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Positive solutions of semipositone singular fractional differential systems with a parameter and integral boundary conditions

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Abstract: In this paper, the existence of positive solutions for systems of semipositone singular fractional differential equations with a parameter and integral boundary conditions is investigated. By using fixed point theorem in cone, sufficient conditions which guarantee the existence of positive solutions are obtained. An example is given to illustrate the results.

Keywords: Positive solutions, Semipositone, Singular fractional differential systems, Integral boundary conditions

MSC: 26A33, 34A08, 34B18

1 Introduction

The subject of fractional calculus has gained considerable popularity and importance during the past decades, mainly due to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. In recent years, fractional differential equations have been widely used in optics and thermal systems, electromagnetics, control engineering and robotic, and many other fields, see [1-6] and the references therein. The research on fractional differential equations is very important in both theory and applications. By using nonlinear analysis tools, some scholars established the existence, uniqueness, multiplicity and qualitative properties of solutions, we refer the readers to [7-20] and the references therein for fractional differential equations, and [21-33] for fractional differential systems.

Boundary value problems (BVPs for short) with integral boundary conditions for ordinary differential equations represent a very interesting and important class of problems, and arise in the study of various biological, physical and chemical processes [34-37], such as heat conduction, thermo-elasticity, chemical engineering, underground water flow, and plasma physics. The existence of solutions or positive solutions for such class of problems has attracted much attention (see [38-52] and the references therein).

In this paper, we study the systems of semipositone singualr fractional differential equations with a parameter and integral boundary conditions

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$$
\begin{aligned}
&D_{0^+}^{\alpha-\beta}(D_{0^+}^{\beta}u(t)) + \lambda f_1(t, u(t), D_{0^+}^{\beta}u(t), v(t)) = 0, \quad 0 < t < 1,\\
&D_{0^+}^{\gamma-\delta}(D_{0^+}^{\delta}v(t)) + \lambda f_2(t, u(t), v(t), D_{0^+}^{\delta}v(t)) = 0, \quad 0 < t < 1,\\
&D_{0^+}^{\beta}u(0) = D_{0^+}^{\beta+1}u(0) = 0, \quad D_{0^+}^{\beta+1}u(1) = \int_0^1 D_{0^+}^{\beta}u(s)dA(s),\\
&D_{0^+}^{\delta}v(0) = D_{0^+}^{\delta+1}v(0) = 0, \quad D_{0^+}^{\delta+1}v(1) = \int_0^1 D_{0^+}^{\delta}v(s)dB(s),
\end{aligned}
\tag{1}
$$

where $D_{0^+}^{\alpha-\beta}$, $D_{0^+}^{\beta}$, $D_{0^+}^{\gamma-\delta}$ and $D_{0^+}^{\delta}$ are the standard Riemann-Liouville fractional derivatives, $\lambda > 0$ is a parameter, $2 < \alpha, \gamma \le 3$, $0 < \beta, \delta < 1$, $\alpha - \beta > 2$, $\gamma - \delta > 2$. f_1, $f_2 : (0,1) \times [0,+\infty)^3 \to (-\infty,+\infty)$ are continuous and may be singular at $t = 0, 1$. A and B are nondecreasing functions of bunded variations, $\int_0^1 D_{0^+}^{\beta}u(s)dA(s)$ and $\int_0^1 D_{0^+}^{\delta}v(s)dB(s)$ are Riemann-Stieltjes integrals.

The study of nonlinear fractional differential systems is important as this kind of systems occur in various problems of applied mathematics. Recently, Wu et al. [26] considered the fractional differential systems involving nonlocal boundary conditions

$$
\begin{aligned}
&D_{0^+}^{\alpha}u(t) + f(t, u(t), v(t)) = 0, \quad 0 < t < 1,\ 1 < \alpha \le 2,\\
&D_{0^+}^{\beta}v(t) + g(t, u(t), v(t)) = 0, \quad 0 < t < 1,\ 1 < \beta \le 2,\\
&u(0) = 0, \quad u(1) = \int_0^1 u(s)dA(s),\\
&v(0) = 0, \quad v(1) = \int_0^1 v(s)dB(s),
\end{aligned}
\tag{2}
$$

where $D_{0^+}^{\alpha}$ and $D_{0^+}^{\beta}$ are the standard Riemann-Liouville fractional derivatives, A, B are nondecreasing functions of bunded variations, $\int_0^1 u(s)dA(s)$ and $\int_0^1 v(s)dB(s)$ are Riemann-Stieltjes integrals, $f(t,x,y)$, $g(t,x,y) : (0,1) \times (0,\infty)^2 \to [0,\infty)$ are two continuous functions and may be singular at $t = 0, 1$ and $x = y = 0$. The existence of positive solutions is established by the upper and lower solutions technique and Schauder fixed point theorem. For the special boundary conditions $u(1) = \int_0^1 \phi(s)u(s)ds$, $v(1) = \int_0^1 \varphi(s)v(s)ds$, where $\phi, \varphi \in L(0,1)$ are nonnegative, Liu et al. [27] investigated the existence of a pair of positive solutions for nonlocal fractional differential systems (2) by constructing two cones and computing the fixed point index in product cone. For the case $f = a(t)\widetilde{f}(t, u(t))$, $g = b(t)\widetilde{g}(t, v(t))$, $u(1) = \int_0^1 \phi(s)u(s)ds$, $v(1) = \int_0^1 \varphi(s)v(s)ds$, Yang [28] established sufficient conditions for the existence and nonexistence of positive solutions to fractional differential systems (2) by the Banach fixed point theorem, nonlinear differentiation of Leray-Schauder type and the fixed point theorems of cone expansion and compression of norm type.

In [29], Henderson, Luca and Tudorache discussed the systems of nonlinear fractional differential equations with integral boundary conditions

$$
\begin{aligned}
&D_{0^+}^{\alpha}u(t) + \lambda f(t, u(t), v(t)) = 0, \quad 0 < t < 1,\ n-1 < \alpha \le n,\\
&D_{0^+}^{\beta}v(t) + \mu g(t, u(t), v(t)) = 0, \quad 0 < t < 1,\ m-1 < \beta \le m,\\
&u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = \int_0^1 u(s)dA(s),\\
&v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \quad v(1) = \int_0^1 v(s)dB(s)
\end{aligned}
\tag{3}
$$

where $D_{0^+}^{\alpha}$ and $D_{0^+}^{\beta}$ are the standard Riemann-Liouville fractional derivatives, $f, g : [0,1] \times [0,\infty)^2 \to [0,\infty)$ are continuous. Under different combinations of superlinearity and sublinearity of the functions f and g, various existence and nonexistence results for positive solutions are derived in terms of different value of

λ and μ via the Guo-Krasnosel'skii fixed point theorem. For the multi-point boundary conditions $u(1) = \sum_{i=1}^{p} a_i u(\xi_i)$, $v(1) = \sum_{i=1}^{q} b_i v(\eta_i)$, Henderson and Luca [30] proved the existence theorems for the positive solutions with respect to a cone for nonlinear fractional differential systems (3) by the Guo-Krasnosel'skii fixed point theorem. Under sufficient conditions on functions f and g, the authors [31] studied the existence and multiplicity of positive solutions of nonlinear fractional differential systems (3) when $\lambda = \mu = 1$ and $f = \widetilde{f}(t, v)$, $g = \widetilde{g}(t, u)$ by using the Guo-Krasnosel'skii fixed point theorem and some theorems from the fixed point index theory.

In [32], Wang et al. investigated the fractional differential systems involving integral boundary conditions arising from the study of HIV infection models

$$\begin{aligned}
&D_{0^+}^{\alpha} u(t) + \lambda f(t, u(t), D_{0^+}^{\beta} u(t), v(t)) = 0, \quad 0 < t < 1,\ 2 < \alpha \le 3,\\
&D_{0^+}^{\gamma} v(t) + \lambda g(t, u(t)) = 0, \quad 0 < t < 1,\ 2 < \gamma \le 3,\\
&D_{0^+}^{\beta} u(0) = D_{0^+}^{\beta+1} u(0) = 0, \quad D_{0^+}^{\beta} u(1) = \int_0^1 D_{0^+}^{\beta} u(s)\, dA(s),\\
&v(0) = v'(0) = 0, \quad v(1) = \int_0^1 v(s)\, dB(s),
\end{aligned} \tag{4}$$

where $\lambda > 0$ is a parameter, $0 < \beta < 1$, $\alpha - \beta > 2$, $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$ and $D_{0^+}^{\gamma}$ are the standard Riemann-Liouville fractional derivatives, A, B are nondecreasing functions of bunded variations, $\int_0^1 D_{0^+}^{\beta} u(s)\, dA(s)$ and $\int_0^1 v(s)\, dB(s)$ are Riemann-Stieltjes integrals, $f : (0, 1) \times [0, +\infty)^3 \to (-\infty, +\infty)$ and $g : (0, 1) \times [0, +\infty) \to (-\infty, +\infty)$ are two continuous functions and may be singular at $t = 0, 1$. By using the fixed point theorem in cone, existence results of positive solutions for systems (4) are established.

In [33], Jiang, Liu and Wu considered the following semipositone singular fractional differential systems:

$$\begin{aligned}
&D_{0^+}^{\alpha} u(t) + p(t) f(t, u(t), v(t)) - q_1(t) = 0, \quad 0 < t < 1,\ 2 < \alpha \le 3,\\
&D_{0^+}^{\beta} v(t) + q(t) g(t, u(t), v(t)) - q_2(t) = 0, \quad 0 < t < 1,\ 2 < \beta \le 3,\\
&u(0) = u'(0) = 0, \quad u'(1) = \int_0^1 u(s)\, dA(s),\\
&v(0) = v'(0) = 0, \quad v'(1) = \int_0^1 v(s)\, dB(s),
\end{aligned} \tag{5}$$

where $D_{0^+}^{\alpha}$ and $D_{0^+}^{\beta}$ are the standard Riemann-Liouville fractional derivatives, $f, g : [0, 1] \times [0, \infty)^2 \to [0, \infty)$ are continuous, $q_1, q_2 : (0, 1) \to [0, +\infty)$ are Lebesgue integrable, A, B are suitable functions of bounded variation, $\int_0^1 u(t)\, dA(t)$ and $\int_0^1 v(t)\, dB(t)$ involving Stieltjes integrals with signed measures. The existence and multiplicity of positive solutions to systems (5) are obtained by using a well known fixed point theorem.

It should be noted that the nonlinearity in most of the previous works needs to be nonnegative to get the positive solutions [22-32]. When the nonlinearity is allowed to take on both positive and negative values, such problems are called semipositone problems in the literature. Motivated by the works mentioned above, we consider the semipositone singular fractional differential systems (1). The existence of positive solutions is established by applying the fixed point theorem in cone. In comparison with previous works, this paper has several new features. Firstly, nonlinearities are allowed to change sign and tend to negative infinity. Secondly, systems (1) involves a parameter and f_1, f_2 involve fractional derivatives of unknown functions. Finally, the nonlocal conditions are given by Riemann-Stieltjes integrals, which include two-point, three-point, multi-point and some nonlocal conditions as special cases.

2 Preliminaries and lemmas

For convenience of the reader, we present here some necessary definitions and properties about fractional calculus theory.

Definition 2.1. *The Riemann-Liouville fractional integral of order α of a function $u : (0, +\infty) \to (-\infty, +\infty)$ is given by*

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

provided the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2 ([1,2])**.** *The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $u : (0, +\infty) \to (-\infty, +\infty)$ is given by*

$$D_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s) ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided the right-hand side is pointwise defined on $(0, +\infty)$.

Lemma 2.3 ([1,2])**.** *If $u \in L(0, 1)$, $\rho > \sigma > 0$ and n is a natural number, then*

$$I_{0^+}^{\rho} I_{0^+}^{\sigma} u(t) = I_{0^+}^{\rho+\sigma} u(t), \quad D_{0^+}^{\sigma} I_{0^+}^{\rho} u(t) = I_{0^+}^{\rho-\sigma} u(t),$$

$$D_{0^+}^{\sigma} I_{0^+}^{\sigma} u(t) = u(t), \quad \left(\frac{d}{dt}\right)^n (D_{0^+}^{\sigma} u(t)) = D_{0^+}^{n+\sigma} u(t).$$

Lemma 2.4 ([1,2])**.** *Assume that $u \in C(0, 1) \cap L(0, 1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0, 1) \cap L(0, 1)$. Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N},$$

for some $c_1, c_2, \cdots, c_N \in (-\infty, +\infty)$, where N is the smallest integer greater than or equal to α.

Lemma 2.5 ([33])**.** *Given $h \in C(0, 1) \cap L(0, 1)$, then the BVP*

$$\begin{aligned} &D_{0^+}^{\alpha-\beta} x(t) + h(t) = 0, \quad 0 < t < 1,\ 2 < \alpha - \beta \le 3, \\ &x(0) = x'(0) = 0, \quad x'(1) = 0, \end{aligned}$$

has a unique solution

$$x(t) = \int_0^1 G_1^*(t, s) h(s) ds,$$

where

$$G_1^*(t, s) = \frac{1}{\Gamma(\alpha-\beta)} \begin{cases} t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-2}, & 0 \le t \le s \le 1, \\ t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-2} - (t-s)^{\alpha-\beta-1}, & 0 \le s \le t \le 1, \end{cases}$$

and the Green function $G_1^*(t, s)$ has the following properties:

(1) $G_1^*(t, s) > 0$, $t, s \in (0, 1)$.

(2) $k_1(t) G_1^*(1, s) \le G_1^*(t, s) \le G_1^*(1, s)$, $t, s \in [0, 1]$.

(3) $G_1^*(t, s) \le \frac{k_1(t)}{\Gamma(\alpha-\beta)}$, $t, s \in [0, 1]$, *where* $k_1(t) = t^{\alpha-\beta-1}$.

By Lemma 2.4, the unique solution of the BVP

$$D_{0^+}^{\alpha-\beta}x(t) = 0, \quad 0 < t < 1,$$
$$x(0) = x'(0) = 0, \quad x'(1) = 1,$$

is $\gamma_1(t) = \frac{t^{\alpha-\beta-1}}{\alpha-\beta-1}$. By [33], the Green function for the BVP

$$D_{0^+}^{\alpha-\beta}x(t) + h(t) = 0, \quad 0 < t < 1,$$
$$x(0) = x'(0) = 0, \quad x'(1) = \int_0^1 x(s)dA(s),$$

is given by

$$G_1(t,s) = G_1^*(t,s) + \frac{\gamma_1(t)}{1-\jmath_1}\ell_1(s), \tag{6}$$

where

$$\jmath_1 = \int_0^1 \gamma_1(t)dA(t) \neq 1, \quad \ell_1(s) = \int_0^1 G_1^*(t,s)dA(t), \quad s \in [0,1].$$

Similarly, the Green function for the BVP

$$D_{0^+}^{\gamma-\delta}y(t) + h(t) = 0, \quad 0 < t < 1,$$
$$y(0) = y'(0) = 0, \quad y'(1) = \int_0^1 y(s)dB(s),$$

is given by

$$G_2(t,s) = G_2^*(t,s) + \frac{\gamma_2(t)}{1-\jmath_2}\ell_2(s), \tag{7}$$

where

$$G_2^*(t,s) = \frac{1}{\Gamma(\gamma-\delta)}\begin{cases} t^{\gamma-\delta-1}(1-s)^{\gamma-\delta-2}, & 0 \le t \le s \le 1, \\ t^{\gamma-\delta-1}(1-s)^{\gamma-\delta-2} - (t-s)^{\gamma-\delta-1}, & 0 \le s \le t \le 1, \end{cases}$$

$$\jmath_2 = \int_0^1 \gamma_2(t)dB(t) \neq 1, \quad \ell_2(s) = \int_0^1 G_2^*(t,s)dB(t), \quad \gamma_2(t) = \frac{t^{\gamma-\delta-1}}{\gamma-\delta-1}.$$

Lemma 2.6 ([33]). *The Green function $G_2^*(t,s)$ has the following properties:*
(1) $G_2^*(t,s) > 0,\ t,s \in (0,1)$.
(2) $k_2(t)G_2^*(1,s) \le G_2^*(t,s) \le G_2^*(1,s),\ t,s \in [0,1]$.
(3) $G_2^*(t,s) \le \frac{k_2(t)}{\Gamma(\gamma-\delta)},\ t,s \in [0,1]$, *where* $k_2(t) = t^{\gamma-\delta-1}$.

Lemma 2.7 ([33]). *Let $\jmath_1,\ \jmath_2 \in [0,1)$ and $\ell_1(s),\ \ell_2(s) \ge 0$ for $s \in [0,1]$, the functions $G_1(t,s)$ and $G_2(t,s)$ given by* (6) *and* (7) *satisfy:*
(1) $G_i(t,s) \ge G_i^*(t,s) > 0,\ t,s \in (0,1),\ i = 1,2$.
(2) $k_i(t)G_i^*(1,s) \le G_i(t,s) \le \rho_i G_i^*(1,s),\ t,s \in [0,1],\ i = 1,2$.
(3) $G_1(t,s) \le \frac{\rho_1}{\Gamma(\alpha-\beta)}k_1(t),\ G_2(t,s) \le \frac{\rho_2}{\Gamma(\gamma-\delta)}k_2(t),\ t,s \in [0,1]$, *where*

$$\rho_1 = 1 + \frac{\int_0^1 dA(t)}{1-\jmath_1}, \quad \rho_2 = 1 + \frac{\int_0^1 dB(t)}{1-\jmath_2}.$$

Now let us consider the following modified problem of systems (1)

$$\begin{aligned}
&D_{0^+}^{\alpha-\beta}\omega(t)+\lambda f_1(t,I_{0^+}^{\beta}\omega(t),\omega(t),I_{0^+}^{\delta}z(t))=0,\quad 0<t<1,\\
&D_{0^+}^{\gamma-\delta}z(t)+\lambda f_2(t,I_{0^+}^{\beta}\omega(t),I_{0^+}^{\delta}z(t),z(t))=0,\quad 0<t<1,\\
&\omega(0)=\omega'(0)=0,\quad \omega'(1)=\int_0^1\omega(s)dA(s),\\
&z(0)=z'(0)=0,\quad z'(1)=\int_0^1 z(s)dB(s).
\end{aligned}\tag{8}$$

Lemma 2.8. *If* $(\omega,z)\in C[0,1]\times C[0,1]$ *is a positive solution of systems* (8)*, then* $(I_{0^+}^{\beta}\omega,\ I_{0^+}^{\delta}z)$ *is a positive solution of systems* (1).

Proof. Suppose $(\omega,z)\in C[0,1]\times C[0,1]$ is a positive solution of systems (8), denote $u(t)=I_{0^+}^{\beta}\omega(t),\ v(t)=I_{0^+}^{\delta}z(t)$, then

$$\begin{aligned}
&D_{0^+}^{\alpha-\beta}(D_{0^+}^{\beta}u(t))=D_{0^+}^{\alpha-\beta}(D_{0^+}^{\beta}I_{0^+}^{\beta}\omega(t))=D_{0^+}^{\alpha-\beta}\omega(t)\\
&=-\lambda f_1(t,I_{0^+}^{\beta}\omega(t),\omega(t),I_{0^+}^{\delta}z(t))\\
&=-\lambda f_1(t,u(t),D_{0^+}^{\beta}u(t),v(t)),\\
&D_{0^+}^{\gamma-\delta}(D_{0^+}^{\delta}v(t))=D_{0^+}^{\gamma-\delta}(D_{0^+}^{\delta}I_{0^+}^{\delta}z(t))=D_{0^+}^{\gamma-\delta}z(t)\\
&=-\lambda f_2(t,I_{0^+}^{\beta}\omega(t),I_{0^+}^{\delta}z(t),z(t))\\
&=-\lambda f_2(t,u(t),v(t),D_{0^+}^{\delta}v(t)),
\end{aligned}$$

$$D_{0^+}^{\beta}u(0)=\omega(0)=0,\quad D_{0^+}^{\beta+1}u(0)=\frac{d}{dt}(D_{0^+}^{\beta}u(t))|_{t=0}=\omega'(0)=0,$$

$$D_{0^+}^{\beta+1}u(1)=\frac{d}{dt}(D_{0^+}^{\beta}u(t))|_{t=1}=\omega'(1)=\int_0^1\omega(s)dA(s)=\int_0^1 D_{0^+}^{\beta}u(s)dA(s),$$

$$D_{0^+}^{\delta}v(0)=z(0)=0,\quad D_{0^+}^{\delta+1}v(0)=\frac{d}{dt}(D_{0^+}^{\delta}v(t))|_{t=0}=z'(0)=0,$$

$$D_{0^+}^{\delta+1}v(1)=\frac{d}{dt}(D_{0^+}^{\delta}v(t))|_{t=1}=z'(1)=\int_0^1 z(s)dB(s)=\int_0^1 D_{0^+}^{\delta}v(s)dB(s).$$

On the other hand, if $\omega(t)>0,\ z(t)>0$, by Definition 2.1, we have $u(t)>0,\ v(t)>0,\ t\in(0,1)$, then $(u,v)=(I_{0^+}^{\beta}\omega,I_{0^+}^{\delta}z)$ is a positive solution of systems (1). □

We impose the following assumptions:

(H_1) A,B are increasing functions of bounded variations such that $\ell_1(s)\ge 0,\ \ell_2(s)\ge 0$ for $s\in[0,1]$ and $0\le\jmath_1,\ \jmath_2<1$.

(H_2) $f_1,f_2:(0,1)\times[0,+\infty)^3\to(-\infty,+\infty)$ are continuous and satisfy

$$-q_1(t)\le f_1(t,u_1,u_2,u_3)\le p_1(t)g_1(t,u_1,u_2,u_3),$$

$$-q_2(t)\le f_2(t,v_1,v_2,v_3)\le p_2(t)g_2(t,v_1,v_2,v_3),$$

where $g_i\in C([0,1]\times[0,+\infty)^3,[0,+\infty)),\ q_i,p_i\in C((0,1),[0,+\infty))$ and

$$0<\int_0^1 p_i(s)ds<+\infty,\quad 0<\int_0^1 q_i(s)ds<+\infty,\quad i=1,2.$$

(H_3) There exists a constant

$$r > \max\left\{\frac{2\rho_1^2}{\Gamma(\alpha-\beta)}\int_0^1 q_1(s)ds,\ \frac{2\rho_2^2}{\Gamma(\gamma-\delta)}\int_0^1 q_2(s)ds\right\}$$

such that

$$f_1(t,u_1,u_2,u_3)\geq 0,\quad (t,u_1,u_2,u_3)\in(0,1)\times\left[0,\frac{r}{2\Gamma(\beta+1)}\right]\times\left[0,\frac{r}{2}\right]\times[0,+\infty),$$

$$f_2(t,v_1,v_2,v_3)\geq 0,\quad (t,v_1,v_2,v_3)\in(0,1)\times[0,+\infty)\times\left[0,\frac{r}{2\Gamma(\delta+1)}\right]\times\left[0,\frac{r}{2}\right].$$

Define a modified function $[z(t)]^*$ for any $z\in C[0,1]$ by

$$[z(t)]^* = \begin{cases} z(t), & z(t)\geq 0,\\ 0, & z(t)<0.\end{cases}$$

Next we consider the following systems:

$$\begin{aligned}
&D_{0^+}^{\alpha-\beta}x(t)+\lambda[f_1(t,I_{0^+}^{\beta}[x(t)-a(t)]^*,[x(t)-a(t)]^*,I_{0^+}^{\delta}[y(t)-b(t)]^*)+q_1(t)]=0,\ 0<t<1,\\
&D_{0^+}^{\gamma-\delta}y(t)+\lambda[f_2(t,I_{0^+}^{\beta}[x(t)-a(t)]^*,I_{0^+}^{\delta}[y(t)-b(t)]^*,[y(t)-b(t)]^*)+q_2(t)]=0,\ 0<t<1,\\
&x(0)=x'(0)=0,\quad x'(1)=\int_0^1 x(s)dA(s),\\
&y(0)=y'(0)=0,\quad y'(1)=\int_0^1 y(s)dB(s),
\end{aligned}\tag{9}$$

where $a(t)=\lambda\int_0^1 G_1(t,s)q_1(s)ds$ and $b(t)=\lambda\int_0^1 G_2(t,s)q_2(s)ds$ are the solutions of the following BVPs (10) and (11), respectively,

$$\begin{aligned}
&D_{0^+}^{\alpha-\beta}a(t)+\lambda q_1(t)=0,\quad 0<t<1,\\
&a(0)=a'(0)=0,\quad a'(1)=\int_0^1 a(s)dA(s),
\end{aligned}\tag{10}$$

$$\begin{aligned}
&D_{0^+}^{\gamma-\delta}b(t)+\lambda q_2(t)=0,\quad 0<t<1,\\
&b(0)=b'(0)=0,\quad b'(1)=\int_0^1 b(s)dB(s).
\end{aligned}\tag{11}$$

Lemma 2.9 ([33]). *Assume that condition (H_1) holds. Then the positive solutions $a(t)$ and $b(t)$ of BVPs* (10) *and* (11) *satisfy*

$$a(t)\leq\frac{\lambda\rho_1}{\Gamma(\alpha-\beta)}k_1(t)\int_0^1 q_1(t)dt,\quad b(t)\leq\frac{\lambda\rho_2}{\Gamma(\gamma-\delta)}k_2(t)\int_0^1 q_2(t)dt,\quad t\in[0,1].$$

Lemma 2.10. *If $(x,y)\in C[0,1]\times C[0,1]$ with $x(t)>a(t)$, $y(t)>b(t)$ for any $t\in(0,1)$ is a positive solution of systems* (9)*, then $(\omega(t)=x(t)-a(t),\ z(t)=y(t)-b(t))$ is a positive solution of systems* (8)*, and $(u(t)=I_{0^+}^{\beta}\omega(t),\ v(t)=I_{0^+}^{\delta}z(t))$ is a positive solution of systems* (1).

Proof. In fact, if $(x, y) \in C[0,1] \times C[0,1]$ is a solution of systems (9) with $x(t) > a(t),\ y(t) > b(t)$, then from systems (9) and the definition of $[\,]^*$, we get

$$\begin{aligned}
&D_{0^+}^{\alpha-\beta}\omega(t) = D_{0^+}^{\alpha-\beta}(x(t)-a(t)) = D_{0^+}^{\alpha-\beta}x(t) - D_{0^+}^{\alpha-\beta}a(t)\\
&= -\lambda[f_1(t, I_{0^+}^{\beta}[x(t)-a(t)]^*, [x(t)-a(t)]^*, I_{0^+}^{\delta}[y(t)-b(t)]^*) + q_1(t)] - [-\lambda q_1(t)]\\
&= -\lambda f_1(t, I_{0^+}^{\beta}[x(t)-a(t)]^*, [x(t)-a(t)]^*, I_{0^+}^{\delta}[y(t)-b(t)]^*)\\
&= -\lambda f_1(t, I_{0^+}^{\beta}\omega(t), \omega(t), I_{0^+}^{\delta}z(t)),
\end{aligned}$$

$$\omega(0) = x(0) - a(0) = 0, \quad \omega'(0) = x'(0) - a'(0) = 0,$$

$$\omega'(1) = x'(1) - a'(1) = \int_0^1 x(s)dA(s) - \int_0^1 a(s)dA(s) = \int_0^1 \omega(s)dA(s),$$

and

$$\begin{aligned}
&D_{0^+}^{\gamma-\delta}z(t) = D_{0^+}^{\gamma-\delta}(y(t)-b(t)) = D_{0^+}^{\gamma-\delta}y(t) - D_{0^+}^{\gamma-\delta}b(t)\\
&= -\lambda[f_2(t, I_{0^+}^{\beta}[x(t)-a(t)]^*, I_{0^+}^{\delta}[y(t)-b(t)]^*, [y(t)-b(t)]^*) + q_2(t)] - [-\lambda q_2(t)]\\
&= -\lambda f_2(t, I_{0^+}^{\beta}[x(t)-a(t)]^*, I_{0^+}^{\delta}[y(t)-b(t)]^*, [y(t)-b(t)]^*)\\
&= -\lambda f_2(t, I_{0^+}^{\beta}\omega(t), I_{0^+}^{\delta}z(t), z(t)),
\end{aligned}$$

$$z(0) = y(0) - b(0) = 0, \quad z'(0) = y'(0) - b'(0) = 0,$$

$$z'(1) = y'(1) - b'(1) = \int_0^1 y(s)dB(s) - \int_0^1 b(s)dB(s) = \int_0^1 z(s)dB(s).$$

So (ω, z) is a positive solution of systems (8). It follows from Lemma 2.8 that $(u(t) = I_{0^+}^{\beta}\omega(t),\ v(t) = I_{0^+}^{\beta}z(t))$ is a positive solution of systems (1). □

Let $X = C[0,1] \times C[0,1]$, then X is a Banach space with the norm

$$\|(u,v)\|_1 = \|u\| + \|v\|, \quad \|u\| = \max_{0\le t\le 1}|u(t)|, \quad \|v\| = \max_{0\le t\le 1}|v(t)|, \quad (u,v) \in X.$$

Let

$$P = \left\{(u,v) \in X : u(t) \ge \rho_1^{-1}k_1(t)\|u\|,\ v(t) \ge \rho_2^{-1}k_2(t)\|v\|,\ t \in [0,1]\right\},$$

then P is a cone of X. Define an operator $A : P \to X$ by

$$A(x,y) = (A_1(x,y), A_2(x,y)),$$

where $A_1, A_2 : P \to C[0,1]$ are defined by

$$\begin{cases}
A_1(x,y)(t) = \lambda \displaystyle\int_0^1 G_1(t,s)(f_1(s, I_{0^+}^{\beta}[x(s)-a(s)]^*, [x(s)-a(s)]^*, I_{0^+}^{\delta}[y(s)-b(s)]^*) + q_1(s))ds,\\
A_2(x,y)(t) = \lambda \displaystyle\int_0^1 G_2(t,s)(f_2(s, I_{0^+}^{\beta}[x(s)-a(s)]^*, I_{0^+}^{\delta}[y(s)-b(s)]^*, [y(s)-b(s)]^*) + q_2(s))ds.
\end{cases}$$

Clearly, if $(x,y) \in P$ is a fixed point of A, then (x,y) is a solution of systems (9).

Lemma 2.11. *Assume that conditions $(H_1) - (H_3)$ hold, then $A : P \to P$ is a completely continuous operator.*

Proof. For any $(x,y) \in P$, there exists a constant $L > 0$ such that $\|(x,y)\|_1 \le L$, then

$$[x(s)-a(s)]^* \le x(s) \le \|x\| \le \|(x,y)\|_1 \le L, \quad s \in [0,1],$$

$$[y(s)-b(s)]^* \le y(s) \le \|y\| \le \|(x,y)\|_1 \le L, \quad s \in [0,1],$$

$$I_{0^+}^{\beta}[x(s)-a(s)]^* = \int_0^s \frac{(s-t)^{\beta-1}[x(t)-a(t)]^*}{\Gamma(\beta)}dt \le \frac{L}{\Gamma(\beta+1)},$$

$$I_{0^+}^{\delta}[y(s)-b(s)]^* = \int_0^s \frac{(s-t)^{\delta-1}[y(t)-b(t)]^*}{\Gamma(\delta)}dt \le \frac{L}{\Gamma(\delta+1)}.$$

It follows from Lemma 2.7 that

$$\begin{aligned}
&A_1(x,y)(t)\\
\le&\lambda\int_0^1 \rho_1 G_1^*(1,s)(p_1(s)g_1(s, I_{0^+}^{\beta}[x(s)-a(s)]^*, [x(s)-a(s)]^*, I_{0^+}^{\delta}[y(s)-b(s)]^*)+q_1(s))ds\\
\le&\lambda\rho_1(M+1)\int_0^1 G_1^*(1,s)(p_1(s)+q_1(s))ds<+\infty,
\end{aligned}$$

$$\begin{aligned}
&A_2(x,y)(t)\\
\le&\lambda\int_0^1 \rho_2 G_2^*(1,s)(p_2(s)g_2(s, I_{0^+}^{\beta}[x(s)-a(s)]^*, I_{0^+}^{\delta}[y(s)-b(s)]^*, [y(s)-b(s)]^*)+q_2(s))ds\\
\le&\lambda\rho_2(M+1)\int_0^1 G_2^*(1,s)(p_2(s)+q_2(s))ds<+\infty,
\end{aligned}$$

where

$$M=\max\left\{\max_{t\in[0,1],\ 0\le u_1\le\frac{L}{\Gamma(\beta+1)},\ 0\le u_2\le L,\ 0\le u_3\le\frac{L}{\Gamma(\delta+1)}} g_1(t,u_1,u_2,u_3),\right.$$
$$\left.\max_{t\in[0,1],\ 0\le v_1\le\frac{L}{\Gamma(\beta+1)},\ 0\le v_2\le\frac{L}{\Gamma(\delta+1)},\ 0\le v_3\le L} g_2(t,v_1,v_2,v_3)\right\}.$$

Thus, $A: P\to X$ is well defined.

Next, we prove $A(P)\subset P$. Denote

$$F_1(s)=f_1(s, I_{0^+}^{\beta}[x(s)-a(s)]^*, [x(s)-a(s)]^*, I_{0^+}^{\delta}[y(s)-b(s)]^*)+q_1(s),\quad s\in[0,1],$$

$$F_2(s)=f_2(s, I_{0^+}^{\beta}[x(s)-a(s)]^*, I_{0^+}^{\delta}[y(s)-b(s)]^*, [y(s)-b(s)]^*)+q_2(s),\quad s\in[0,1].$$

For any $(x,y)\in P$, we have

$$\|A_1(x,y)\|=\max_{0\le t\le 1}|A_1(x,y)(t)|\le\lambda\rho_1\int_0^1 G_1^*(1,s)F_1(s)ds.$$

So

$$A_1(x,y)(t)\ge\lambda\int_0^1 k_1(t)G_1^*(1,s)F_1(s)ds\ge\rho_1^{-1}k_1(t)\|A_1(x,y)\|,\quad t\in[0,1].$$

Similarly, $A_2(x,y)(t)\ge\rho_2^{-1}k_2(t)\|A_2(x,y)\|$, $t\in[0,1]$. Thus, $A(P)\subset P$.

According to the Ascoli-Arzela theorem and the Lebesgue dominated convergence theorem, we can easily get that $A: P\to P$ is completely continuous. □

Lemma 2.12 ([53]). *Let P be a cone in Banach space E, Ω_1 and Ω_2 are bounded open sets in E, $\theta\in\Omega_1$, $\overline{\Omega}_1\subset\Omega_2$, $A: P\cap(\overline{\Omega}_2\backslash\Omega_1)\to P$ is a completely continuous operator. If the following conditions are satisfied:*
$\|Ax\|\le\|x\|$, $\forall x\in P\cap\partial\Omega_1$, $\|Ax\|\ge\|x\|$, $\forall x\in P\cap\partial\Omega_2$, *or*
$\|Ax\|\ge\|x\|$, $\forall x\in P\cap\partial\Omega_1$, $\|Ax\|\le\|x\|$, $\forall x\in P\cap\partial\Omega_2$,
then A has at least one fixed point in $P\cap(\overline{\Omega}_2\backslash\Omega_1)$.

3 Main results

Theorem 3.1. *Assume that conditions* $(H_1)-(H_3)$ *are satisfied. Further assume that the following condition holds:*

(H_4) *There exists* $[a,b]\subset(0,1)$ *such that*

$$\lim_{u_2\to+\infty}\min_{\substack{t\in[a,b]\\ u_1,u_3\geq 0}}\frac{f_1(t,u_1,u_2,u_3)}{u_2}=+\infty,\qquad \lim_{v_3\to+\infty}\min_{\substack{t\in[a,b]\\ v_1,v_2\geq 0}}\frac{f_2(t,v_1,v_2,v_3)}{v_3}=+\infty.$$

Then there exists $\overline{\lambda}>0$ *such that for any* $0<\lambda<\overline{\lambda}$, *systems* (1) *have at least one positive solution* $(\overline{u},\overline{v})$.

Proof. Let

$$\Omega_1=\{(x,y)\in X:\|(x,y)\|_1<r\},$$

$$\overline{\lambda}=\min\left\{1,\frac{r}{2\rho_1(g_1^*+1)\int_0^1 G_1^*(1,s)[p_1(s)+q_1(s)]ds},\frac{r}{2\rho_2(g_2^*+1)\int_0^1 G_2^*(1,s)[p_2(s)+q_2(s)]ds}\right\},$$

where

$$g_1^*=\max\left\{g_1(s,u_1,u_2,u_3):0\leq s\leq 1,0\leq u_1\leq\frac{r}{\Gamma(\beta+1)},0\leq u_2\leq r,0\leq u_3\leq\frac{r}{\Gamma(\delta+1)}\right\},$$

$$g_2^*=\max\left\{g_2(s,v_1,v_2,v_3):0\leq s\leq 1,0\leq v_1\leq\frac{r}{\Gamma(\beta+1)},0\leq v_2\leq\frac{r}{\Gamma(\delta+1)},0\leq v_3\leq r\right\}.$$

Suppose $0<\lambda<\overline{\lambda}$, then for any $(x,y)\in P\cap\partial\Omega_1$, $s\in[0,1]$, we have

$$0\leq[x(s)-a(s)]^*\leq x(s)\leq\|x\|\leq r,\quad 0\leq[y(s)-b(s)]^*\leq y(s)\leq\|y\|\leq r,$$

$$I_{0^+}^{\beta}[x(s)-a(s)]^*=\int_0^s\frac{(s-\tau)^{\beta-1}[x(\tau)-a(\tau)]^*}{\Gamma(\beta)}d\tau\leq\frac{r}{\Gamma(\beta+1)},$$

$$I_{0^+}^{\delta}[y(s)-b(s)]^*=\int_0^s\frac{(s-\tau)^{\delta-1}[y(\tau)-b(\tau)]^*}{\Gamma(\delta)}d\tau\leq\frac{r}{\Gamma(\delta+1)}.$$

Thus,

$$\begin{aligned}
&\|A_1(x,y)\|\\
\leq&\lambda\int_0^1\rho_1G_1^*(1,s)[f_1(s,I_{0^+}^{\beta}[x(s)-a(s)]^*,[x(s)-a(s)]^*,I_{0^+}^{\delta}[y(s)-b(s)]^*)+q_1(s)]ds\\
\leq&\lambda\rho_1\int_0^1G_1^*(1,s)[p_1(s)g_1(s,I_{0^+}^{\beta}[x(s)-a(s)]^*,[x(s)-a(s)]^*,I_{0^+}^{\delta}[y(s)-b(s)]^*)+q_1(s)]ds\\
<&\overline{\lambda}\rho_1(g_1^*+1)\int_0^1G_1^*(1,s)[p_1(s)+q_1(s)]ds\leq\frac{r}{2},
\end{aligned}$$

and

$$\begin{aligned}
&\|A_2(x,y)\| \\
\le&\lambda \int_0^1 \rho_2 G_2^*(1,s)[f_2(s, I_{0^+}^{\beta}[x(s)-a(s)]^*, I_{0^+}^{\delta}[y(s)-b(s)]^*, [y(s)-b(s)]^*) + q_1(s)]ds \\
\le&\lambda\rho_2 \int_0^1 G_2^*(1,s)[p_2(s)g_2(s, I_{0^+}^{\beta}[x(s)-a(s)]^*, I_{0^+}^{\delta}[y(s)-b(s)]^*, [y(s)-b(s)]^*) + q_2(s)]ds \\
<&\overline{\lambda}\rho_2(g_2^*+1)\int_0^1 G_2^*(1,s)[p_2(s)+q_2(s)]ds \le \frac{r}{2}.
\end{aligned}$$

So

$$\|A(x,y)\|_1 = \|A_1(x,y)\| + \|A_2(x,y)\| \le r = \|(x,y)\|_1, \quad \forall (x,y) \in P \cap \partial\Omega_1. \tag{12}$$

On the other hand, let $\widetilde{k_i} = \min_{t\in[a,b]} k_i(t)$ and $\widetilde{L} > \max\left\{\frac{4\rho_1}{\lambda\widetilde{k_1}^2\int_a^b G_1^*(1,s)ds}, \frac{4\rho_2}{\lambda\widetilde{k_2}^2\int_a^b G_2^*(1,s)ds}\right\}$. By (H_4), there exists $N > 0$ such that

$$f_1(t,u_1,u_2,u_3) \ge \widetilde{L}u_2, \quad t\in[a,b],\ u_2 \ge N,\ u_1,u_3 \ge 0,$$

$$f_2(t,v_1,v_2,v_3) \ge \widetilde{L}v_3, \quad t\in[a,b],\ v_3 \ge N,\ v_1,v_2 \ge 0.$$

Let

$$R > \max\left\{2r, \frac{4\rho_1 N}{\widetilde{k_1}}, \frac{4\rho_2 N}{\widetilde{k_2}}\right\},$$

$$\Omega_2 = \{(x,y) \in P : \|(x,y)\|_1 < R\}.$$

For any $(x,y) \in P \cap \partial\Omega_2$, $\|(x,y)\|_1 = R$, we have $\|x\| \ge \frac{R}{2}$ or $\|y\| \ge \frac{R}{2}$. If $\|x\| \ge \frac{R}{2}$, we deduce

$$\begin{aligned}
x(t)-a(t) \ge&\rho_1^{-1}k_1(t)\|x\| - \frac{\rho_1}{\Gamma(\alpha-\beta)}k_1(t)\int_0^1 q_1(s)ds \\
=&k_1(t)\left[\frac{\|x\|}{\rho_1} - \frac{\rho_1}{\Gamma(\alpha-\beta)}\int_0^1 q_1(s)ds\right] \\
\ge&\widetilde{k_1}\left(\frac{R}{2\rho_1} - \frac{R}{4\rho_1}\right) \\
=&\frac{\widetilde{k_1}}{4\rho_1}R \ge N, \quad t\in[a,b],
\end{aligned}$$

then

$$\begin{aligned}
&f_1(s, I_{0^+}^{\beta}[x(s)-a(s)]^*, [x(s)-a(s)]^*, I_{0^+}^{\delta}[y(s)-b(s)]^*) \\
=&f_1(s, I_{0^+}^{\beta}(x(s)-a(s)), (x(s)-a(s)), I_{0^+}^{\delta}[y(s)-b(s)]^*) \\
\ge&\widetilde{L}[x(s)-a(s)] \ge \frac{\widetilde{k_1}\widetilde{L}R}{4\rho_1}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|A_1(x,y)\| \\
&= \max_{t\in[0,1]}\left|\lambda\int_0^1 G_1(t,s)[f_1(s, I_{0^+}^{\beta}[x(s)-a(s)]^*, [x(s)-a(s)]^*, I_{0^+}^{\delta}[y(s)-b(s)]^*)+q_1(s)]ds\right| \\
&\geq \max_{t\in[0,1]}\lambda\int_a^b k_1(t)G_1^*(1,s)\widetilde{L}(x(s)-a(s))ds \\
&\geq \lambda\widetilde{k_1}\widetilde{L}\int_a^b G_1^*(1,s)\frac{\widetilde{k_1}}{4\rho_1}Rds \\
&= \frac{\lambda\widetilde{k_1}^2\widetilde{L}}{4\rho_1}R\int_a^b G_1^*(1,s)ds \geq R = \|(x,y)\|_1.
\end{aligned}$$

If $\|y\| \geq \frac{R}{2}$, in a similar manner, we have $y(t)-b(t) \geq \frac{\widetilde{k_2}}{4\rho_2}R \geq N,\ t\in[a,b]$, and

$$\|A_2(x,y)\| \geq \frac{\lambda\widetilde{k_2}^2\widetilde{L}}{4\rho_2}R\int_a^b G_2^*(1,s)ds \geq R = \|(x,y)\|_1.$$

Thus

$$\|A(x,y)\|_1 = \|A_1(x,y)\| + \|A_2(x,y)\| \geq \|(x,y)\|_1, \quad \forall (x,y)\in P\cap\partial\Omega_2. \tag{13}$$

By (12), (13) and Lemma 2.12, A has a fixed point $(\widetilde{x},\widetilde{y})$ with $r \leq \|(\widetilde{x},\widetilde{y})\|_1 \leq R$. Now, we will prove $\widetilde{x}(t) > a(t),\ \widetilde{y}(t) \geq b(t)$(or $\widetilde{x}(t) \geq a(t),\ \widetilde{y}(t) > b(t)$), $t\in(0,1)$. We shall divide the proof into three cases: (i) $\|\widetilde{x}\| \geq \frac{r}{2},\ \|\widetilde{y}\| \geq \frac{r}{2}$; (ii) $\|\widetilde{x}\| > \frac{r}{2},\ \|\widetilde{y}\| < \frac{r}{2}$; (iii) $\|\widetilde{x}\| < \frac{r}{2},\ \|\widetilde{y}\| > \frac{r}{2}$.

Case (i). If $\|\widetilde{x}\| \geq \frac{r}{2}$, from (H_3) and Lemma 2.9, we get

$$\begin{aligned}
\widetilde{x}(t) &\geq \rho_1^{-1}k_1(t)\|\widetilde{x}\| \geq \rho_1^{-1}k_1(t)\frac{r}{2} > \rho_1^{-1}k_1(t)\frac{\rho_1^2}{\Gamma(\alpha-\beta)}\int_0^1 q_1(s)ds \\
&= \frac{\rho_1}{\Gamma(\alpha-\beta)}k_1(t)\int_0^1 q_1(s)ds \geq a(t), \quad t\in(0,1).
\end{aligned}$$

Similarly, if $\|\widetilde{y}\| \geq \frac{r}{2}$ we obtain $\widetilde{y}(t) > b(t),\ t\in(0,1)$.

Case (ii). For $\|\widetilde{x}\| > \frac{r}{2}$, similar to (i), we have $\widetilde{x}(t) > a(t),\ t\in(0,1)$. For $\|\widetilde{y}\| < \frac{r}{2}$, we have

$$0 \leq [\widetilde{y}(s)-b(s)]^* \leq \widetilde{y}(s) \leq \|\widetilde{y}\| \leq \frac{r}{2},$$

$$0 \leq I_{0_+}^{\beta}[\widetilde{x}(s)-a(s)]^* = I_{0_+}^{\beta}(\widetilde{x}(s)-a(s)) \leq \frac{R}{\Gamma(\beta+1)},$$

$$0 \leq I_{0_+}^{\delta}[\widetilde{y}(s)-b(s)]^* \leq \frac{r}{2\Gamma(\delta+1)}.$$

It follows from (H_2) that

$$f_2(t,v_1,v_2,v_3) \geq 0, \quad (t,v_1,v_2,v_3)\in(0,1)\times\left[0,\frac{R}{\Gamma(\beta+1)}\right]\times\left[0,\frac{r}{2\Gamma(\delta+1)}\right]\times\left[0,\frac{r}{2}\right],$$

then

$$\begin{aligned}
\widetilde{y}(t) &= \lambda\int_0^1 G_2(t,s)[f_2(s, I_{0^+}^{\beta}[\widetilde{x}(s)-a(s)]^*, I_{0^+}^{\delta}[\widetilde{y}(s)-b(s)]^*, [\widetilde{y}(s)-b(s)]^*)+q_2(s)]ds \\
&\geq \lambda\int_0^1 G_2(t,s)q_2(s)ds = b(t), \quad t\in(0,1).
\end{aligned}$$

Case (iii). If $\|\widetilde{x}\| < \frac{r}{2}$ and $\|\widetilde{y}\| > \frac{r}{2}$, similar to (ii), we have $\widetilde{x}(t) \geq a(t),\ \widetilde{y}(t) > b(t),\ t \in (0,1)$.

So by Lemma 2.10 we know that $(\overline{u}(t), \overline{v}(t)) = (I_{0^+}^{\beta}(\widetilde{x}(t) - a(t)), I_{0^+}^{\delta}(\widetilde{y}(t) - b(t)))$ is a positive solution of systems (1). □

4 An Example

Example 4.1. *Consider the following problem:*

$$\begin{aligned}
&D_{0^+}^{\frac{5}{2}}(D_{0^+}^{\frac{1}{4}}u(t)) + \frac{\lambda\sqrt{\pi}}{\sqrt{t(1-t)}}\left[\frac{(u(t)-90)^2 + (D_{0^+}^{\frac{1}{4}}u(t)-3)^4 + v^2(t)}{685\sqrt[4]{t}} - \frac{1}{48}\right] = 0,\ t \in (0,1),\\
&D_{0^+}^{\frac{9}{4}}(D_{0^+}^{\frac{1}{8}}v(t)) + \frac{\lambda\Gamma(\frac{9}{4})}{\sqrt[4]{1-t}}\left[\frac{u^2(t) + (v(t)-30)^2 + (D_{0^+}^{\frac{1}{8}}v(t)-3)^2}{30\sqrt{t}} - \frac{1}{12}\right] = 0,\ t \in (0,1),\\
&D_{0^+}^{\frac{1}{4}}u(0) = D_{0^+}^{\frac{5}{4}}u(0) = 0, \quad D_{0^+}^{\frac{5}{4}}u(1) = \frac{96}{97}D_{0^+}^{\frac{1}{4}}u\left(\frac{1}{16}\right),\\
&D_{0^+}^{\frac{1}{8}}v(0) = D_{0^+}^{\frac{9}{8}}v(0) = 0, \quad D_{0^+}^{\frac{9}{8}}v(1) = \frac{40}{41}D_{0^+}^{\frac{1}{8}}v\left(\frac{1}{16}\right).
\end{aligned} \tag{14}$$

Problem (14) can be regarded as a problem of the form (1) with $\alpha = \frac{11}{4},\ \beta = \frac{1}{4},\ \gamma = \frac{19}{8},\ \delta = \frac{1}{8}$,

$$A(t) = \begin{cases} 0, & t \in \left[0, \frac{1}{16}\right),\\ \frac{96}{97}, & t \in \left[\frac{1}{16}, 1\right], \end{cases} \qquad B(t) = \begin{cases} 0, & t \in \left[0, \frac{1}{16}\right),\\ \frac{40}{41}, & t \in \left[\frac{1}{16}, 1\right], \end{cases}$$

$$f_1(t, u_1, u_2, u_3) = \frac{\sqrt{\pi}}{\sqrt[4]{t^3}\sqrt{1-t}}\frac{(u_1-90)^2 + (u_2-3)^4 + u_3^2}{685} - \frac{\sqrt{\pi}}{48\sqrt{t(1-t)}},$$

$$f_2(t, v_1, v_2, v_3) = \frac{\Gamma(\frac{9}{4})}{\sqrt{t}\sqrt[4]{1-t}}\frac{v_1^2 + (v_2-30)^2 + (v_3-3)^2}{30} - \frac{\Gamma(\frac{9}{4})}{12\sqrt[4]{1-t}}.$$

Evidently, $f_1, f_2 : (0,1) \times [0,+\infty)^3 \to (-\infty,+\infty)$ are continuous and singular at $t = 0, 1$,

By direct calculations, we get

$$k_1(t) = t^{\frac{3}{2}},\ k_2(t) = t^{\frac{5}{4}},\ \gamma_1(t) = \frac{2}{3}t^{\frac{3}{2}},\ \gamma_2(t) = \frac{4}{5}t^{\frac{5}{4}},\ \rho_1 = 2,\ \rho_2 = 2,$$

$$\jmath_1 = \int_0^1 \gamma_1(t)dA(t) = \frac{96}{97} \times \frac{1}{3} \times \left(\frac{1}{16}\right)^{\frac{3}{2}} = \frac{1}{97}, \quad \ell_1(s) = \frac{96}{97}G_1^*\left(\frac{1}{16}, s\right) \geq 0,$$

$$\jmath_2 = \int_0^1 \gamma_2(t)dB(t) = \frac{40}{41} \times \frac{4}{5} \times \left(\frac{1}{16}\right)^{\frac{5}{4}} = \frac{1}{41}, \quad \ell_2(s) = \frac{40}{41}G_2^*\left(\frac{1}{16}, s\right) \geq 0,$$

so condition (H_1) is satisfied. Let

$$q_1(t) = \frac{\sqrt{\pi}}{48\sqrt{t(1-t)}},\ p_1(t) = \frac{\sqrt{\pi}}{\sqrt[4]{t^3}\sqrt{1-t}},\ g_1(t, u_2, u_2, u_3) = \frac{(u_1-90)^2 + (u_2-3)^4 + u_3^2}{685},$$

$$q_2(t) = \frac{\Gamma(\frac{9}{4})}{12\sqrt[4]{1-t}},\ p_2(t) = \frac{\Gamma(\frac{9}{4})}{\sqrt{t}\sqrt[4]{1-t}},\ g_2(t, v_2, v_2, v_3) = \frac{v_1^2 + (v_2-30)^2 + (v_3-3)^2}{30},$$

then

$$\int_0^1 q_1(t)dt = \frac{\sqrt{\pi}}{48}B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\sqrt{\pi}\pi}{48}, \quad \int_0^1 q_2(t)dt = \frac{\Gamma(\frac{9}{4})}{9},$$

$$\int_0^1 p_1(t)dt = \sqrt{\pi}B\left(\frac{1}{4},\frac{1}{2}\right), \quad \int_0^1 p_2(t)dt = \Gamma\left(\frac{9}{4}\right)B\left(\frac{3}{4},\frac{1}{2}\right),$$

so condition (H_2) is satisfied. On the other hand,

$$\max\left\{\frac{2\rho_1^2}{\Gamma(\alpha-\beta)}\int_0^1 q_1(s)ds,\ \frac{2\rho_2^2}{\Gamma(\gamma-\delta)}\int_0^1 q_2(s)ds\right\} = \frac{8}{9}.$$

Select $r = 2$, then

$$f_1(t,u_1,u_2,u_3) \geq \frac{\sqrt{\pi}}{\sqrt{t(1-t)}}\left[\frac{(u_1-90)^2+(u_2-3)^4+u_3^2}{685} - \frac{1}{48}\right] > 0,$$
$$(t,u_1,u_2,u_3) \in (0,1)\times\left[0,\frac{1}{\Gamma(\frac{5}{4})}\right]\times[0,1]\times[0,+\infty),$$

$$f_2(t,v_1,v_2,v_3) \geq \frac{\Gamma(\frac{9}{4})}{\sqrt[4]{1-t}}\left[\frac{v_1^2+(v_2-30)^2+(v_3-3)^2}{30} - \frac{1}{12}\right] > 0,$$
$$(t,v_1,v_2,v_3) \in (0,1)\times[0,+\infty)\times\left[0,\frac{1}{\Gamma(\frac{9}{8})}\right]\times[0,1].$$

Hence, condition (H_3) is satisfied. In addition, for any $[a,b] \subset (0,1)$,

$$\lim_{u_2\to+\infty}\min_{\substack{t\in[a,b]\\ u_1,u_3\geq 0}}\frac{f_1(t,u_1,u_2,u_3)}{u_2} = +\infty, \quad \lim_{v_3\to+\infty}\min_{\substack{t\in[a,b]\\ v_1,v_2\geq 0}}\frac{f_2(t,v_1,v_2,v_3)}{v_3} = +\infty,$$

condition (H_4) is satisfied. Hence, problem (14) has at least one positive solution by Theorem 3.1.

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