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POSITIVE SOLUTIONS FOR SOME SEMI-POSITONE PROBLEMS VIA BIFURCATION THEORY

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Dedicated to the memory of Peter Hess

Abstract. Bifurcation Theory is used to prove the existence of positive solutions of some classes of semi-positone problems.

1. Introduction. In this paper we deal with the existence of positive solutions of Dirichlet boundary value problems like

$$\begin{cases} -\Delta u = \lambda f(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$ and $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}$. If $f(x, 0) \geq 0$ then (1) is called a positone problem and has been extensively studied; see, e.g., [3], [10], [11], and the survey [1].

On the contrary we deal here with the so called *semi-positone* (or *non-positone*) problem, when f is such that

(f_1) $f(x, 0) < 0, \quad \forall x \in \Omega.$

Recently some existence results concerning semi-positone problems have been proved; see [4], [6], [7], and [12]. With the exception of [6] (that deals with sub-linear problems and uses sub and super-solutions) the common feature of the papers mentioned above is that they are obtained by means of ODE techniques, such as the shooting method, and hence they handle the case where Ω is an annulus, a ball or a set close to a ball and $f(x, u) = f(u)$.

The main purpose of the present paper is to show that Bifurcation theory can be easily used to study semi-positone problems, like the positone ones. The same abstract setting is employed to handle both asymptotically linear, superlinear as well as sublinear problems on general domains (hence genuine partial differential equations).

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Roughly, we first show that there exists a global branch of solutions of (1) "emanating from infinity"; next we prove that for λ near the bifurcation value, solutions of large norm are indeed positive.

After some notation and preliminaries listed in Section 2, we deal in Section 3 with asymptotically linear problems and use techniques close to [3] to prove an existence result that is new in the frame of semi-positone problems. In Section 4 we discuss superlinear problems. A 'blow-up' argument jointly with some a priori estimates of [9] allows one to show that (1) possesses positive solutions for $0 < \lambda < \lambda_*$. Similar arguments can be used in the sublinear case, discussed in Section 5, to show that (1) has positive solutions provided λ is large enough.

2. Notation and preliminaries. Standard notation will be used for Lebesgue and Sobolev spaces. The norm in $L^r(\Omega)$ will be denoted by $|\cdot|_r$ and the scalar product in $L^2(\Omega)$ by $(\cdot \mid \cdot)$. We will work in $X = C(\overline{\Omega})$ or $Y = C^{1,\nu}(\overline{\Omega})$, the space of continuous, C^1 with Hölder continuous first derivative respectively, functions. The usual norm in such spaces will be denoted by $\|\cdot\| = |\cdot|_\infty$ and $\|\cdot\|_{1,\nu}$; we also set $B_r = \{u \in X : \|u\| < r\}$. The first eigenvalue of $-\Delta$ with zero Dirichlet boundary conditions is denoted by λ_1; φ_1 is the corresponding eigenfunction such that $\varphi_1 > 0$ in Ω and $|\varphi_1|_2 = 1$. We also set $\mathbb{R}^+ = [0, \infty)$.

We let $K : X \to X$ denote the Green operator of $-\Delta$ with zero Dirichlet boundary conditions, i.e., $v = Ku$ if and only if

$$\begin{cases} -\Delta u = v, & x \in \Omega, \\ \quad u = 0, & x \in \partial\Omega. \end{cases}$$

In Section 5 we will consider K as a map into Y.

By a solution of (1) we mean a $u \in H^{2,r}(\Omega) \cap H_0^{1,2}(\Omega)$, $\forall\, r \geq 1$, which solves (1) weakly. With the above notation, problem (1) is equivalent to

$$u - \lambda K f(u) = 0, \quad u \in X. \tag{2}$$

Hereafter we will use the same symbol to denote both the function and the associated Nemitski operator.

We say that λ_∞ is a *bifurcation from infinity* for (2) if there exist $\mu_n \to \lambda_\infty$ and $u_n \in X$, such that $u_n - \mu_n K f(u_n) = 0$ and $\|u_n\| \to \infty$. Extending the preceding definition, we will say that $\lambda_\infty = +\infty$ is a bifurcation from infinity for (2) if solutions (μ_n, u_n) of (2) exist with $\mu_n \to +\infty$ and $\|u_n\| \to +\infty$. This is the case we will meet in Section 5.

In some situations, like the specific ones we will discuss later, an appropriate rescaling permits one to find bifurcation from infinity by means of Leray-Schauder topological degree $\deg(\cdot, \cdot, \cdot)$. Recall that $K : X \to X$ is (continuous and) compact, and hence it makes sense to consider the topological degree of $I - \lambda K f$, I identity map.

3. Asymptotically linear problems. In this section we suppose that $f \in C(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R})$ satisfies (f_1) and

(f_2) $\exists\, m > 0$ such that

$$\lim_{u \to +\infty} \frac{f(x,u)}{u} = m.$$

Let $\lambda_\infty = \frac{\lambda_1}{m}$ and define

$$a(x) = \liminf_{u \to +\infty} (f(x,u) - mu)\,, \quad A(x) = \limsup_{u \to +\infty} (f(x,u) - mu)\,.$$

We will show

Theorem 1. *Suppose that f satisfies (f_1) and (f_2). Then there exists $\epsilon > 0$ such that* (1) *has positive solutions provided either*

(i) *$a > 0$ (possibly $+\infty$) in Ω and $\lambda \in [\lambda_\infty - \epsilon, \lambda_\infty[$; or*
(ii) *$A < 0$ (possibly $-\infty$) in Ω and $\lambda \in]\lambda_\infty, \lambda_\infty + \epsilon]$.*

Actually, Theorem 1 is a particular case of a somewhat more general result; see the Remarks at the end of this section.

The proof of Theorem 1 will be carried out in several steps. First of all, we extend $f(x,\cdot)$ to all of $\mathbb{R}$ by setting

$$F(x,u) = f(x,|u|).$$

Let $X = C(\overline{\Omega})$ and set, for $u \in X$,

$$\Phi(\lambda, u) := u - \lambda K F(u).$$

Plainly, any $u > 0$ such that $\Phi(\lambda, u) = 0$ is a positive solution of (1). The following two Lemmas are closely related to [3].

Lemma 2. *For every compact interval $\Lambda \subset \mathbb{R}^+ \setminus \{\lambda_\infty\}$ there exists $r > 0$ such that $\Phi(\lambda, u) \neq 0$, $\forall\, \lambda \in \Lambda$, $\forall\, \|u\| \geq r$. Moreover,*

(i) *if $a > 0$ then we can also take $\Lambda = [\lambda_\infty, \lambda]$, $\forall \lambda > \lambda_\infty$;*
(ii) *if $A < 0$ then we can also take $\Lambda = [0, \lambda_\infty]$.*

Proof. Let $\mu_n \to \mu \geq 0$, $\mu \neq \lambda_\infty$, and $\|u_n\| \to \infty$ be such that

$$u_n = \mu_n K F(u_n).$$

Setting $w_n = u_n \|u_n\|^{-1}$, we find

$$w_n = \mu_n \|u_n\|^{-1} K F(u_n).$$

Assumption (f_2), elliptic theory and the definition of F yield that, up to a subsequence, $w_n \to w$ in $C^{1,\nu}(\overline{\Omega})$, where w is such that $\|w\| = 1$ and satisfies

$$\begin{cases} -\Delta w = \mu m |w|, & x \in \Omega, \\ w = 0, & x \in \partial\Omega. \end{cases}$$

By the maximum principle it follows that $w \geq 0$. Since $\|w\| = 1$ we infer that $\mu m = \lambda_1$, namely $\mu = \lambda_\infty$, a contradiction that proves the first statement.

The remaining arguments are the same as that of Lemmas 3.1, 3.6 of [3] and we will only give a short sketch of (i). Taking $\mu_n \downarrow \lambda_\infty$, it follows that $w \geq 0$ satisfies $-\Delta w = \lambda_1 w$ in Ω and $w = 0$ on $\partial\Omega$ and hence there exists $\beta > 0$ such that $w = \beta\phi_1$. Then one has $u_n = \|u_n\| w_n \to +\infty$ for all $x \in \Omega$ and $F(u_n) = f(u_n)$ for n large. From $\Phi(\lambda_n, u_n) = 0$ it follows that

$$\lambda_1 (u_n \mid \phi_1) = \mu_n(f(u_n) - mu_n \mid \phi_1) + \mu_n m(u_n \mid \phi_1).$$

Since $\mu_n > \lambda_\infty$ and $(u_n \mid \phi_1) > 0$ for n large, we infer that $(f(u_n) - mu_n \mid \phi_1) < 0$ for n large and the Fatou Lemma yields

$$0 \geq \liminf(f(u_n) - mu_n \mid \phi_1) \geq (a \mid \phi_1),$$

a contradiction if $a > 0$.

Lemma 3. *If $\lambda > \lambda_\infty$ there exists $r > 0$ such that*

$$\Phi(\lambda, u) \neq t\phi_1, \quad \forall t \geq 0, \ \|u\| \geq r.$$

Proof. Taking into account that $F(x, u) \simeq m|u|$ as $|u| \to \infty$, one can repeat the arguments of Lemma 3.3 of [3] with some minor changes. □

For $u \neq 0$, we set $z = u\|u\|^{-2}$. Letting

$$\Psi(\lambda, z) = \|u\|^{-2}\Phi(\lambda, u) = z - \lambda\|z\|^2 \, KF\Big(\frac{z}{\|z\|^2}\Big),$$

one has that λ_∞ is a bifurcation from infinity for (2) if and only if it is a bifurcation from the trivial solution $z = 0$ for $\Psi = 0$. From Lemma 2 it follows by homotopy that

$$\begin{aligned} \deg(\Psi(\lambda, \cdot), B_{1/r}, 0) &= \deg(\Psi(0, \cdot), B_{1/r}, 0) \\ &= \deg(I, B_{1/r}, 0) = 1, \quad \forall\lambda < \lambda_\infty. \end{aligned} \tag{3}$$

Similarly, by Lemma 3 one infers, for all $\tau \in [0, 1]$ and for all $\lambda > \lambda_\infty$,

$$\begin{aligned} \deg(\Psi(\lambda, \cdot), B_{1/r}, 0) &= \deg(\Psi(\lambda, \cdot) - \tau\phi_1, B_{1/r}, 0) \\ &= \deg(\Psi(\lambda, \cdot) - \phi_1, B_{1/r}, 0) = 0. \end{aligned} \tag{4}$$

Let us set

$$\Sigma = \{(\lambda, u) \in \mathbb{R}^+ \times X \ : \ u \neq 0, \ \Phi(\lambda, u) = 0\}.$$

From (3) and (4) and the preceding discussion we deduce

Lemma 4. *λ_∞ is a bifurcation from infinity for* (2). *More precisely there exists an unbounded closed connected set $\Sigma_\infty \subset \Sigma$ that bifurcates from infinity. Moreover, Σ_∞ bifurcates to the left* (*to the right*) *provided $a > 0$* (*respectively $A < 0$*).

Proof of Theorem 1. By the above Lemmas, it suffices to show that if $\mu_n \to \lambda_\infty$ and $\|u_n\| \to \infty$ then $u_n > 0$ in Ω for n large. Setting $w_n = u_n \, \|u_n\|^{-1}$ and using the preceding arguments, we find that, up to subsequence, $w_n \to w$ in $C^{1,\nu}$, and $w = \beta\phi_1$, $\beta > 0$. Then, it follows that $u_n > 0$ in Ω, for n large.

Remarks.

(1) The proof of Theorem 1 actually shows there exists $k > 0$ such that for all $(\lambda, u) \in \Sigma_\infty$ with $\|u\| \geq k$ one has that $u > 0$ in Ω. Thus such (λ, u) are solutions of (1).

(2) It is clear that the Laplace operator can be substituted by any uniformly elliptic second order operator with smooth coefficients. Moreover we can allow that m depends on x.

(3) Suppose that $f(x, u) = f(u)$ and there exists $c > 0$ such that $f(c) = 0$, $f(u) < 0$ and $f(u) > 0$ for all $0 \leq u < c, u > c$ respectively. The maximum principle implies that $u(x_{\max}) > c$ whenever u is positive somewhere in Ω. Since solutions on the branch Σ_∞ are indeed positive for $\lambda \to \lambda_\infty$, then a continuity argument shows that $\|u\| > c$ for all $(\lambda, u) \in \Sigma_\infty$.

(4) In general, solutions on Σ_∞ can change sign and the behavior of Σ_∞ depends on the definition of f for $u < 0$. However, independently of such definition, there exists a branch $\Sigma_0 \subset \Sigma$ emanating from $(0, 0) \in \mathbb{R} \times X$, consisting of negative solutions of (1), and such that $\Sigma_\infty \cap \Sigma_0 = \emptyset$. Let us point out that this is in contrast with the positone case.

(5) The following example shows that, in general, (1) does not possess positive solutions for λ large. Let us consider the equation $-u'' = \lambda(u - 1)$, $x \in (0, \pi)$, $u(0) = u(\pi) = 0$. An explicit calculation shows that a *positive* solution exists if and only if either $1 < \lambda \leq 4$ or $\lambda = 4k^2$ ($k \in \mathbb{N}$) (although such a problem has solutions for all $\lambda > 1$).

4. Superlinear problems. We will study the existence of positive solutions of problem (1) when $f(x, .)$ is superlinear. Precisely, we suppose that $f \in C(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R})$ satisfies (f_1) and

(f_3) $\exists\, b \in C(\overline{\Omega}), b > 0$, such that $\lim_{u\to\infty} u^{-p} f(x, u) = b$, uniformly in $x \in \overline{\Omega}$, with $1 < p < 2^* - 1$,

where $2^* = \frac{2N}{N-2}$ if $N \geq 3$, $2^* = +\infty$ if $N = 1, 2$.

Our main result is

Theorem 5. *Let $f \in C(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R})$ satisfy (f_1) and (f_3). Then $\exists\, \lambda_* > 0$ such that* (1) *has positive solutions for all $0 < \lambda \leq \lambda_*$. More precisely, there exists a connected set of positive solutions of* (1) *bifurcating from infinity at $\lambda_\infty = 0$.*

Proof. As before we set

$$F(x, u) = f(x, |u|)$$

and let

$$G(x, u) = F(x, u) - b|u|^p.$$

For the remainder of the proof, we will omit the dependence with respect to $x \in \overline{\Omega}$.

In order to prove that $\lambda_\infty = 0$ is a bifurcation from infinity for

$$u - \lambda K F(u) = 0, \tag{5}$$

we use the rescaling $w = \gamma u$, $\lambda = \gamma^{p-1}$, $\gamma > 0$. A direct calculation shows that (λ, u), $\lambda > 0$, is a solution of (5) if and only if

$$w - K\widetilde{F}(\gamma, w) = 0, \tag{6}$$

where

$$\widetilde{F}(\gamma, w) := b|w|^p + \gamma^p G(\gamma^{-1} w). \tag{7}$$

We can extend $\widetilde{F}$ to $\gamma = 0$ by setting

$$\widetilde{F}(0, w) = b|w|^p$$

and, by (f_3), such an extension is continuous. We set

$$S(\gamma, w) = w - K\widetilde{F}(\gamma, w), \quad \gamma \in \mathbb{R}^+.$$

Let us point out explicitly that $S(\gamma, \cdot) = I - \mathcal{K}$, with $\mathcal{K}$ compact. For $\gamma = 0$, solutions of $S_0(w) := S(0, w) = 0$ are nothing but solutions of

$$\begin{cases} -\Delta w = b|w|^p, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{8}$$

Since $1 < p < 2^* - 1$, we can use the a priori estimates of [9] and the degree theoretic arguments of [8] to infer the existence of $R > r > 0$ such that

$$S_0(w) \neq 0, \quad \forall \|w\| \in \{r, R\} \tag{9}$$

and

$$\deg\left(S_0, B_R - \overline{B}_r, 0\right) = -1. \tag{10}$$

Next we show

Lemma 6. *There exists $\gamma_0 > 0$ such that*

(i) $\deg\big(S(\gamma, \cdot), B_R - \overline{B}_r, 0\big) = -1, \quad \forall\, 0 \le \gamma \le \gamma_0$;

(ii) *if $S(\gamma, w) = 0$, $\gamma \in [0, \gamma_0]$, $r \le \|w\| \le R$, then $w > 0$ in Ω.*

Proof. Clearly (i) follows if we show that $S(\gamma, w) \neq 0$ for all $\|w\| \in \{r, R\}$ and all $0 \le \gamma \le \gamma_0$. Otherwise, there exists a sequence (γ_n, w_n) with $\gamma_n \to 0$, $\|w_n\| \in \{r, R\}$ and $w_n = K\tilde{F}(\gamma_n, w_n)$. Since K is compact then, up to a subsequence, $w_n \to w$ and $S_0(w) = 0$, $\|w\| \in \{r, R\}$, a contradiction with (9).

To prove (ii), we argue again by contradiction. As in the preceding argument, we find a sequence $w_n \in X$, with $\{x \in \Omega : w_n(x) \le 0\} \neq \emptyset$, such that $w_n \to w$, $\|w\| \in [r, R]$ and $S_0(w) = 0$; namely, w solves (8). By the maximum principle $w > 0$ on Ω and $\partial w/\partial n < 0$ on $\partial\Omega$. Moreover, elliptic theory implies that, without relabeling, $w_n \to w$ in $C^{1,\nu}$. Therefore $w_n > 0$ on Ω for n large, a contradiction.

Proof of Theorem 5 completed. By Lemma 6, problem (6) has a positive solution w_γ for all $0 \le \gamma \le \gamma_0$. As remarked before, for $\gamma > 0$, the rescaling $\lambda = \gamma^{p-1}$, $u = w/\gamma$ gives a solution (λ, u_λ) of (5) for all $0 < \lambda < \lambda_* := \gamma_0^{p-1}$. Since $w_\gamma > 0$, (λ, u_λ) is a positive solution of (1). Finally $\|w_\gamma\| \ge r$ for all $\gamma \in [0, \gamma_0]$ implies that $\|u_\lambda\| = \|w_\gamma\|/\gamma \to \infty$ as $\gamma \to 0$. This completes the proof.

Remarks.

(1) The result of [5] shows that, in general, (1) has no positive solutions for λ large.

(2) As in Remarks 3 and 4 concerning the asymptotically linear case, also here a branch Σ_0 emanating from $(0, 0) \in \mathbb{R} \times X$ exists and consists of negative solutions. This branch is disconnected from the one emanating from infinity, in contrast with the case $f(x, 0) > 0$; see, for example, the survey [1].

5. Sublinear problems. In this final section we deal with sublinear f, namely $f \in C(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R})$ that satisfy (f_1) and

(f_4) $\exists\, b \in C(\overline{\Omega})$, $b > 0$, such that $\lim_{u\to\infty} u^{-q} f(x, u) = b$, uniformly in $x \in \overline{\Omega}$, with $0 \le q < 1$.

We will show that in this case positive solutions of (1) branch off from ∞ for $\lambda_\infty = +\infty$. First, some preliminaries are in order. It is convenient to work on $Y = C^{1,\nu}(\overline{\Omega})$. Following the same procedure as for the superlinear case, we employ the rescaling $w = \gamma u$, $\lambda = \gamma^{q-1}$ and use the same notation, with q instead of p and Y instead of X. As before, (λ, u) solves (5) if and only if (γ, w) satisfies (6). Note that now, since $0 \le q < 1$, one has that

$$\lambda \to +\infty \quad \Leftrightarrow \quad \gamma \to 0. \tag{11}$$

For future reference, we recall that it is well known that the problem

$$\begin{cases} -\Delta u = bw^q, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases} \tag{12}$$

has a unique positive solution w_0. Moreover, letting $\lambda_1[bw_0^{q-1}]$ denote the first eigenvalue of the linearized problem

$$\begin{cases} -\Delta v = \lambda b w_0^{q-1} v, & x \in \Omega \\ v = 0, & x \in \partial\Omega, \end{cases} \tag{13}$$

(12) implies that $v = w_0$ is an eigenfunction corresponding to

$$\lambda_1[bw_0^{q-1}] = 1. \tag{14}$$

Concerning (13), it is worth pointing out that, although $q < 1$, the spectral theory can be carried over; see, for example, Remark 3.1 of [2].

We set $D_\delta = \{w \in Y : \|w - w_0\|_{1,\nu} \leq \delta\}$ and extend $\widetilde{F}$ to $\gamma = 0$ by

$$\widetilde{F}_0(w) = \widetilde{F}(0, w) := b|w|^q.$$

Lemma 7. *There exists $\delta > 0$ such that $K\widetilde{F} : [0, \infty) \times D_\delta \to Y$ is (compact and) continuous.*

Proof. When $0 < q < 1$ the same arguments used for $p > 1$ show that $\widetilde{F}$ is continuous. Let $q = 0$ and let $\delta > 0$ be such that $w > 0$ for all $w \in D_\delta$. Plainly, it suffices to show that $K\widetilde{F}(\gamma_n, w_n) \to K\widetilde{F}_0(w)$ whenever $\gamma_n \to 0$ and $w_n \to w$ in Y. Since $w > 0$ then $\gamma_n^{-1} w_n \to +\infty$, pointwise in Ω. By (f_4) it follows that

$$G(\gamma_n^{-1} w_n) \to 0 \quad \text{in } L^r, \ \forall\, r \geq 1.$$

Then the elliptic theory yields

$$K\widetilde{F}(\gamma_n, w_n) = K\widetilde{F}_0(w_n) + KG(\gamma_n^{-1} w_n) \to K\widetilde{F}_0(w),$$

in the Sobolev space $H^{2,r}$, $\forall\, r \geq 1$, and the result follows in a standard way.

Theorem 8. *Let $f \in C(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R})$ satisfy (f_1) and (f_4). Then $\exists\, \lambda^* > 0$ such that (1) has positive solutions for all $\lambda \geq \lambda^*$. More precisely, there exists a connected set of positive solutions of (1) bifurcating from infinity for $\lambda_\infty = +\infty$.*

Proof. By Lemma 7, degree theoretic arguments apply to $S(\gamma, w) = w - K\widetilde{F}(\gamma, w)$. Moreover, note that $S_0(w) = S(0, w) = w - K\widetilde{F}_0(w)$ is C^1 on D_δ and its Fréchet derivative $S_0'(w_0)$ is given by

$$S_0'(w_0)v = \begin{cases} v - K[qbw_0^{q-1}v], & \text{if} \quad 0 < q < 1 \\ v, \quad \text{if} \quad q = 0. \end{cases}$$

In particular, for $0 < q < 1$, (14) implies that all the characteristic values of $I - S_0'(w_0)$ are greater than 1. Therefore, we infer that

$$\deg(S_0, D_\delta, 0) = 1, \quad \forall\, q \in [0, 1).$$

By continuation, we deduce that there exists a connected subset Γ of solutions of $S(\gamma, w) = 0$ ($\gamma > 0$), such that $(0, w_0) \in \overline{\Gamma}$. Moreover, $\exists\ \gamma_0 > 0$ such that these solutions are positive provided $0 < \gamma \leq \gamma_0$. By the rescaling $\lambda = \gamma^{q-1}$, $u = w/\gamma$, Γ is transformed into a connected subset Σ_∞ of solutions of (1). These solutions are indeed positive for all $\lambda > \lambda^* := \gamma_0^{q-1}$ and, according to (11), Σ_∞ bifurcates from infinity for $\lambda_\infty = +\infty$.

Remark. We suspect that, if Σ_∞ crosses $\lambda = \lambda_1$, then (1) has positive solutions for $\lambda \in (\lambda_1, \lambda_1 + \epsilon]$. For a result in this direction, see [6].

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