

POSITIVE TOPOLOGICAL ENTROPY IMPLIES CHAOS DC2

T. DOWNAROWICZ

(Communicated by Byrna Kra)

ABSTRACT. Using methods of entropy in ergodic theory, we prove that positive topological entropy implies chaos DC2. That is, if a system (X, T) has positive topological entropy, then there exists an uncountable set E such that for any two distinct points x, y in E ,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \text{dist}(T^i x, T^i y) = 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \text{dist}(T^i x, T^i y) > 0.$$

1. INTRODUCTION

The notion of chaos was invented in 1975 by Li and Yorke in their seminal paper [LY] in the context of continuous transformations of the interval. Since then several refinements of chaos have been introduced and extensively studied, mainly in connection with interval maps or variants of the Sharkovsky Theorem. There are in fact very few implications between chaos and other topological or measure-theoretic invariants that hold for general topological dynamical systems (understood as continuous transformations of compact metric spaces); most of results are either counterexamples or require strong assumptions. Depending on the attitude, this fact may be accounted as a disadvantage of the notions of chaos (they are somewhat artificial) or as an advantage (they capture new phenomena). We skip a wider historical review of the topic, as this is done in many papers and survey papers devoted to the subject, and restrict ourselves to the basic facts that concern the subject of this paper. We refer the reader to the survey [BHS] for more information.

One of the few general recent affirmative results says that positive topological entropy implies Li-Yorke chaos ([BGKM]). That the converse need not hold has been known since 1986 [S1].

A strengthening of the notion of chaos was introduced in 1994 by Schweizer and Smítal (so-called Schweizer–Smítal chaos; [SS]). Since then it has evolved into three variants of so-called *distributional chaos*: DC1, DC2 and DC3 (ordered from strongest to weakest). Around 2000 Pikula showed that positive topological entropy does not imply DC1 (published in 2007 [P]). Smítal conjectured that it does imply DC2 (see e.g. [S2]). This problem has been around for several years, and several papers have proposed partial solutions (see e.g. [O1] for UPE systems or [S2] for some other cases).

Received by the editors October 8, 2011 and, in revised form, February 20, 2012.

2010 *Mathematics Subject Classification.* Primary 37A35.

Key words and phrases. Distributional chaos, chaos DC2, ergodic process, scrambled set.

The author's research was supported from resources for science in years 2009–2012 as research project grant MENII N N201 394537, Poland.

©2013 American Mathematical Society
 Reverts to public domain 28 years from publication

In this paper we solve the problem posed by Smítal by proving his conjecture in the general case; in particular, we strengthen the result of [BGKM]:

Theorem 1.1. *Assume that a topological dynamical system (X, T) has positive topological entropy $h_{\text{top}}(T) > 0$. Then the system possesses an uncountable DC2-scrambled set.*

Additionally, the scrambled set can be obtained as perfect, which resolves another conjecture of Smítal (see the final remarks of this paper).

Recall that a set $E \subset X$ is *DC2-scrambled* if so is any pair $x, y \in E$, $x \neq y$. It remains to recall the definition of a DC2-scrambled pair:

Definition 1.2. A pair $x, y \in X$ is DC2-scrambled if the following two conditions hold:

$$\begin{aligned} \forall_{t>0} \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \# \{i \in [0, n-1] : \text{dist}(T^i x, T^i y) < t\} &= 1, \\ \exists_{t_0>0} \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \# \{i \in [0, n-1] : \text{dist}(T^i x, T^i y) < t_0\} &< 1. \end{aligned}$$

In other words, the orbits of x and y are arbitrarily close with upper density one, but for some distance, with lower density smaller than one.

The inspiration for the proof presented in this paper came from the interpretation of the definition. It is not hard to see that the above two conditions are equivalent to the following ones:

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \text{dist}(T^i x, T^i y) = 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \text{dist}(T^i x, T^i y) > 0.$$

(We leave the easy verification of the equivalence to the interested reader.) In this manner DC2 is expressed in terms of ergodic averages (of a function on the product space), which makes it a very natural ergodic-theoretic object. (DC1 or DC3 do not translate so well, not to mention Li-Yorke chaos.) Although we are not going to use this rephrased definition directly in the proof, it helped us realize that the key to the solution lies in ergodic theory and measure-theoretic entropy. Indeed, topological properties play a marginal role in the proof.

Most of the time we will be concerned with an ergodic measure-theoretic dynamical system $(X, \mathfrak{B}, \mu, T)$, where (X, \mathfrak{B}, μ) is a standard probability space and $T : X \rightarrow X$ is a measure-preserving transformation (endomorphism). Any finite measurable partition \mathcal{P} of X generates a factor of the system obtained by projecting onto the subinvariant sub-sigma algebra $\mathcal{P}^{[0, \infty)}$ of \mathfrak{B} , where we use the following notational convention: for $D \subset \mathbb{N} \cup \{0\}$ we set

$$\mathcal{P}^D = \bigvee_{n \in D} T^{-n}(\mathcal{P})$$

(if D is finite this is a finite partition; otherwise it is a sigma-algebra). Abusing the notation slightly, we will denote such a factor by (X, \mathcal{P}, μ, T) and call it *the process generated by \mathcal{P}* . This process is isomorphic to a symbolic system in which \mathcal{P} plays the role of the alphabet and in which μ becomes a shift-invariant measure.

This and the rest of our notation (which is relatively standard) matches roughly that in [D]. Most of the facts concerning entropy of processes and other facts in ergodic theory and topological dynamics can be found in that book; in most cases we provide statement reference.

2. SOME TOOLS

2.1. Roughly equal partitions. The Shannon–McMillan Theorem (McMillan, 1953) asserts that given an ergodic process (X, \mathcal{P}, μ, T) with positive entropy $h = h_\mu(T, \mathcal{P})$ and some $\varepsilon > 0$, if n is large enough, then $1 - \varepsilon$ of the space is covered by cylinders of length n whose measures range within $2^{-n(h \pm \varepsilon)}$. (In 1957 Breiman strengthened this result, proving almost everywhere convergence of the corresponding information function; this version is known as the *Shannon–McMillan–Breiman Theorem* and can be found in any textbook on ergodic theory.)

The above property of the partition $\mathcal{P}^{[0, n-1]}$ formed by the cylinders of length n becomes the main tool in our proof. For notational convenience, let us isolate this property in a separate definition.

Definition 2.1. We will say that a probability vector $\mathbf{P} = (p_1, p_2, \dots, p_l)$ is *roughly equal* with parameters $\varepsilon > 0$, $n \in \mathbb{N}$ and $h > 0$ if

$$\sum_{i \in I} p_i > 1 - \varepsilon,$$

where $I = \{i : p_i \in (2^{-n(h+\varepsilon)}, 2^{-n(h-\varepsilon)})\}$.

The condition says that the majority of the mass of \mathbf{P} is carried by entries approximately equal to 2^{-nh} (hence there is approximately 2^{nh} of them). The meaning of “approximately” is very rough, as the tolerance is up to the (large) multiplicative factor $2^{n\varepsilon}$.

It is a fairly obvious observation that the condition of being roughly equal with some fixed parameters ε, n, h is open in ℓ^1 .

Definition 2.2. Let $\mathcal{P} = \{P_1, P_2, \dots, P_l\}$ be a finite measurable partition of a standard probability space (X, \mathfrak{B}, μ) . We will say that the partition \mathcal{P} is *μ -roughly equal with parameters ε, n, h* whenever $\mathbf{P}_{\mu, \mathcal{P}}$ is roughly equal (with the same parameters), where $\mathbf{P}_{\mu, \mathcal{P}}$ denotes the vector of values assigned to \mathcal{P} by the measure μ , i.e.,

$$\mathbf{P}_{\mu, \mathcal{P}} = (\mu(P_1), \mu(P_2), \dots, \mu(P_l)).$$

The atoms of \mathcal{P} of measure ranging within $2^{-n(h \pm \varepsilon)}$ will be referred to as *good*.

We can now rephrase the assertion of the Shannon–McMillan Theorem: for large enough n the partition $\mathcal{P}^{[0, n-1]}$ is μ -roughly equal with parameters ε, n, h . Obviously, by invariance of the measure, the same holds for any partition of the form $\mathcal{P}^{[a, b-1]}$, where $b - a = n$.

Further, in the same context, if \mathfrak{F} is some subinvariant sigma-algebra, then the conditional version of the Shannon–McMillan Theorem implies that for a long enough interval $[a, b - 1]$ there is a set of measure at least $1 - \varepsilon$ of atoms y of \mathfrak{F} such that the partition $\mathcal{P}^{[a, b-1]}$ is μ_y -roughly equal with parameters $\varepsilon, n, h_\mu(T, \mathcal{P}|\mathfrak{F})$, where μ_y is the disintegration measure of μ supported by the atom y . This conditional version (with a proof and in the stronger, almost everywhere version) can be found e.g. in Appendix B of [D].

The following fact is completely standard, so we leave the proof as an exercise.

Fact 1. Let X' be a subset of positive measure of a probability space (X, \mathfrak{B}, μ) . Fix some ε , h and $\varepsilon' > \frac{\varepsilon}{\mu(X')}$. Then, for n large enough the following implication holds: if the partition \mathcal{P} is μ -roughly equal with parameters ε, n, h , then the same partition is $\mu_{X'}$ -roughly equal with parameters ε', n, h , where $\mu_{X'}$ is the conditional measure μ on X' (i.e., $\mu_{X'} = \frac{\mu(\cdot \cap X')}{\mu(X')}$). \square

2.2. Conditionally independent partitions. We assume that the reader has some familiarity with the notion of Shannon entropy $H(\cdot)$ for probability vectors and $H_\mu(\cdot)$ for partitions.

Definition 2.3. Let \mathcal{P} be a finite partition of a probability space (X, \mathfrak{B}, μ) and let \mathfrak{F} be a sub-sigma algebra of \mathfrak{B} . We will say that \mathcal{P} is δ -independent of \mathfrak{F} (which we will write as $\mathcal{P} \perp^\delta \mathfrak{F}$) if

$$H_\mu(\mathcal{P}|\mathfrak{F}) > H_\mu(\mathcal{P}) - \delta.$$

In particular, if \mathfrak{F} is the algebra generated by a finite partition \mathcal{Q} we will say that \mathcal{P} is δ -independent of \mathcal{Q} (and write $\mathcal{P} \perp^\delta \mathcal{Q}$).

Since we can write the above as $H_\mu(\mathcal{P} \vee \mathcal{Q}) > H_\mu(\mathcal{P}) + H_\mu(\mathcal{Q}) - \delta$, it is immediately seen that the relation \perp^δ applied to partitions is symmetric. A very important (and easy) fact is that if $\mathcal{P}_1 \preceq \mathcal{P}$ and $\mathfrak{F}_1 \preceq \mathfrak{F}$, then $\mathcal{P} \perp^\delta \mathfrak{F}$ implies $\mathcal{P}_1 \perp^\delta \mathfrak{F}_1$. Another important fact is that if (\mathcal{Q}_k) is a sequence of partitions that generates \mathfrak{F} and $\mathcal{P} \perp^\delta \mathcal{Q}_k$ for every k , then $\mathcal{P} \perp^\delta \mathfrak{F}$ (see [D], Section 3.1).

What will be very important for us is the interpretation of δ -independence. Using disintegration, we can rewrite the definition as

$$\int H(\mathbf{P}_{\mu_z, \mathcal{P}}) d\nu(z) > H(\mathbf{P}_{\mu, \mathcal{P}}) - \delta,$$

where μ_z is the disintegration measure of μ on the atom z of \mathfrak{F} and ν is the marginal of μ on \mathfrak{F} . We also have $\mathbf{P}_{\mu, \mathcal{P}} = \int \mathbf{P}_{\mu_z, \mathcal{P}} d\nu(z)$; hence the above inequality compares a generalized convex combination of values of the entropy attained on some l -dimensional probability vectors with the entropy of the corresponding generalized convex combination of these vectors. Because entropy is a continuous and strictly convex function on probability vectors of dimension l and the set of all such vectors is compact, we can deduce that if we fix the partition \mathcal{P} and $\varepsilon > 0$, then we can find δ so small that for any sigma-algebra \mathfrak{F} , $\mathcal{P} \perp^\delta \mathfrak{F}$ implies that

$$(2.1) \quad \|\mathbf{P}_{\mu, \mathcal{P}} - \mathbf{P}_{\mu_z, \mathcal{P}}\|_1 < \varepsilon \quad \text{for atoms } z \text{ of } \mathfrak{F} \text{ joint measure at least } 1 - \varepsilon$$

(here $\|\cdot\|_1$ denotes the norm in ℓ_1). We refer the reader to Fact 1.1.11 in [D] for a more complete proof.

Combining this fact with the ℓ^1 openness of the property of being roughly equal, we obtain the following statement:

Fact 2. Suppose \mathcal{P} is a finite measurable partition of a probability space (X, \mathfrak{B}, μ) , which is μ -roughly equal with parameters α, n, h . Then for any $\beta > 0$ there exists $\gamma > 0$ such that whenever a sub-sigma algebra \mathfrak{F} satisfies $\mathcal{P} \perp^\gamma \mathfrak{F}$, then \mathcal{P} is μ_z -roughly equal with the same parameters α, n, h and the same good atoms, for atoms z of \mathfrak{F} of joint measure at least $1 - \beta$. \square

We will need a bit more intricate reversed statement:

Lemma 2.4. *Suppose \mathcal{P} is a finite measurable partition of a probability space (X, \mathfrak{B}, μ) , which is μ -roughly equal with parameters α, n, h . Fix some positive $\beta < \alpha$ and $h' > 0$. Then there exist a positive $\gamma < \beta$ and m such that whenever \mathcal{Q} is μ -roughly equal with parameters γ, m, h' and $\mathcal{P} \vee \mathcal{Q}$, then*

- $\mathcal{P} \vee \mathcal{Q}$ is μ -roughly equal with parameters $\alpha + \beta + \gamma, m, h'$, and
- for a collection of atoms A of \mathcal{P} , of joint measure at least $1 - \alpha - \beta$, \mathcal{Q} is μ_A -roughly equal with parameters β, m, h' .

Proof. We choose γ according to Fact 2, and then, for atoms B of \mathcal{Q} of joint measure at least $1 - \beta$, \mathcal{P} is μ_B -roughly equal with parameters α, n, h . Moreover, \mathcal{P} has the same good atoms for all such B . Let V' denote the union of bad atoms B of \mathcal{Q} and atoms B which do not satisfy the above. This set has measure μ at most $\beta + \gamma$. Every other atom B of \mathcal{Q} splits into a set of measure μ_B at most α and atoms $A \cap B$ of $\mathcal{P} \vee \mathcal{Q}$ of measures μ ranging within $2^{-m(h' \pm \gamma) - n(h \pm \alpha)}$. If m is much larger than n , this range is contained within $2^{-m(h' \pm 2\gamma)}$. The joint measure μ of these atoms $A \cap B$ is at least $1 - \alpha - \beta - \gamma$. This implies (with sacrifice of precision) that $\mathcal{P} \vee \mathcal{Q}$ is μ -roughly equal with the parameters $\alpha + \sqrt{2\beta} + \gamma, m, h'$ (we have an artificially enlarged β for cosmetic reasons; we will adjust it in a moment).

In what follows, for shorter writing, we replace $\beta + \gamma$ by the larger term 2β . It is clear that the collection of atoms A of \mathcal{P} such that $\mu_A(V') > \sqrt{2\beta}$ has jointly the measure μ smaller than $\sqrt{2\beta}$. Among the remaining atoms A of \mathcal{P} we select only the good ones and call them “very good”. The joint measure μ of these atoms A is at least $1 - \alpha - \sqrt{2\beta}$. The set $A \setminus V'$ has, for a very good A , the measure μ_A at least $1 - \sqrt{2\beta}$, and it consists of good atoms $A \cap B$ (their measures μ range within $2^{-m(h' \pm \delta) - n(h \pm \alpha)}$). Notice that the measure μ_A of $A \cap B$ cannot be smaller than its measure μ , while it can be larger by at most the multiplicative factor $2^{n(h + \alpha)}$. So, the measure μ_A of a good $A \cap B$ ranges between $2^{-m(h' + \gamma) - n(h + \alpha)}$ and $2^{-m(h' - \gamma) + 2n\alpha}$, which, for large enough m , lies within $2^{-m(h' \pm 2\gamma)}$. We obtain (again, with sacrifice of precision) that \mathcal{Q} is μ_A -roughly equal with parameters $\sqrt{2\beta}, m, h'$, and this is true on a subset of good atoms A whose joint measure μ is at least $1 - \alpha - \sqrt{2\beta}$.

Since β is arbitrary, we can replace it with $\frac{\beta^2}{2}$, and we obtain both hypotheses of the lemma. \square

Remark 1. Notice in the proof that the intersection of the union of these good atoms of \mathcal{P} on which \mathcal{Q} is roughly equal (as specified in the lemma) with the union of good atoms of $\mathcal{P} \vee \mathcal{Q}$ has measure at least $1 - \alpha - 2\sqrt{2\beta}$. These atoms alone allow one to classify $\mathcal{P} \vee \mathcal{Q}$ as μ -roughly equal with the parameters $\alpha + 2\sqrt{2\beta}, m, h'$. Adjusting β and sacrificing some good atoms of $\mathcal{P} \vee \mathcal{Q}$ (treating them as “not good”), we obtain that the lemma holds with the additional property $V_0 \supset V_1 \supset V_2$, where V_0 is the union of good atoms of \mathcal{P} , V_1 is the union of these atoms of \mathcal{P} on which \mathcal{Q} is roughly equal, and V_2 is the union of good atoms of $\mathcal{P} \vee \mathcal{Q}$. We will use this fact later.

2.3. Pinsker factor. Let $(X, \mathfrak{B}, \mu, T)$ be some ergodic system with positive entropy $h_\mu(T)$. By Π we will denote the Pinsker sigma-algebra of the system. It is obtained as $\bigvee_{\mathcal{P}} \Pi_{\mathcal{P}}$, where \mathcal{P} ranges over all finite partitions of X and $\Pi_{\mathcal{P}} = \bigcap_{n \geq 1} \mathcal{P}^{[n, \infty)}$ ($\Pi_{\mathcal{P}}$ is sometimes called the *remote future* of the process generated

by \mathcal{P}). It is known that the sigma-algebra Π is invariant (i.e., $T^{-1}(\Pi) = \Pi$) and the factor associated with Π has entropy zero (in fact it is the largest factor with entropy zero). In particular, for any finite partition \mathcal{P} we have $h_\mu(T, \mathcal{P}|\Pi) = h_\mu(T, \mathcal{P})$. These facts originate from [RS]. We refer also to [D], sections 3.2 and 4.2, for a more recent exposition.

Lemma 2.5. *Let (X, μ, T) be an ergodic system with positive entropy and let \mathcal{P} and \mathcal{Q} be some finite measurable partitions of X . Then for every $\varepsilon > 0$ there exists $n > 1$ such that for every $m > n$ we have*

$$H_\mu(\mathcal{P}|\mathcal{Q}^{[n, m-1]} \vee \Pi) > H_\mu(\mathcal{P}|\Pi) - \varepsilon.$$

Proof. The subtlety is that in general it is not true that if \mathfrak{F}_n is a decreasing sequence of sigma-algebras and \mathfrak{G} is another sigma-algebra, then $\bigcap_n (\mathfrak{G} \vee \mathfrak{F}_n) = \mathfrak{G} \vee \bigcap_n \mathfrak{F}_n$. However, in our case it will be so; the conditioning sigma-algebras $\mathcal{Q}^{[n, \infty)} \vee \Pi$ decrease to Π . Here is why: the Pinsker factor, as an invertible zero-entropy system, always admits a finite bilateral generator \mathcal{P}_0 (we are using Krieger's Generator Theorem) and then $\Pi = \bigcap_n \mathcal{P}_0^{[n, \infty)}$. Thus

$$\mathcal{Q}^{[n, \infty)} \vee \Pi \preceq \mathcal{Q}^{[n, \infty)} \vee \mathcal{P}_0^{[n, \infty)} = (\mathcal{Q} \vee \mathcal{P}_0)^{[n, \infty)}$$

and

$$\Pi \preceq \bigcap_n (\mathcal{Q}^{[n, \infty)} \vee \Pi) \preceq \bigcap_n (\mathcal{Q} \vee \mathcal{P}_0)^{[n, \infty)} \preceq \Pi.$$

Further, since conditional entropy respects decreasing limits of the conditioning sigma-algebras (see [D], Fact 1.7.11 (1.7.14)), we have

$$H_\mu(\mathcal{P}|\Pi) = H_\mu(\mathcal{P}|\bigcap_n (\mathcal{Q}^{[n, \infty)} \vee \Pi)) = \lim_n H_\mu(\mathcal{P}|\mathcal{Q}^{[n, \infty)} \vee \Pi),$$

which implies that for large enough n , $H_\mu(\mathcal{P}|\mathcal{Q}^{[n, \infty)} \vee \Pi) > H_\mu(\mathcal{P}|\Pi) - \varepsilon$. Obviously, this inequality will be maintained if we replace the sigma-algebra $\mathcal{Q}^{[n, \infty)}$ by the partition $\mathcal{Q}^{[n, m-1]}$. \square

3. THE MAIN PROOF

3.1. Measure-theoretic part of the proof. Consider an arbitrary topological dynamical system (X, T) with positive topological entropy. By the variational principle, there exists an ergodic T -invariant Borel probability measure μ on X with positive Kolmogorov–Sinai entropy. We fix μ and choose a sequence of measurable partitions \mathcal{P}_k with the following properties:

- the partitions with even indices $2k$ are chosen so that $\mathcal{P}_{2k+2} \succcurlyeq \mathcal{P}_{2k}$ and $\text{diam}(\mathcal{P}_{2k}) \rightarrow 0$;
- for all odd indices $2k-1$, \mathcal{P}_{2k-1} equals one fixed finite partition \mathcal{P} which has positive dynamic entropy $h = h_\mu(T, \mathcal{P})$ and contains one special set P_0 of measure smaller than δ such that if x, y belong to different atoms of \mathcal{P} other than P_0 , then $\text{dist}(x, y) > t_0$ (t_0 and δ are some positive numbers).

Getting a partition \mathcal{P} with the above property (the “separating atom” P_0) is an easy exercise: One should start with an arbitrary partition \mathcal{P}' (having positive dynamical entropy) and then, using regularity of the measure, find inside each atom of \mathcal{P}' a compact set of nearly the same measure, and finally define \mathcal{P} as the collection of these compact sets and one open set P_0 of small measure δ . Now t_0 is the minimal distance between the compact members of \mathcal{P} . By continuity of the

dynamical entropy, it remains positive for \mathcal{P} . The above established parameters t_0 and δ will eventually play the role of t_0 and the gap between 1 and the lower density in the second requirement in the definition of a scrambled pair. Note that the term δ can be chosen as small as we wish (then t_0 will be rather small, too).

We let h_k denote $h_\mu(T, \mathcal{P}_k)$ (of course, $h_k = h$ for odd k). Next we will select a strictly increasing sequence S of integers starting with $a_1 = 0$. We will write this sequence as two alternating sequences, $S = (a_1, b_1, a_2, b_2, a_3, b_3, \dots)$. The sequence should grow so fast that the ratios $\frac{b_k}{a_k}$ tend to infinity. Many more conditions on the growth of S will be imposed during the construction. We introduce some more notation. For $k \geq 1$ and $m \geq k$ (including $m = \infty$) we will write

$$\mathcal{R}_k = \mathcal{P}_k^{[a_k, b_k-1]} \quad \text{and} \quad \mathcal{R}_{k,m} = \bigvee_{i=k}^m \mathcal{R}_i$$

(for $m = \infty$ the latter is a sigma-algebra rather than a partition). We will also write

$$\mathcal{R}_{1,2k-1}^{\text{odd}} = \bigvee_{i=1}^k \mathcal{R}_{2i-1}, \quad \mathcal{R}_{2,2k}^{\text{even}} = \bigvee_{i=1}^k \mathcal{R}_{2i} \quad \text{and} \quad \mathfrak{R} = \mathcal{R}_{2,\infty}^{\text{even}}.$$

Let $(\varepsilon_k)_{k \geq 0}$ be a summable sequence of positive numbers with a small sum ε_0 . Again, many more conditions on the decay of this sequence will be imposed later. As we have mentioned in the introduction, the conditional Shannon–McMillan Theorem implies that by choosing the length $n_k = b_k - a_k$ large enough, we can assure that for a family of atoms y of the Pinsker sigma-algebra Π , of joint measure at least $1 - \varepsilon_k$, the partition \mathcal{R}_k is μ_y -roughly equal with parameters ε_k, n_k, h_k . For atoms y from a set Y of measure $1 - \varepsilon_0$, this holds for every k .

By our Lemma 2.5, we can arrange our sequence S to grow so fast that for every $k > 1$,

$$(3.1) \quad H(\mathcal{R}_{1,k-1} | \mathcal{R}_k \vee \Pi) > H(\mathcal{R}_{1,k-1} | \Pi) - \varepsilon_k^2.$$

Using disintegration, we can rewrite the above as

$$\int H_y(\mathcal{R}_{1,k-1} | \mathcal{R}_k) d\nu(y) > \int H_y(\mathcal{R}_{1,k-1}) d\nu(y) - \varepsilon_k^2,$$

where H_y is the entropy with respect to the disintegration measure μ_y of μ over the atoms y of Π , and where ν denotes the marginal of μ on this sigma-algebra. Because the integrated function on the left is, at every y , not larger than that on the right, we conclude that for y 's of joint measure at least $1 - \varepsilon_k$ we have

$$(3.2) \quad H_y(\mathcal{R}_{1,k-1} | \mathcal{R}_k) > H_y(\mathcal{R}_{1,k-1}) - \varepsilon_k,$$

i.e., $\mathcal{R}_{1,k-1} \perp^{\varepsilon_k} \mathcal{R}_k$ with respect to the measure μ_y . There exists a set Y' of atoms y , of joint measure at least $1 - \varepsilon_0$, where the above holds for all k . Of course, $Y \cap Y'$ has measure at least $1 - 2\varepsilon_0$ which is positive. We fix an atom $y \in Y \cap Y'$ (the fact that the y belongs to Y will be used later).

By elementary identities concerning conditional entropy, we have, for $m > k$,

$$\begin{aligned}
 H_y(\mathcal{R}_{1,m}) &= H_y(\mathcal{R}_{1,m-1}|\mathcal{R}_m) + H_y(\mathcal{R}_m) > H_y(\mathcal{R}_{1,m-1}) + H_y(\mathcal{R}_m) - \varepsilon_m \\
 &= H_y(\mathcal{R}_{1,m-2}|\mathcal{R}_{m-1}) + H_y(\mathcal{R}_{m-1}) + H_y(\mathcal{R}_m) - \varepsilon_m \\
 &> H_y(\mathcal{R}_{1,m-2}) + H_y(\mathcal{R}_{m-1}) + H_y(\mathcal{R}_m) - \varepsilon_{m-1} - \varepsilon_m \\
 &\vdots \\
 &= H_y(\mathcal{R}_{1,k-1}) + H_y(\mathcal{R}_k) + H_y(\mathcal{R}_{k+1}) + \cdots + H_y(\mathcal{R}_m) - \varepsilon_k - \varepsilon_{k+1} - \cdots - \varepsilon_m.
 \end{aligned}$$

Denoting by ξ_k^2 the tail $\sum_{i \geq k} \varepsilon_i$, we draw two conclusions:

$$H_y(\mathcal{R}_{1,m}) = H_y(\mathcal{R}_{1,k-1} \vee \mathcal{R}_{k,m}) \geq H_y(\mathcal{R}_{1,k-1}) + H(\mathcal{R}_{k,m}) - \xi_k^2,$$

i.e., $\mathcal{R}_{1,k-1} \perp_{\xi_k^2}^{\mathcal{R}_{k,m}}$ with respect to μ_y , and

$$H_y(\mathcal{R}_{1,m}) = H_y(\mathcal{R}_k \vee (\mathcal{R}_{1,k-1} \vee \mathcal{R}_{k+1,m})) \geq H_y(\mathcal{R}_k) + H_y(\mathcal{R}_{1,k-1} \vee \mathcal{R}_{k+1,m}) - \xi_k^2,$$

i.e., $\mathcal{R}_k \perp_{\xi_k^2}^{\mathcal{R}_{1,k-1} \vee \mathcal{R}_{k+1,m}}$ with respect to μ_y . Because this holds for all $m > k$, we get

$$(3.3) \quad \mathcal{R}_{1,k-1} \perp_{\xi_k^2}^{\mathcal{R}_{k,\infty}} \quad \text{with respect to } \mu_y$$

(which we will use a bit later) and $\mathcal{R}_k \perp_{\xi_k^2}^{\mathcal{R}_{1,k-1} \vee \mathcal{R}_{k+1,\infty}}$. Applying the latter to $2k+1$ rather than k and using the fact that the independence passes to sub-sigma algebras, we obtain that $\mathcal{R}_{2k+1} \perp_{\xi_{2k+1}^2}^{\mathcal{R}_{1,2k-1}^{\text{odd}} \vee \mathfrak{R}}$ with respect to μ_y , i.e.,

$$H_y(\mathcal{R}_{2k+1}|\mathcal{R}_{1,2k-1}^{\text{odd}} \vee \mathfrak{R}) > H_y(\mathcal{R}_{2k+1}) - \xi_{2k+1}^2,$$

which obviously implies

$$(3.4) \quad H_y(\mathcal{R}_{2k+1}|\mathcal{R}_{1,2k-1}^{\text{odd}} \vee \mathfrak{R}) > H_y(\mathcal{R}_{2k+1}|\mathfrak{R}) - \xi_{2k+1}^2.$$

By using exactly the same calculation as we used to pass from (3.1) to (3.2) we deduce from (3.4) that there exists a set of atoms z of the sigma-algebra \mathfrak{R} , of joint measure μ_y at least $1 - \xi_{2k+1}$, such that

$$(3.5) \quad H_{yz}(\mathcal{R}_{2k+1}|\mathcal{R}_{1,2k-1}^{\text{odd}}) > H_{yz}(\mathcal{R}_{2k+1}) - \xi_{2k+1},$$

i.e., $\mathcal{R}_{1,2k-1}^{\text{odd}} \perp_{\xi_{2k+1}}^{\mathcal{R}_{2k+1}}$ with regard to the measure μ_{yz} (the disintegration measure of μ_y with respect to \mathfrak{R} on the atom z , the same as the disintegration of μ with respect to $\Pi \vee \mathfrak{R}$ at $y \cap z$). We can require that the sequence ξ_k is summable to a number smaller than $\frac{1}{2}$ (this holds for sufficiently fast decreasing parameters ε_k) and then there exists a set Z of measure ν_y larger than $\frac{1}{2}$ of atoms z of \mathfrak{R} satisfying for all k ,

$$(a) \quad \mathcal{R}_{1,2k-1}^{\text{odd}} \perp_{\xi_{2k+1}}^{\mathcal{R}_{2k+1}} \quad \text{with regard to the measure } \mu_{yz}.$$

Notice the obvious fact that (a) holds if we replace the sequence of partitions (\mathcal{R}_k) by any subsequence (\mathcal{R}_{j_k}) such that $k \rightarrow j_k$ preserves parity. By passing to a subsequence we can control the speed of growth of the parameters n_k and of decay of the parameters ε_k .

Because $y \in Y$, each partition \mathcal{R}_{2k} is μ_y -roughly equal with the parameters ε_{2k} , n_{2k} , h_{2k} , and it follows from (3.2) that with respect to μ_y it is ε_{2k} -independent of the preceding even-indexed partitions.

We are now going to apply the first part of Lemma 2.4 inductively (for the first time; we will do it again, later). For $k = 1$ we apply it with $\alpha = \varepsilon_2, \beta = \varepsilon_3, \gamma = \varepsilon_4, n = n_2$ and $m = n_4$ and obtain that if ε_4 is small and n_4 large enough (this can be achieved by passing to a subsequence of (\mathcal{R}_k)), then $\mathcal{R}_2 \vee \mathcal{R}_4$ is μ_y -roughly equal with the parameters $\varepsilon_2 + \varepsilon_3 + \varepsilon_4, n_4, h_4$. Repeating this argument we easily deduce that, by passing to a subsequence, we can arrange $\mathcal{R}_{2,2k}^{\text{even}}$ to be μ_y -roughly equal with parameters $\varepsilon_2 + \varepsilon_3 + \cdots + \varepsilon_{2k}, n_{2k}, h_{2k}$, for every k . We denote the first parameter (i.e., $\sum_{i=2}^{2k} \varepsilon_i$) by s_k . The following three properties (which we have obtained):

- (1) $\mathcal{R}_{2,2k}^{\text{even}}$ is μ_y -roughly equal with parameters s_k, n_{2k}, h_{2k} ,
- (2) \mathcal{R}_{2k+1} is μ_y -roughly equal with parameters $\varepsilon_{2k+1}, n_{2k+1}, h$,
- (3) $\mathcal{R}_{2,2k}^{\text{even}} \perp^{\varepsilon_{2k+1}} \mathcal{R}_{2k+1}$ with respect to μ_y (a consequence of (3.2)),

are open properties of the probability vector assigned by μ_y to the finite partition $\mathcal{R}_{1,2k+1}$. It follows from (3.3) that $\mathcal{R}_{1,2k+1} \perp^{\xi_{2k+2}^2} (\mathcal{R}_{2k+2} \vee \mathcal{R}_{2k+4} \vee \cdots)$ with respect to μ_y , so, by (2.1), if we arrange ξ_{2k+2}^2 small enough, we will have these properties on a large (say, larger than $1 - \varepsilon_{2k}$) set of atoms v of $(\mathcal{R}_{2k+2} \vee \mathcal{R}_{2k+4} \vee \cdots)$:

- (1) $\mathcal{R}_{2,2k}^{\text{even}}$ is μ_{yv} -roughly equal with parameters s_k, n_{2k}, h_{2k} ,
- (2) \mathcal{R}_{2k+1} is μ_{yv} -roughly equal with parameters $\varepsilon_{2k+1}, n_{2k+1}, h$,
- (3) $\mathcal{R}_{2,2k}^{\text{even}} \perp^{\varepsilon_{2k+1}} \mathcal{R}_{2k+1}$ with respect to μ_{yv} .

We can now apply the second part of Lemma 2.4 (with $\alpha = s_k, \beta = \varepsilon_{2k}, \gamma = \varepsilon_{2k+1}$) to deduce that (with a good choice of ε_{2k+1}) \mathcal{R}_{2k+1} is μ_{yvu} -roughly equal with the parameters $\varepsilon_{2k}, n_{2k+1}, h$ on a set of atoms u of $\mathcal{R}_{2,2k}^{\text{even}}$ of joint measure μ_{yv} at least $1 - s_k - \varepsilon_{2k}$ (which is larger than $1 - \varepsilon_0$). Notice that the intersection $v \cap u$ is an atom z of \mathfrak{R} . In this manner we have derived:

- (b) \mathcal{R}_{2k+1} is μ_{yz} -roughly equal with parameters $\varepsilon_{2k}, n_{2k+1}, h$ on a set of atoms z of \mathfrak{R} of joint measure μ_y larger than $1 - \varepsilon_0$.

It is now obvious that there exists a set Z' of measure μ_y larger than $1 - \varepsilon_0$ consisting of atoms z of \mathfrak{R} satisfying (b) for infinitely many indices k . We choose an atom z from the nonempty intersection $Z \cap Z'$.

At this point we intrude on our proof with an ingredient needed for the topological requirements. We address the “separating set” P_0 . By the Ergodic Theorem, we can arrange the lengths n_k so large that for a set of points $x \in X$ of measure $1 - \varepsilon_k$, $T^n x$ visits P_0 for no more than $n_k \delta$ indices $n \in [a_k, b_k - 1]$. In a set X' of measure at least $1 - \varepsilon_0$, this happens for every k . It is clear that the collection of atoms $y \cap z$ of $\Pi \vee \mathfrak{R}$ for which $\mu_{yz}(X') < 1 - \sqrt{\varepsilon_0}$ has joint measure at most $\sqrt{\varepsilon_0}$. Thus, for ε_0 small enough, we can adjust the choices of $y \in Y \cap Y'$ and $z \in Z \cap Z'$ so that $\mu_{yz}(X') \geq 1 - \sqrt{\varepsilon_0}$. Now we can use our Fact 1. If we replace μ_{yz} by its conditional measure on X' , then for every k , \mathcal{R}_{2k+1} remains roughly equal with the parameters $\varepsilon'_{2k+1}, n_{2k+1}, h$ (where ε'_{2k+1} is slightly larger than $\frac{\varepsilon_{2k+1}}{1 - \sqrt{\varepsilon_0}}$). By an adjustment of the parameters we will skip the “prime” and still write ε_{2k+1} . We will also write μ_{yz} (rather than $\mu_{yz|X'}$), remembering that we are dealing with the restriction to X' . It is also easy to see that this restriction does not affect the independences arranged in (a) (again, a small adjustment of ε_{2k+1} does the job). From now on we will think that all our points belong to the atom $y \cap z$ (which supports the conditional measure μ_{yz}) intersected with X' ; in particular, all our partitions are treated as partitions of $y \cap z \cap X'$.

We would like to pass to the subsequence of S along which (b) is fulfilled for z . This time, however, we cannot affect the choice of z (and thus neither the construction of \mathfrak{R}). So, we keep all even-indexed partitions \mathcal{R}_{2k} and we select the subsequence only from the odd-indexed partitions. In this manner we can assume that the atom z satisfies both (a) and (b) for all k .

We shall inductively use the first part of Lemma 2.4 for the second time, now with respect to the measure μ_{yz} . In order to have the same parameter in (a) and (b), we set $\delta_k = \max\{\xi_k, \varepsilon_{k-1}\}$ and require that these numbers form a summable sequence with a small sum δ_0 . Now we have an almost identical situation as in the first inductive application, except that we look at the odd-indexed (rather than even-indexed) partitions \mathcal{R}_{2k+1} and have the parameters δ_{2k+1} in place of the former ε_{2k} . As a result we get that, by passing to a subsequence, we can arrange each of the partitions $\mathcal{R}_{1,2k+1}^{\text{odd}}$ to be μ_{yz} -roughly equal with the parameters r_k, n_{2k+1}, h , where $r_k = \sum_{i=1}^{2k+1} \delta_i$.

The second part of Lemma 2.4 (with $\alpha = r_{k-1}, \beta = \delta_{2k}$ and $\gamma = \delta_{2k+1}$) now gives: on atoms B of the partition $\mathcal{R}_{1,2k-1}^{\text{odd}}$, of joint measure μ_{yz} at least $1 - r_k - \delta_{2k}$ (which is larger than $1 - 2\delta_0$), the partition \mathcal{R}_{2k+1} is $\mu_{yz|B}$ -roughly equal with the parameters δ_{2k}, n_{2k+1}, h . Let V_k denote the union of the good atoms B for which this holds. Using Remark 1, we can arrange that the sets V_k decrease.

We can now draw the following conclusion (h' denotes a constant slightly smaller than h):

- If B is an atom of $\mathcal{R}_{1,2k-1}^{\text{odd}}$ contained in V_k , then it contains at least $2^{n_{2k+1}h'}$ (nonempty) atoms of $\mathcal{R}_{1,2k+1}^{\text{odd}}$ contained in V_{k+1} .

3.2. Counting Hamming balls. Let l denote the cardinality of \mathcal{P} . In the symbolic representation of the process (X, \mathcal{P}, μ, T) , the partition \mathcal{R}_{2k+1} depends on the interval of coordinates $[a_{2k+1}, b_{2k+1} - 1]$ of length n_{2k+1} , and its atoms can be identified with blocks over the alphabet \mathcal{P} . The *Hamming distance* between two such blocks is the number of positions where the blocks disagree divided by n_{2k+1} .

It is a standard fact that any ball of radius δ in the Hamming distance around a block of length n has at most $2^{n(H(\delta, 1-\delta) + \delta \log l)}$ elements (other blocks). For a small enough δ the term $\beta(\delta) = H(3\delta, 1 - 3\delta) + 3\delta \log l$ is smaller than h' (the constants h' and δ depend only on h , and they can be established at the start of the proof).

We will now indicate a family B_κ , where κ ranges over all finite binary blocks, such that B_κ is a (nonempty) atom of $\mathcal{R}_{1,2k+1}^{\text{odd}}$ contained in V_{k+1} , where k is the length of κ (for $k = 0$, κ is the empty word). We will assure that if ι extends κ to the right, then $B_\iota \subset B_\kappa$. Moreover, we will arrange that if $\kappa \neq \kappa'$, both have the same length k , and differ at a position i_0 , then the Hamming distance between A_i and A'_i is at least 3δ for all $i = i_0, i_0 + 1, \dots, k$, where A_i and A'_i are the blocks corresponding to the coordinates $[a_{2i+1}, b_{2i+1} - 1]$ appearing in the representation of the atoms B_κ and $B_{\kappa'}$, respectively. We will do it by induction on k ; in each step we choose two “children” of every thus far constructed atom of $\mathcal{R}_{1,2k-1}^{\text{odd}}$.

In step $k = 0$ we assign B_\emptyset to be an arbitrarily selected atom of \mathcal{R}_1 contained in V_1 . Suppose the task has been completed for some $k - 1$, i.e., that we have selected 2^{k-1} atoms B_κ of $\mathcal{R}_{1,2k-1}^{\text{odd}}$, contained in V_{k-1} , and pairwise 3δ -separated as required. We know that every such B_κ contains at least $2^{n_{2k+1}h'}$ atoms of $\mathcal{R}_{1,2k+1}^{\text{odd}}$ contained in V_{k+1} . Every such atom has the form $B_\kappa \cap A$ where A is an atom of \mathcal{R}_{2k+1} . We will call the atoms A such that $B_\kappa \cap A$ is contained in V *good continuations* of B_κ .

Let $\mathcal{A}(B_\kappa)$ denote the family of all good continuations of B_κ . Take the first selected atom B_{κ_1} (for $\kappa_1 = 000\dots 0$). Choose one good continuation A_0 of B_{κ_1} . From every family $\mathcal{A}(B_\kappa)$ (including $\kappa = \kappa_1$) we eliminate (for future choices) all the atoms A that belong to the Hamming ball of radius 3δ around A_0 . Every family $\mathcal{A}(B_\kappa)$ has “lost” at most $2^{n_{2k+1}\beta(\delta)}$ elements (out of $2^{n_{2k+1}h'}$), so a vast majority remains. Next we choose A_1 from the remaining good continuations of B_{κ_1} and, again, from each of the families $\mathcal{A}(B_\kappa)$ we eliminate all the atoms A that belong to the Hamming ball of radius 3δ around A_1 . Again, the losses are negligibly small. We assign $B_{\kappa_1 0} = B_{\kappa_1} \cap A_0$ and $B_{\kappa_1 1} = B_{\kappa_1} \cap A_1$ (here $\kappa_1 0$ and $\kappa_1 1$ denote the two continuations of κ_1).

Next we abandon B_{κ_1} and pass to B_{κ_2} (earlier we need to order the κ 's somehow), and we repeat the procedure choosing two of its good continuations, say A'_0, A'_1 , not eliminated in the preceding steps, each time eliminating for future choices all members of the respective Hamming balls. We proceed until we choose two good continuations for every κ of length $k - 1$. Note that near the end of this procedure we will have eliminated less than $2^{k+n_{2k+1}\beta(\delta)}$ blocks from each family $\mathcal{A}(B_\kappa)$, which is still a negligible fraction of the cardinality of $\mathcal{A}(B_\kappa)$; hence the procedure will be possible until the end.

3.3. Verification of DC2. From now on we use the letter κ to denote infinite binary sequences. For every κ we define a set C_κ as the decreasing intersection $\bigcap_k \overline{B_{\kappa_k}}$, where κ_k is the initial string of length k of κ , and the closure is taken in X (the sets B_{κ_k} are understood as subsets of $y \cap z \cap X'$; still, we close them in the entire compact space X). In this manner we have defined a family of nonempty closed sets indexed by the binary sequences. The scrambled set E is obtained by selecting one point from every set C_κ (a priori we have no guarantee that the sets C_κ are pairwise disjoint, but this will become obvious if we show that every pair of points in E is scrambled).

Take two points, x_1, x_2 from E . They belong to the closure of the same atom z of \mathfrak{R} (also to the closure of the same atom y of Π , but this fact has no importance). Any pair of points x, x' belonging to z (without the closure) satisfy $\text{dist}(T^n x, T^n x') \leq \varepsilon_k$ for all k and $n \in [a_{2k}, b_{2k} - 1]$. This property obviously passes to all pairs in the closure of the atom z , in particular to x_1, x_2 . Since $\frac{b_{2k}}{a_{2k}}$ tends to infinity, it is very easy to see that for every $t > 0$ the orbits of our two points are at least t apart along a set of times of upper density 1. We have shown that E satisfies the first requirement of being DC2-scrambled.

On the other hand, our two points belong to the sets C_κ for two κ 's that differ at some position i_0 . This implies that for each $k \geq i_0$ they belong to the closures of two atoms $B_{\kappa_k}, B_{\kappa'_k}$, where κ_k and κ'_k are binary strings of length k differing at i_0 . We will first analyze the behavior of a pair x, x' belonging to the atoms $B_{\kappa_k}, B_{\kappa'_k}$, respectively (without the closures). By the construction, we know that the Hamming distance between the blocks A_{κ_k} and $A_{\kappa'_k}$, representing the atoms of \mathcal{R}_{2k+1} to which x and x' belong, is at least 3δ . This means that $T^n x, T^n x'$ belong to different atoms of \mathcal{P} for at least $n_{2k+1}3\delta$ times $n \in [a_{2k+1}, b_{2k+1} - 1]$. Because we treat these atoms as subsets of X' , the situation, where at least one of the aforementioned atoms of \mathcal{P} is P_0 , may happen at most $n_{2k+1}2\delta$ times, leaving at least $n_{2k+1}\delta$ times n when the two points fall in two different atoms of \mathcal{P} , different from P_0 . By the property of P_0 , the distance between these points is at least t_0 .

Now observe that the property

- $\text{dist}(T^n x, T^n x') \geq t_0$ for at least $n_{2k+1}\delta$ times $n \in [a_{2k+1}, b_{2k+1} - 1]$

is closed; hence it passes to the pairs of points selected from the closures of the respective atoms. We deduce that our points x_1, x_2 have this property for every $k \geq i_0$. Since $\frac{b_{2k+1}}{a_{2k+1}}$ tends to infinity, this easily proves that $\text{dist}(T^n x, T^n x') \geq t_0$ with upper density at least δ . In particular, the points x_1, x_2 satisfy the second requirement in the definition of a DC2-scrambled pair, completing the proof. \square

4. FINAL REMARKS

Remark 2. Notice that our scrambled set has additional properties: the parameters of scrambling (i.e., the number t_0 and the lower density δ) are common for all pairs (which is good, in the sense that the chaos is strong). Such a situation is called *uniform DC2* (see [BS]; see also *uniform DC1* in [O2] and *uniform Li-Yorke chaos* in [AGHSY]).

On the other hand, the scrambling is “synchronic”; i.e., all points in the scrambled set get together at the same times (at least we do not control how often they get together pairwise at other times). Such behavior is somewhat too regular to satisfy what one intuitively expects from “chaos” (one expects some kind of independence in the behavior of points). But this is how chaos is defined – the definition admits “synchronic chaos”. It might be interesting to invent and study a stronger notion of chaos which imposes (with positive upper density) the condition that some pairs be together at the same time as other pairs are apart. The author is aware that such definitions are around the corner. In particular, one might ask whether such a strengthened chaos follows from positive entropy.

Remark 3. This remark is due to Piotr Oprocha. The author thanks him for the contribution. Šmítal also conjectured that in the positive entropy case the DC2-scrambled set can be obtained as perfect (see e.g. [S2]). In fact, we can prove this conjecture as well. The sets C_κ constructed in Section 3.2 are disjoint (this follows from the separation by the distance t_0 proved later), compact and have a compact union. It is also seen from the construction that “belonging in the same C_κ ” is a Borel equivalence relation on the union. A special case of Silver’s Theorem (see [Sr], Theorem 5.13.9) implies that if the classes of a Borel equivalence relation are of type F_σ and there are uncountably many of them, then there exists a perfect set having at most one element in every class. It is easy to see (by the rules of separation) that this perfect set is a Cantor set.

REFERENCES

- [AGHSY] E. Akin, E. Glasner, W. Huang, S. Shao, and X. Ye, *Sufficient conditions under which a transitive system is chaotic*, Ergodic Theory Dynam. Systems **30** (2010), no. 5, 1277–1310, DOI 10.1017/S0143385709000753. MR2718894 (2011j:37014)
- [BS] Marta Babilonová-Štefánková, *Extreme chaos and transitivity*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **13** (2003), no. 7, 1695–1700, DOI 10.1142/S02181274030007540. Dynamical systems and functional equations (Murcia, 2000). MR2015619 (2004i:37075)
- [BHS] François Blanchard, Wen Huang, and L’ubomír Snoha, *Topological size of scrambled sets*, Colloq. Math. **110** (2008), no. 2, 293–361, DOI 10.4064/cm110-2-3. MR2353910 (2008j:37013)
- [BGKM] François Blanchard, Eli Glasner, Sergiï Kolyada, and Alejandro Maass, *On Li-Yorke pairs*, J. Reine Angew. Math. **547** (2002), 51–68, DOI 10.1515/crll.2002.053. MR1900136 (2003g:37014)

- [D] Tomasz Downarowicz, *Entropy in dynamical systems*, New Mathematical Monographs, vol. 18, Cambridge University Press, Cambridge, 2011. MR2809170 (2012k:37001)
- [LY] Tien Yien Li and James A. Yorke, *Period three implies chaos*, Amer. Math. Monthly **82** (1975), no. 10, 985–992. MR0385028 (52 #5898)
- [O1] P. Oprocha: *Minimal systems and distributionally scrambled sets*, Bull. Soc. Math. France **140** (2012), no. 3, 401–439. MR3059120
- [O2] Piotr Oprocha, *Distributional chaos revisited*, Trans. Amer. Math. Soc. **361** (2009), no. 9, 4901–4925, DOI 10.1090/S0002-9947-09-04810-7. MR2506431 (2010d:37019)
- [P] Rafał Pikuła, *On some notions of chaos in dimension zero*, Colloq. Math. **107** (2007), no. 2, 167–177, DOI 10.4064/cm107-2-1. MR2284159 (2007m:37021)
- [RS] V. A. Rohlin and Ja. G. Sinaĭ, *The structure and properties of invariant measurable partitions*, Dokl. Akad. Nauk SSSR **141** (1961), 1038–1041 (Russian). MR0152629 (27 #2604)
- [SS] B. Schweizer and J. Smítal, *Measures of chaos and a spectral decomposition of dynamical systems on the interval*, Trans. Amer. Math. Soc. **344** (1994), no. 2, 737–754, DOI 10.2307/2154504. MR1227094 (94k:58091)
- [S1] J. Smítal, *Chaotic functions with zero topological entropy*, Trans. Amer. Math. Soc. **297** (1986), no. 1, 269–282, DOI 10.2307/2000468. MR849479 (87m:58107)
- [S2] Jaroslav Smítal, *Topological entropy and distributional chaos*, Real Anal. Exchange **30th Summer Symposium Conference** (2006), 61–65. MR2323823 (2008b:37015)
- [Sr] S. M. Srivastava, *A course on Borel sets*, Graduate Texts in Mathematics, vol. 180, Springer-Verlag, New York, 1998. MR1619545 (99d:04002)

INSTITUTE OF MATHEMATICS AND COMPUTER SCIENCE, WROCLAW UNIVERSITY OF TECHNOLOGY, WYBRZEŻE WYSPIAŃSKIEGO 27, 50-370 WROCLAW, POLAND

E-mail address: `downar@pwr.wroc.pl`