functions $\varphi_i, \psi_i : \Omega \to \mathbb{R}$, $i = 1,2$, defined as

$$\varphi_1(x,u,v) \equiv M_1(x)u + f(x,u,v) \equiv M_1(x)\psi_1(x,u,v), \tag{2.31}$$

$$\varphi_2(x,u,v) \equiv M_2(x)v + g(x,u,v) \equiv M_2(x)\psi_2(x,u,v) \tag{2.32}$$

are such that

$$\text{if } u_0 \leq u < \tilde u \leq u^0, \text{ then } \varphi_1(x,u,v) < \varphi_1(x,\tilde u,v) \text{ for all } v \in [v_0, v^0], \tag{2.33}$$

$$\text{if } v_0 \leq v < \tilde v \leq v^0, \text{ then } \varphi_2(x,u,v) < \varphi_2(x,u,\tilde v) \text{ for all } u \in [u_0, u^0], \tag{2.34}$$

$$\psi_1(x,u,v), \psi_2(x,u,v) \in C_0^1(\bar\Omega) \quad \text{for all } (u,v) \in [u_0,u^0] \times [v_0,v^0], \tag{2.35}$$

$$\|\psi_i(x,u,v)\|_{C^1(\bar\Omega)} \leq k_i(\|u\|_{C^1(\bar\Omega)} + \|v\|_{C^1(\bar\Omega)}) \tag{2.36}$$

$$\text{for all } (u,v) \in [u_0,u^0] \times [v_0,v^0] \text{ for } i = 1,2, \tag{2.37}$$

$$\text{if } u, \tilde u \in [u_0,u^0], \text{ then } \|\psi_i(x,u,v) - \psi_i(x,\tilde u,v)\|_{C^1(\bar\Omega)} \leq \tag{2.38}$$

$$k_{1i}\|u - \tilde u\|_{C^1(\bar\Omega)} \text{ for all } v \in [v_0,v^0],\ i = 1,2, \tag{2.39}$$

$$\text{if } v, \tilde v \in [v_0,v^0], \text{ then } \|\psi_i(x,u,v) - \psi_i(x,u,\tilde v)\|_{C^1(\bar\Omega)} \leq \tag{2.40}$$

$$k_{2i}\|v - \tilde v\|_{C^1(\bar\Omega)} \text{ for all } u \in [u_0,u^0],\ i = 1,2. \tag{2.41}$$

Let $D \equiv B_R \cap ([u_0,u^0] \times [v_0,v^0])$, where $B_R = \{(u,v) \in C_0^1(\bar\Omega) \times C_0^1(\bar\Omega) : \|u\|_{C^1(\bar\Omega)} + \|v\|_{C^1(\bar\Omega)} \leq R\}$, with R large enough . Let us consider the nonlinear operator $\mathcal{K} : D \to C_0^1(\bar\Omega) \times C_0^1(\bar\Omega)$ defined by $\mathcal{K}(u,v) = (w,z)$ where, by Theorem 1, (w, z) is the unique solution of the system

$$\mathcal{L}_1 w + M_1(x)w = \varphi_1(x, u, v) \quad \text{in } \Omega, \tag{2.42}$$

$$\mathcal{L}_2 z + M_2(x)z = \varphi_2(x, u, v) \quad \text{in } \Omega, \tag{2.43}$$

$$w = z = 0 \quad \text{on } \partial\Omega. \tag{2.44}$$

Note that (2.42)-(2.44) is equivalent to (2.1)-(2.3) if $w = u$ and $z = v$, see (2.29).

In order to prove that the operator $\mathcal{K}$ is such that $\mathcal{K}(D) \subset D$ we will proceed in two steps.

i)If $(u, v) \in [u_0, u^0] \times [v_0, v^0]$ then $\mathcal{K}(u, v) \equiv (w, z) \in [u_0, u^0] \times [v_0, v^0]$. Indeed, according to (2.27) and (2.42)

$$\mathcal{L}_1(w - u_0) + M_1(x)(w - u_0) \geq \tag{2.45}$$

$$M_1(x)\psi_1(x, u, v) - M_1(x)u_0 - f(x, u_0, v) \geq 0 \quad \text{in } \Omega, \tag{2.46}$$

$$\mathcal{L}_1(w - u^0) + M_1(x)(w - u^0) \leq \tag{2.47}$$

$$M_1(x)\psi_1(x, u, v) - M_1(x)u^0 - f(x, u^0, v) \leq 0 \quad \text{in } \Omega; \tag{2.48}$$

taking into account that $\psi_1(x, \cdot, v)$ is increasing for all $x \in \Omega$ and for all $v \in [v_0, v^0]$, that M_1 satisfies (H.4), (2.33) and the maximum principle it follows that $u_0 \leq w \leq u^0$. The inequality $v_0 \leq z \leq v^0$ is proved in a completely analogous way.

ii)Every $(u, v) \in B_R$ satisfies $\mathcal{K}(u, v) \equiv (w, z) \in B_R$. Indeed, according to Proposition 1, (H.3) and (2.37), the functions w and z defined by (2.42)-(2.44) are such that

$$\|w\|_{C^{1,\delta}(\bar{\Omega})} \leq K_1 k_1(\|u\|_{C^1(\bar{\Omega})} + \|v\|_{C^1(\bar{\Omega})}), \tag{2.49}$$

$$\|z\|_{C^{1,\delta}(\bar{\Omega})} \leq K_2 k_2(\|u\|_{C^1(\bar{\Omega})} + \|v\|_{C^1(\bar{\Omega})}). \tag{2.50}$$

From (2.25), for $q > N + 2$ fixed, (2.6) and the above considerations we get

$$\begin{aligned} \|w\|_{C^1(\bar{\Omega})} + \|z\|_{C^1(\bar{\Omega})} \leq \varepsilon(Kk_1 + K_2k_2)(\|u\|_{C^1(\bar{\Omega})} + \|v\|_{C^1(\bar{\Omega})}) + \\ C_{\varepsilon,q}(\|u\|_{L_q(\Omega)} + \|v\|_{L_q(\Omega)}) \leq \\ \varepsilon(Kk_1 + K_2k_2)(\|u\|_{C^1(\bar{\Omega})} + \|v\|_{C^1(\bar{\Omega})}) + C_{\varepsilon,q}K_3. \end{aligned} \tag{2.51}$$

Hence, for $\varepsilon = 1/(2K_1k_1 + 2K_2k_2)$, the inclusion $\mathcal{K}(B_R) \subset B_R$ is satisfied for $R > 0$ large enough.

Finally, since w and z are actually in $C_0^{1,\delta}(\bar{\Omega})$ and the imbedding of this space into $C^1(\bar{\Omega})$ is compact, the operator $\mathcal{K}$ is compact. It is easily seen that D is convex and hence by Schauder's fixed point theorem there is a solution of the equation $\mathcal{K}(u, v) = (u, v)$.

Uniqueness of positive solutions in sublinear elliptic equations follows in the regular case using a "concavity" argument which was extended to singular problems in [19] invoking a generalization of the strong maximum principle in [18]. Here, we extend this argument to the singular system (2.1)-(2.3).

We say the nonlinearity (f, g) is concave if we have

$$f(x, tu, tv) > tf(x, u, v) \quad \text{for all } x \in \Omega, \text{and for all } u > 0,\ v > 0, \tag{2.52}$$

$$g(x, tu, tv) > tg(x, u, v) \quad \text{for all } x \in \Omega, \text{and for all } u > 0,\ v > 0 \tag{2.53}$$

for any $t \in (0, 1)$.

Theorem 2.3. *Assume that conditions* (H.1)-(H.3) *are satisfied and that the nonlinear term* (f, g) *is cooperative and concave. If* (u_1, v_1) *and* (u_2, v_2) *are solutions to* (2.1)-(2.3) *with the regularity in Theorem* 2.1 *or* 2.2, *then* $u_1 \equiv u_2$ *and* $v_1 \equiv v_2$.

Proof. If (u_1, v_1) and (u_2, v_2) are two such solutions, we define

$$\Lambda = \{t \in [0, 1] : tu_1 \leq u_2, tv_1 \leq v_2\}. \tag{2.54}$$

From the smoothness of the solutions it follows that $[0, \delta[\subset \Lambda$ for some $\delta > 0$. We claim that $1 \in \Lambda$. Assume, for contradiction, that $t_0 = \sup\{t \in \Lambda\} < 1$. Then, according to (2.13), we have

$$\mathcal{L}_1(u_2 - t_0 u_1) = f(x, u_2, v_2) - t_0 f(x, u_1, v_1) > \\ f(x, u_2, v_2) - f(x, t_0 u_1, t_0 v_1) \geq 0 \quad \text{in } \Omega, \tag{2.55}$$

$$u_2 - t_0 u_1 = 0 \quad \text{on } \partial\Omega. \tag{2.56}$$

From the strong maximum principle in [18] it follows that either $u_2 - t_0 u_1 \equiv 0$ in Ω or $u_2 - t_0 u_1 > 0$ in Ω, $\partial(u_2 - t_0 u_1)/\partial n < 0$ on $\partial\Omega$. In the latter case $u_2 - t_0 u_1 > \varepsilon u_1$ in Ω and $\partial(u_2 - t_0 u_1)/\partial n < \varepsilon \partial u_1/\partial n$ on $\partial\Omega$ for some $\varepsilon > 0$ small enough, a contradiction. A similar argument works for $v_2 - t_0 v_1$. Finally, if $u_2 - t_0 u_1 \equiv v_2 - t_0 v_1 \equiv 0$ in Ω then, from (2.1) and concavity,

$$0 = f(x, u_2, v_2) - t_0 f(x, u_1, v_1) = f(x, t_0 u_1, t_0 v_1) - t_0 f(x, u_1, v_1) > 0, \tag{2.57}$$

which is again a contradiction, and completes the proof.

Remark 3. For the case of one equation, see [19] and the references therein (see also [16, 17, 28, 30]). An extension of Krasnoselski's idea (see [22])to cooperative systems was given in [7]. An abstract version of this result can be found in [1, Theorem 24.2].

3. APPLICATIONS

In this section we apply the general existence theorems in Section 2 to several examples.

Example 1. We first consider the system

$$\mathcal{L}_1 u = \lambda u^p v^q \quad \text{in} \quad \Omega, \tag{3.1}$$

$$\mathcal{L}_2 v = \mu u^r v^s \quad \text{in} \quad \Omega, \tag{3.2}$$

$$u = v = 0 \quad \text{on} \quad \partial\Omega, \tag{3.3}$$

where Ω satisfies (H.1), $\mathcal{L}_1$ and $\mathcal{L}_2$ are differential operators satisfying (H.2), λ and μ are real parameters and p, q, r, s are such that

$$-1 < p + q < 1, \tag{3.4}$$

$$-1 < r + s < 1. \tag{3.5}$$

As is well known (see [18], Theorem 2.6), the following linear eigenvalue problems posses principal eigenvalues λ_1, $\mu_1 > 0$ with eigenfunctions φ_1, $\psi_1 \in C_0^1(\bar{\Omega})$ such that φ_1, $\psi_1 > 0$ in Ω, and $\frac{\partial \varphi_1}{\partial n}$, $\frac{\partial \psi_1}{\partial n} < 0$ on $\partial\Omega$, satisfying

$$\mathcal{L}_1 \varphi_1 = \lambda_1 \varphi_1 \qquad \text{in} \qquad \Omega, \tag{3.6}$$

$$\mathcal{L}_2 \psi_1 = \mu_1 \psi_1 \qquad \text{in} \qquad \Omega, \tag{3.7}$$

$$\varphi_1 = \psi_1 = 0 \quad \text{on} \quad \partial\Omega. \tag{3.8}$$

Now we consider the intended coupled sub-super-solution (u_0, u^0) - (v_0, v^0), defined as

$$u_0 \equiv a\varphi_1 \qquad u^0 \equiv c u_1, \tag{3.9}$$

$$v_0 \equiv b\psi_1, \qquad v^0 \equiv d v_1, \tag{3.10}$$

where u_1 and v_1 are the unique solutions of the linear problems

$$\mathcal{L}_1 u_1 = \varphi_1^{p+q} \quad \text{in} \quad \Omega, \tag{3.11}$$

$$\mathcal{L}_2 v_1 = \psi_1^{r+s} \quad \text{in} \quad \Omega, \tag{3.12}$$

$$u_1 = v_1 = 0 \quad \text{on} \quad \partial\Omega, \tag{3.13}$$

given in [18](or Corollary 2.1 in [19]) since (3.4)-(3.5) are satisfied. Notice that u_1, $v_1 > 0$ in Ω, and $\frac{\partial u_1}{\partial n}$, $\frac{\partial v_1}{\partial n} < 0$ on $\partial\Omega$. Notice also that under (3.4)-(3.5), the system (3.1)-(3.3) is not cooperative in general.

Let us check that $(u_0, u^0) - (v_0, v^0)$ is actually a sub-super-solution. To this end, we note the following.

i) For any $v \in [v_0, v^0]$ we have

$$\mathcal{L}_1 u_0 - \lambda u_0^p v^q = a\lambda_1 \varphi_1 - \lambda a^p \varphi_1^p v^q = a^p \varphi_1^p (\lambda_1 a^{1-p} \varphi_1^{1-p} - \lambda v^q) \le 0 \tag{3.14}$$

provided that

$$\lambda_1 a^{1-p}\varphi_1^{1-p} \le \lambda v^q, \tag{3.15}$$

thus, it is then sufficient to have

$$\lambda_1 a^{1-p}\varphi_1^{1-p} \le \lambda b^q \psi_1^q. \tag{3.16}$$

It follows from the smoothness of φ_1 and ψ_1 that there exists $k_1 > 0$ such that $\psi_1 \ge k_1\varphi_1$, and we only need to show that

$$\lambda_1 a^{1-p}\varphi_1^{1-p} \le \lambda b^q k_1^q \varphi_1^q. \tag{3.17}$$

If we set $a = b$, then the condition (3.17) can be written as

$$\lambda_1 a^{1-p-q}\varphi_1^{1-p-q} \le \lambda k_1^q, \tag{3.18}$$

which (by (3.4)), for any $\lambda > 0$ fixed, is satisfied for $a = b > 0$ small enough.

ii) Again, for any $v \in [v_0, v^0]$ we have

$$\begin{aligned}\mathcal{L}_1 u^0 - \lambda(u^0)^p v^q = c\mathcal{L}_1 u_1 - \lambda c^p u_1^p v^q = c\varphi_1^{p+q} - \lambda c^p u_1^p v^q = \\ c^p(c^{1-p}\varphi_1^{p+q} - \lambda u_1^p v^q) \ge 0,\end{aligned} \tag{3.19}$$

provided that v is such that

$$\lambda_1 u_1^p v^q \le c^{1-p}\varphi_1^{p+q}, \tag{3.20}$$

which in turn holds whenever

$$\lambda_1 u_1^p d^q v_1^q \le c^{1-p}\varphi_1^{p+q}. \tag{3.21}$$

Setting $c = d$, this inequality is written as

$$\lambda_1 u_1^p v_1^q \le c^{1-p-q}\varphi_1^{p+q}. \tag{3.22}$$

As above, it follows from the smoothness of u_1 and v_1 that there exists $\rho_1 > 0$ such that $\varphi_1 \ge \rho_1 u_1$, $\psi_1 \ge \rho_1 v_1$, which yields

$$\lambda_1 u_1^p v_1^q \le c^{1-p-q}\rho_1^{p+q} u_1^p v_1^q, \tag{3.23}$$

or, equivalently,

$$\lambda \le c^{1-p-q}\rho_1^{p+q}. \tag{3.24}$$

For any $\lambda > 0$ fixed, this condition is satisfied for $c = d > 0$ large enough.

iii) Now for any $u \in [u_0, u^0]$ we have (with $a = b$)

$$\mathcal{L}_2 v_0 - \mu u^r v_0^s = b\mu_1\psi_1 - \mu u^r b^s \psi_1^s = b^s\psi_1^s(\mu_1 b^{1-s}\psi_1^{1-s} - \mu u^r) \le 0 \tag{3.25}$$

provided that

$$\mu_1 a^{1-s}\psi_1^{1-s} \le \mu u^r, \tag{3.26}$$

and it is enough to get

$$\mu_1 a^{1-s}\psi_1^{1-s} \le \mu a^r \varphi_1^r. \tag{3.27}$$

Noting that there exists $k_2 > 0$ such that $\psi_1 \le k_2\varphi_1$, condition (3.27) reads

$$k_2^{1-s}\mu_1 a^{1-r-s}\varphi_1^{1-r-s} \le \mu, \tag{3.28}$$

which, invoking (3.5), is satisfied for any $\mu > 0$ fixed, for $a = b > 0$ small enough.

iv) Again for $u \in [u_0, u^0]$, we have, with $c = d$,

$$\mathcal{L}_2 v^0 - \mu u^r (v^0)^s = c\mathcal{L}_2 v_1 - \mu u^r c^s v_1^s = c\psi_1^{r+s} - \mu c^s u^r v_1^s = \\ c^s(c^{1-s}\psi_1^{r+s} - \mu u^r v_1^s) \ge 0 \tag{3.29}$$

if

$$\mu u^r v_1^s \le c^{1-s}\psi_1^{r+s}, \tag{3.30}$$

and it is enough to get

$$\mu c^r u_1^r v_1^s \le c^{1-s}\psi_1^{r+s}, \tag{3.31}$$

or in turn

$$\mu u_1^r v_1^s \le c^{1-r-s}\psi_1^{r+s}. \tag{3.32}$$

Using again the smoothness of ψ_1, u_1 and v_1, there exists $\sigma_1 > 0$ such that $\psi_1 \ge \sigma_1 u_1$, $\psi_1 \ge \sigma_1 v_1$ and then it is enough that

$$\mu \le c^{1-r-s}\sigma_1^{r+s}, \tag{3.33}$$

which, for $\mu > 0$ given, is satisfied for $c = d > 0$ large.

Note that we can always pick the constant a (respectively b) sufficiently small and c (respectively d) large enough in such a way that $u_0 \le u^0$, $v_0 \le v^0$. Invoking Theorem 2.2, we have the following.

Theorem 3.1. *Assume that* (3.4) *and* (3.5) *are satisfied. Then for any* $\lambda > 0$ *and any* $\mu > 0$ *there exists a solution* (u, v) *to* (3.1)-(3.3) *such that* $u, v \in C_0^{1,\delta}(\Omega)$ *for some* $0 < \delta < 1$, $u, v > 0$ *in* Ω *and* $\frac{\partial u}{\partial n}$, $\frac{\partial v}{\partial n} < 0$ *on* $\partial\Omega$. *Moreover, if* $q > 0$ *and* $r > 0$, *the solution is unique.*

Proof. Existence follows from Theorem 2.2. If $q > 0$ and $r > 0$, then the system is cooperative and since, for $f(u, v) = \lambda u^p v^q$, $g(u, v) = \mu u^r v^s$ we have

$$f(tu, tv) = t^{p+q} f(u, v) > f(u, v), \quad g(tu, tv) = t^{r+s} f(u, v) > g(u, v) \tag{3.34}$$

for any $t \in]0, 1[$ by (3.4) (3.5) and we apply Theorem 2.3.

Remark 4. A very particular instance of problem (3.1)-(3.3), namely $\mathcal{L}_1 u = -\Delta u + u$, $\mathcal{L}_2 v = -\Delta v + \alpha v$, $p = -q = r = -s = 1$, was studied in [4]. Existence was proved by using Schauder's fixed point theorem and different α's were treated separately ($\alpha > 1, \alpha = 1$, $\alpha < 1$) with very specific arguments. Theorem 3.1 also extends the results in [12, 20, 21], which are obtained for $\mathcal{L}_1 u = -\Delta u + \alpha u$, $\mathcal{L}_2 v = -\Delta v + \beta v$ with $\alpha, \beta > 0$, $p, r > 0$ $q, s < 0$, $p + q = r + s$, by using an approximation argument. Uniqueness was proved only in the one-dimensional case, using intricate properties of the zeroes of the solutions.

Remark 5. Systems like (3.1)-(3.3) arise in the Gierer-Meinhardt [13] models for morphogenesis. Choi and McKenna [5] prove existence for $N = 1$ and $N = 2$, Ω a ball, using Schauder's theorem in the case (in our notation) $p = r$, $q = -1$, $s = 0$, with $r > 1$. These results can be extended to general domains if some essential estimate (in Lemma 9 in [5]) holds. Notice that Theorem 3.1 covers this result only for $0 < r < 1$. On the other side, the main instances in [13], namely $p = 2$, $q = -1$, $r = 2$, $s = 0$ and $p = 2$, $q = -4$, $r = 2$, $s = -4$ are covered neither by Theorem 3.1 nor in [5].

Remark 6. Some more general results in [19] allow us to deal with more general situations; for example, Theorem 3.2 in [19] allows us to replace in (3.1) $u^p v^q$ by $d(x)^\delta u^p v^q$ with $-1 < \delta + p + q < 1$, etc. Similar remarks apply for all examples in this section and will not be repeated for each considered problem.

Let us now treat a series of variants of the usual Lotka-Volterra systems for population dynamics. The first one is the following.

Example 2. We first consider the system

$$\mathcal{L}_1 u = u(\lambda - \alpha u^r - \beta v) \quad \text{in} \quad \Omega, \tag{3.35}$$

$$\mathcal{L}_2 v = \mu v - \frac{v^s}{u^\delta} \quad \text{in} \quad \Omega, \tag{3.36}$$

$$u = v = 0 \quad \text{on} \quad \partial\Omega, \tag{3.37}$$

where Ω, $\mathcal{L}_1$, and $\mathcal{L}_2$ are as above, $\alpha, \beta > 0$, λ, μ are real parameters and

$$r > 0, \qquad s - 1 > 0, \qquad s - \delta > 1. \tag{3.38}$$

We prove first the following.

Lemma 3.2. *Under the assumptions above,if there is a nontrivial positive solution of* (3.35)-(3.37), *then*

$$\lambda > \lambda_1, \quad \mu > \mu_1. \tag{3.39}$$

Proof. This follows from well-known comparison arguments on eigenvalues of linear problems (see [18]). (In the case of operators in divergence form, the argument reduces to an integration by parts.)

Now, we can prove the following result.

Theorem 3.3. *If* (3.38) *is satisfied then for any* $\lambda > \lambda_1$ *any* α *and any* $\mu > \mu_1$ *there exists a* $\beta_0 > 0$ *such that for* $0 < \beta < \beta_0$ *there exists a positive solution* $u, v > 0$ *to* (3.35)-(3.37) *with the same regularity as in Theorem* 3.1.

Proof. We use again Theorem 2.2 this time with

$$u_0 \equiv c\varphi_1, \qquad u^0 \equiv w, \tag{3.40}$$

$$v_0 \equiv c\psi_1, \qquad v^0 \equiv Cz, \tag{3.41}$$

where $\varphi_1, \psi_1 > 0$ are as above, the constants c, $C > 0$ will be chosen later, $w > 0$ is the unique positive solution for $\lambda > \lambda_1$ to the logistic equation

$$\begin{aligned} \mathcal{L}_1 w + \alpha w^{r+1} = \lambda w \quad \text{in} \quad \Omega, \\ w = 0 \quad \text{on } \partial\Omega \end{aligned} \tag{3.42}$$

and $z > 0$ is the unique positive solution (for $\mu > \mu_1$) of

$$\begin{aligned} \mathcal{L}_2 z + z^{s-\delta} = \mu z \quad \text{in} \quad \Omega, \\ z = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.43}$$

(Notice that $s - \delta > 1$.)

We check the conditions:

i) If $u_0 \equiv a\varphi_1$, then for all $v \in [v_0, v^0]$

$$\begin{aligned} \mathcal{L}_1 u_0 - \lambda u_0 + \alpha(u_0)^{r+1} + \beta u_0 v = \\ c\lambda_1\varphi_1 - c\lambda\varphi_1 + \alpha c^{r+1}\varphi_1^{r+1} + \beta c\varphi_1 v \leq 0, \end{aligned} \tag{3.44}$$

which follows, for $\lambda > \lambda_1$ fixed, for c, $\beta > 0$ small enough if v^0 is bounded independently of β, which will be the case (see point iv) below).

ii) If $u^0 \equiv w$, then for any $\beta > 0$, $v \geq 0$

$$\mathcal{L}_1 w + \alpha w^{r+1} - \lambda w + \beta w v \geq 0. \tag{3.45}$$

iii) If $v_0 \equiv c\psi_1$ and $u \in [u_0, u^0]$, then

$$\begin{aligned} \mathcal{L}_2 v_0 - \mu v_0 + \frac{(v_0)^s}{u^\delta} = c\mu_1\psi_1 - \mu c\psi_1 + \frac{c^s\psi_1^s}{u^\delta} = \\ c\psi_1(\mu_1 - \mu + \frac{c^{s-1}\psi_1^{s-1}}{u^\delta}) \leq 0, \end{aligned} \tag{3.46}$$

for $\mu > \mu_1$ fixed, and $c > 0$ small, from $s - \delta - 1 > 0$ since $k_1\psi_1 \leq u \leq k_2\psi_1$ for some $k_1, k_2 > 0$, and C as in iv) below.

iv) Now for $v^0 = Cz$ and $u \in [u_0, u^0]$ we obtain

$$\mathcal{L}_2 v^0 - \mu v^0 + \frac{(v^0)^s}{u^\delta} = C(\mu z - z^{s-\delta}) + \frac{C^s z^s}{u^\delta} - \mu C z = \frac{C^s z^s - C z^{s-\delta} u^\delta}{u^\delta} \geq 0 \tag{3.47}$$

if

$$u^\delta z^{s-\delta} \leq C^{s-1} z^s. \tag{3.48}$$

Since, again by the smoothness, $u \leq u^0 \leq w \leq k_3 z$, the condition is satisfied for a small and C large, thus we have a sub-super-solution, and this ends the proof.

Example 3. The following system is a slight variant of the preceding one.

$$\mathcal{L}_1 u = u(\lambda - \alpha u^r - \beta v) \quad \text{in} \quad \Omega, \tag{3.49}$$

$$\mathcal{L}_2 v = \mu v^\alpha - \frac{v^s}{u^\delta} \quad \text{in} \quad \Omega, \tag{3.50}$$

$$u = v = 0 \quad \text{on} \quad \partial\Omega, \tag{3.51}$$

where Ω, $\mathcal{L}_1$ and $\mathcal{L}_2$ are as above, $\beta > 0$, λ and μ are real parameters and

$$r > 0, \qquad -1 < \alpha < 1 < s, \qquad s - \delta > \alpha. \tag{3.52}$$

Reasoning as in Lemma 3.2 we have the following.

Lemma 3.4. *Under the above assumptions, if there is a nontrivial positive solution of* (3.49)-(3.51), *then* $\lambda > \lambda_1$ *and* $\mu > 0$.

Theorem 3.5. *If (3.52) is satisfied, then for any* $\lambda > \lambda_1$ *and any* $\mu > 0$ *there exists a* $\beta_0 > 0$ *such that for any* $0 < \beta < \beta_0$ *there exists a positive solution* $u, v > 0$ *to* (3.49)-(3.51) *with the same regularity as in Theorem* 3.1.

Proof. This is rather similar to the proof of Theorem 3.3, this time with

$$u_0 \equiv c\varphi_1, \qquad u^0 \equiv w, \tag{3.53}$$

$$v_0 \equiv c\psi_1, \qquad v^0 \equiv C\tilde{z}, \tag{3.54}$$

where φ_1, ψ_1, a, C are as in the proof of the preceding theorem, w is again given by (3.42) and $\tilde{z}$ is the unique positive solution (given by Theorem 3.3 in [19]) of

$$\begin{aligned} \mathcal{L}_1 \tilde{z} + \tilde{z}^{s-\delta} &= \mu \tilde{z}^\alpha \quad \text{in} \quad \Omega, \\ \tilde{z} &= 0 \quad \text{on} \ \partial\Omega. \end{aligned} \tag{3.55}$$

i) It is clear that $u^0 \equiv w$ satisfies the condition.

ii) If $u_0 \equiv a\varphi_1$ then the fact that for any $v \in [v_0, v^0]$

$$\mathcal{L}_1 u_0 - \lambda u_0 + u_0^{r+1} + \beta u_0 v \leq 0 \tag{3.56}$$

for any $\beta > 0$ small enough is proved as before.

iii) If $v_0 \equiv a\psi_1$ and $u \in [u_0, u^0]$, then the fact that

$$\mathcal{L}_2 v_0 - \mu(v_0)^\alpha + \frac{v_0^s}{u^\delta} = c^\alpha {\psi_1}^\alpha(\mu_1 c^{1-\alpha}{\psi_1}^{1-\alpha} - \mu + \frac{c^{s-\alpha}{\psi_1}^{s-\alpha}}{u^\delta}) \leq 0 \tag{3.57}$$

for any $\mu > 0$ and $a > 0$ small enough, follows from the fact that $s - \delta > \alpha$ since $k_3\psi_1 \leq u \leq k_4\psi_1$ for some $k_3, k_4 > 0$.

iv) Now for $v^0 \equiv C\tilde{z}$ and $u \in [u_0, u^0]$ we get

$$\mathcal{L}_2 v^0 - \mu(v^0)^\alpha + \frac{(v^0)^s}{u^\delta} = \mu C^\alpha \tilde{z}^\alpha (C^{1-\alpha} - 1) + \frac{C^s \tilde{z}^s - C u^\delta \tilde{z}^{s-\delta}}{u^\delta} \geq 0 \tag{3.58}$$

if

$$u^\delta \tilde{z}^{s-\delta} \leq C^{s-1}\tilde{z}^s \tag{3.59}$$

since, from (3.52), and again by the smoothness $u \leq u^0 \leq w \leq k_5\tilde{z}$, the condition is satisfied for $C > 0$ large (notice that $1 - \alpha > 0$ and $s - 1 > 0$).

Example 4 (singular competition system). We consider next the system

$$\mathcal{L}_1 u = \lambda u^\alpha - u^\delta - a u^p v^q \quad \text{in} \quad \Omega, \tag{3.60}$$

$$\mathcal{L}_2 v = \mu v^\beta - v^\gamma - b u^r v^s \quad \text{in} \quad \Omega, \tag{3.61}$$

$$u = v = 0 \quad \text{on} \quad \partial\Omega, \tag{3.62}$$

where Ω, $\mathcal{L}_1$ and $\mathcal{L}_2$ are as above, $a, b > 0$ are constants, λ and μ are real parameters, and

$$-1 < \alpha < 1, \quad \alpha < \delta, \quad \alpha < p, \quad p - \alpha + q > 0, \tag{3.63}$$

$$-1 < \beta < 1, \quad \beta < \gamma, \quad \beta < s, \quad s - \beta + r > 0. \tag{3.64}$$

Theorem 3.6. *If* (3.63)-(3.64) *are satisfied, then for any* $\lambda > 0$ *and any* $\mu > 0$ *there exists a positive solution* $u, v > 0$ *of* (3.60)-(3.62) *with the regularity of Theorem* 3.1.

Proof. We apply Theorem 2.2 with

$$u_0 \equiv c\varphi_1, \qquad u^0 \equiv w, \tag{3.65}$$

$$v_0 \equiv c\psi_1, \qquad v^0 \equiv z, \tag{3.66}$$

where $\varphi_1, \psi_1 > 0$ are as above and $w > 0$ (respectively $z > 0$) is the unique positive solution of

$$\mathcal{L}_1 w + w^\delta = \lambda w^\alpha \quad \text{in} \quad \Omega, \qquad w = 0 \quad \text{on } \partial\Omega \tag{3.67}$$

(respectively

$$\mathcal{L}_2 z + z^\gamma = \mu z^\beta \quad \text{in} \quad \Omega, \qquad z = 0 \quad \text{on } \partial\Omega) \tag{3.68}$$

given by Theorem 3.3 in [19] (since $\delta > \alpha$ and $\gamma > \beta$).

i) For any $v \in [v_0, v^0]$

$$\mathcal{L}_1 u_0 - \lambda {u_0}^\alpha + {u_0}^\delta + a{u_0}^p v^q =$$

$$c^\delta {\varphi_1}^\delta + ac^p {\varphi_1}^p v^q = \tag{3.69}$$

$$c^\alpha {\varphi_1}^\alpha (\lambda_1 c^{1-\alpha} {\varphi_1}^{1-\alpha} - \lambda + c^{\delta-\alpha} {\varphi_1}^{\delta-\alpha} + ac^{p-\alpha} {\varphi_1}^{p-\alpha} v^q) \leq 0 \tag{3.70}$$

for $c > 0$ small enough by (3.63) since $k_1\varphi_1 \leq v \leq k_2\varphi_1$ with $k_1, k_2 > 0$.

ii) For any $v \in [v_0, v^0]$ we have

$$\mathcal{L}_1 u^0 - \lambda (u^0)^\alpha + (u^0)^\delta + a(u^0)^p v^q \geq \mathcal{L}_1 w - \lambda w^\alpha + w^\delta = 0. \tag{3.71}$$

iii) For any $u \in [u_0, u^0]$ we have

$$\mathcal{L}_2 v_0 - \mu v_0^\beta + v_0^\gamma + bu^r v_0^s =$$

$$c^\beta \psi_1^\beta (\mu_1 c^{1-\beta} \psi_1^{1-\beta} - \mu + c^{\gamma-\beta} \psi_1^{\gamma-\beta} + bc^{s-\beta} \psi_1^{s-\beta} u^r) \leq 0 \tag{3.72}$$

for $c > 0$ small by (3.64) since $k_3\varphi_1 \leq u \leq k_4\varphi_1$ with $k_3, k_4 > 0$.

iv) For any $u \in [u_0, u^0]$

$$\mathcal{L}_2 v^0 - \mu (v^0)^\beta + (v^0)^\gamma + bu^r (v^0)^s \geq \mathcal{L}_2 z - \mu z^\beta - z^\gamma = 0. \tag{3.73}$$

Remark 7. Notice that the proof still works for $\alpha = 1$ if $\delta > 1$, $p > 1$ and $q > 0$. The classical competition Lotka-Volterra system, where $\alpha = 1$, $\delta = 2$, $p = 1$, is not included.

Remark 8. A competition system of this type on the whole space $\mathbb{R}^N$ was studied in [3] for $0 < \alpha < 1$, $\delta \geq 1$, $p \geq 1$ and some q's. In this case, we extend to the singular case for a bounded domain the existence result in [3]. The singular case was studied in [29].

Remark 9. Systems similar to (3.60)-(3.62) (the same observation holds for Examples 5 and 6) arise if linear diffusion is replaced in the usual Lotka-Volterra systems by a nonlinear diffusion $-\Delta u^m$, with $m > 1$. This could give terms such as u^α, $0 < \alpha < 1$, etc., after a change of variables, but not with $-1 < \alpha < 0$.

Example 5 (singular predator-prey system). Consider the system

$$\mathcal{L}_1 u = \lambda u^\alpha - u^\delta - au^p v^q \quad \text{in} \quad \Omega, \tag{3.74}$$

$$\mathcal{L}_2 v = \mu v^\beta - v^\gamma + bu^r v^s \quad \text{in} \quad \Omega, \tag{3.75}$$

$$u = v = 0 \quad \text{on} \quad \partial\Omega, \tag{3.76}$$

where Ω, $\mathcal{L}_1$ and $\mathcal{L}_2$ are as above, $a, b > 0$ are constants and λ and μ are real parameters. Moreover

$$-1 < \alpha < 1, \quad \alpha < \delta, \quad \alpha < p, \quad p - \alpha + q > 0, \tag{3.77}$$

$$-1 < \beta < 1, \quad \beta < s < \gamma < r + s, \quad 1 < \gamma. \tag{3.78}$$

Theorem 3.7. *If* (3.77)-(3.78) *are satisfied, then for any* $\lambda > 0$ *and any* $\mu > 0$ *there exists a positive solution* $u, v > 0$ *of* (3.74)-(3.76) *with the regularity of Theorem* 3.1.

Proof. We apply Theorem 2.2 with

$$u_0 \equiv c\varphi_1, \qquad u^0 \equiv w, \tag{3.79}$$

$$v_0 \equiv c\psi_1, \qquad v^0 \equiv Cz, \tag{3.80}$$

where $\varphi_1, \psi_1 > 0$ are as defined in (3.6)-(3.8), w, z are as defined in (3.67)-(3.68) and $c, C > 0$ will be chosen later.

Proceeding as in in the proof of Theorem 3.6 it is obtained that the two inequalities in (2.27) are satisfied.

Moreover, we have the following.

iii) For any $u \in [u_0, u^0]$ we have

$$\mathcal{L}_2 v_0 - \mu v_0^\beta + v_0^\gamma - bu^r v_0^s =$$
$$c^\beta \psi_1^\beta (\mu_1 c^{1-\beta} \psi_1^{1-\beta} - \mu + c^{\gamma-\beta} \psi_1^{\gamma-\beta} - bc^{s-\beta} \psi_1^{s-\beta} u^r) \le 0 \tag{3.81}$$

for $c > 0$ small by (3.78) since $k_1\psi_1 \le u \le k_2\psi_1$ is bounded for some $k_1, k_2 > 0$, by smoothness. (Notice that the minus sign in the last term is not relevant.)

iv) For any $u \in [u_0, u^0]$ we have

$$\mathcal{L}_2 v^0 - \mu (v^0)^\beta + (v^0)^\gamma - bu^r (v^0)^s =$$
$$C(\mu z^\beta - z^\gamma) - \mu C^\beta z^\beta + C^\gamma z^\gamma - bu^r C^s z^s =$$
$$\mu C^\beta z^\beta (C^{1-\beta} - 1) + C(C^{\gamma-1} - 1) z^\gamma - bu^r C^s z^s \ge 0. \tag{3.82}$$

Taking into account the fact that there exist k_3, k_4 such that $k_3 z \le u \le k_4 z$, and that $\gamma - s > 0$, $\gamma > 1$ and $r + s - \gamma > 0$; i.e., $s < \gamma < r + s$, by (3.78) it is easily seen that the inequality (3.82) is satisfied for $C > 0$ large enough.

Example 6 (singular symbiosis system). Consider now the system

$$\mathcal{L}_1 u = \lambda u^\alpha - u^\delta + a u^p v^q \quad \text{in} \quad \Omega, \tag{3.83}$$

$$\mathcal{L}_2 v = \mu v^\beta - v^\gamma + b u^r v^s \quad \text{in} \quad \Omega, \tag{3.84}$$

$$u = v = 0 \quad \text{on} \quad \partial\Omega, \tag{3.85}$$

where Ω, $\mathcal{L}_1$ and $\mathcal{L}_2$ are as above, $a, b > 0$ are constants and λ and μ are real parameters. Moreover

$$-1 < \alpha < 1, \quad \alpha < p < \delta < p + q, \quad 1 < \delta, \tag{3.86}$$

$$-1 < \beta < 1, \quad \beta < s < \gamma < r + s, \quad 1 < \gamma. \tag{3.87}$$

Theorem 3.8. *If* (3.86)-(3.87) *are satisfied, then for any* $\lambda > 0$ *and any* $\mu > 0$ *there exists a positive solution* $u, v > 0$ *of* (3.83)-(3.85) *with the regularity of Theorem* 3.1. *Moreover, if* $q > 0$ *and* $r > 0$*, it is unique.*

Proof. We apply Theorem 2.2 with

$$u_0 \equiv c\varphi_1, \qquad u^0 \equiv Cw, \tag{3.88}$$

$$v_0 \equiv c\psi_1, \qquad v^0 \equiv Cz, \tag{3.89}$$

where $\varphi_1, \psi_1 > 0, w > 0$ and $z > 0$ are as defined in (3.6)-(3.8), (3.67)-(3.68) and $c, C > 0$ will be chosen as in Theorem 3.7.

Proceeding as in the proof of Theorem 3.7 we obtain the first inequality of (2.27).

Moreover, for any $v \in [v_0, v^0]$ we have

$$\mathcal{L}_1 u^0 - \lambda (u^0)^\alpha + (u^0)^\delta - a(u^0)^p v^q =$$
$$\lambda C^\alpha w^\alpha (C^{1-\alpha} - 1) + C(C^{\delta-1} - 1) w^\delta - aC^p w^p v^q \geq 0. \tag{3.90}$$

By (3.86) the first term on the right-hand side is positive for $C > 1$. Taking into account the fact that $\delta - p > 0$ and $p + q - \delta < 0$; i.e., $p < \delta < p + q$, it is easily seen that the second inequality in (2.27) is satisfied for C large enough.

Finally, proceeding as in the proof of steps iii) and iv) in Theorem 3.7 we obtain the fact that the two inequalities in (2.28) are satisfied. Uniqueness is proved as in Theorem 3.1.

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