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Positively curved manifolds with symmetry

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Abstract

There are very few examples of Riemannian manifolds with positive sectional curvature known. In fact in dimensions above 24 all known examples are diffeomorphic to locally rank one symmetric spaces. We give a partial explanation of this phenomenon by showing that a positively curved, simply connected, compact manifold (M, g) is up to homotopy given by a rank one symmetric space, provided that its isometry group $\mathrm{Iso}(M, g)$ is large. More precisely we prove first that if $\dim(\mathrm{Iso}(M, g)) \geq 2\dim(M) - 6$, then M is tangentially homotopically equivalent to a rank one symmetric space or M is homogeneous. Secondly, we show that in dimensions above $18(k+1)^2$ each M is tangentially homotopically equivalent to a rank one symmetric space, where $k > 0$ denotes the cohomogeneity, $k = \dim(M/\mathrm{Iso}(M, g))$.

Introduction

Studying positively curved manifolds is a classical theme in differential geometry. So far there are very few constraints known. For example there is not a single obstruction known that distinguishes the class of simply connected compact manifolds that admit positively curved metrics from the class of simply connected compact manifolds that admit nonnegatively curved metrics. On the other hand the list of known examples is rather short as well. In particular, in dimensions other than 6, 7, 12, 13 and 24 all known simply connected positively curved examples are diffeomorphic to rank one symmetric spaces. To advance the theory, Grove (1991) proposed to classify positively curved manifolds with a large amount of symmetry. This program may also be viewed as part of a philosophy of W.-Y. Hsiang that in each category one should pay particular attention to those objects with a large amount of symmetry. Another possible motivation is that once one understands the obstructions to positive curvature under symmetry assumptions one might get an idea for a general obstruction. Our investigations here will also give new insights for orbit spaces of linear group actions on spheres which — when applied to slice representations — have consequences for general group actions as well. However, the main hope is that the classifying process will lead toward the construction of new examples.

The three most natural constants measuring the amount of symmetry of a Riemannian manifold (M, g) are:

$$\begin{aligned} \operatorname{symrank}(M,g) &= \operatorname{rank}\big(\operatorname{Iso}(M,g)\big), \\ \operatorname{symdeg}(M,g) &= \dim\big(\operatorname{Iso}(M,g)\big), \\ \operatorname{cohom}(M,g) &= \dim\big((M,g)/\operatorname{Iso}(M,g)\big), \end{aligned}$$

where $\operatorname{Iso}(M,g)$ denotes the isometry group of (M,g). In [22] we analyzed manifolds where the symmetry rank is large, and obtained extensions of results of Grove and Searle [11]. The main new tool was the observation that for a totally geodesic embedded submanifold N^{n-h} of a positively curved manifold M^n the inclusion map $N^{n-h} \to M^n$ is $(n-2h+1)$-connected; see Theorem 1.2 (connectedness lemma) below for a definition and further details. The result is also crucial for the present paper in which we consider positively curved manifolds that have either large symmetry degree or low cohomogeneity. The main results in this context are

Theorem 1. *Let (M^n, g) be a simply connected Riemannian manifold of positive sectional curvature. If* $\operatorname{symdeg}(M^n, g) \geq 2n - 6$, *then M is tangentially homotopically equivalent to a rank one symmetric space or isometric to a homogeneous space of positive sectional curvature.*

Theorem 2. *Let M be a simply connected positively curved manifold. Suppose*

$$\operatorname{symrank}(M,g) > 3\operatorname{cohom}(M,g) + 3.$$

Then M is tangentially homotopically equivalent to a rank one symmetric space or $\operatorname{cohom}(M,g) = 0$.

Corollary 3. *Let $k \geq 1$. In dimensions above $18(k+1)^2$ each simply connected Riemannian manifold M^n of cohomogeneity k with positive sectional curvature is tangentially homotopically equivalent to a rank one symmetric space.*

We recall that a homotopy equivalence between manifolds $f\colon M_1 \to M_2$ is called tangential if the pull back bundle f^*TM_2 is stably isomorphic to the tangent bundle TM_1. It is known that a compact manifold has the tangential homotopy type of $\mathbb{HP}^m$ if and only if it is homeomorphic to $\mathbb{HP}^m$. In general it is known that while there are infinitely many diffeomorphism types of simply connected homotopy $\mathbb{CP}^n$'s in a given even dimension $2n > 4$ there are only finitely many with the tangential homotopy type of a rank one symmetric space. For the case of a nonsimply connected manifold M we refer the reader to the end of Section 13.

In dimension seven Theorem 1 is optimal, as there are nonhomogeneous positively curved Eschenburg examples $\mathsf{SU}(3)/\!/\mathsf{S}^1$ with symmetry degree 7. The simply connected positively curved homogeneous spaces have been classified by Berger [4], Wallach [20] and Berard Bergery [3]. By this classification, exceptional spaces — spaces which are not diffeomorphic to rank one symmetric spaces — only occur in dimensions 6, 7, 12, 13 and 24, and all of these spaces satisfy the hypothesis of Theorem 1.

Of course this classification also implies that Corollary 3 remains valid with $k = 0$ if one replaces the lower bound by 24. Verdiani [19] has shown that an even dimensional simply connected positively curved cohomogeneity one manifold is diffeomorphic to a rank one symmetric space. This fails in odd dimensions where a classification is open. In higher cohomogeneity ($k \geq 2$) only very little is known. A notably exception is the theorem of Hsiang and Kleiner [14] stating that a compact positively curved orientable four manifold is homeomorphic to $\mathbb{S}^4$ or $\mathbb{CP}^2$, provided that it admits a nontrivial isometric action by S^1. Grove and Searle realized that the proof of this result can be phrased naturally in terms of Alexandrov geometry of the orbit space M^4/S^1 which in turn allowed them to classify fixed-point homogeneous manifolds of positive sectional curvature; see Section 1 for a definition.

To the best of the authors knowledge there are no manifolds known which have a large amount of symmetry and which are homotopically equivalent but not diffeomorphic to $\mathbb{CP}^n$ or $\mathbb{HP}^n$. So it is quite possible that one could improve the conclusions of Theorem 1 and Corollary 3 for purely topological reasons. If the manifold M^n in Corollary 3 is a homotopy sphere, we can combine the connectedness lemma (Theorem 1.2) with the work of Davis and Hsiang [7] to strengthen its conclusion. Recall that for suitably chosen p and q the Brieskorn variety

$$\Sigma^{2m-1}(p,q) := \left\{(z_0,\ldots,z_m) \in \mathbb{C}^{m+1} \;\middle|\; z_0^p + z_1^q + z_2^2 + \cdots + z_m^2 = 0\right\} \cap \mathbb{S}^{2m+1}$$

is homeomorphic to a sphere; see Brieskorn [6]. Clearly $\Sigma^{2m-1}(p,q)$ is invariant under an action of $\mathsf{O}(m-1)$.

Theorem 4. *Let (M^n, g) be a homotopy sphere admitting a positively curved cohomogeneity k metric with $n \geq 18(k+1)^2$. Then there is an effective isometric action of $\mathsf{Sp}(d)$ on M with $d \geq \frac{n+1}{4(k+1)} - 2$ such that one of the following holds.*

a) *M^n is equivariantly diffeomorphic to $\mathbb{S}^n$ endowed with an action of $\mathsf{Sp}(d)$, which is induced by a representation $\rho\colon \mathsf{Sp}(d) \to \mathsf{O}(n+1)$.*

b) *The dimension* $n = 2m+1$ *is odd, and* M *is equivariantly diffeomorphic to* $\Sigma^{2m+1}(p,q)$ *endowed with an action of* $\mathsf{Sp}(d)$ *induced by a representation* $\rho\colon \mathsf{Sp}(d) \to \mathsf{O}(m)$.

In either case the representation ρ decomposes as a trivial and r times the $4d$-dimensional standard representation of $\mathsf{Sp}(d)$, where $r \leq d/2$ in case a) and $r \leq d/4$ in case b). In even dimensions the theorem implies that M is diffeomorphic to a sphere. We do not claim that $\mathsf{Sp}(d)$ can be chosen as a normal subgroup of $\mathrm{Iso}(M,g)_0$, but see also Proposition 14.1.

The above results do not provide any evidence for new examples. On the other hand, Theorem 2 suggests that it might be realistic to classify positively curved manifolds of low cohomogeneity (say one or two) in all dimensions. At least the new techniques introduced here should allow one to reduce the problem to a short list of possible candidates.

Next we want to mention some of the new tools that we establish during the proof of the above results. We adopt a philosophy promoted by Grove and Searle and view group actions on positively curved manifolds as generalized representations. The main strategy is to establish a common behavior. In some instances the results might not be trivial for representations either. A central theme is to gain control on the principal isotropy group of the isometric group action. The first crucial new tool in this context is

Lemma 5 (Isotropy Lemma). *Let* G *be a compact Lie group acting isometrically and not transitively on a positively curved manifold* (M,g) *with nontrivial principal isotropy group* H. *Then any nontrivial irreducible subrepresentation of the isotropy representation of* G/H *is equivalent to a subrepresentation of the isotropy representation of* K/H, *where* K *is an isotropy group.*

We will also see that one may choose K such that the orbit type of K has codimension 1 in the orbit space. In that case K/H is a sphere. In particular, the orbit space must have a boundary if H is not trivial. For an orbit space M/G with boundary, a face is the closure of a component of a codimension 1 orbit type. A face is necessarily part of the boundary and the boundary may or may not have more than one face.

It turns out that the lemma is useful for general group actions on manifolds, as well. The lemma applied to slice representations plays a vital role in the proof of the following theorem which does not need curvature assumptions. We recall that for a group action of a Lie group G on a manifold M with principal orbit G/H the core M_{cor} (or principal reduction) is defined as the union of those components of the fixed-point set $\mathrm{Fix}(\mathsf{H})$ of H that project surjectively to M/G. We define a core domain of such a group action as follows. Let $M_{\mathrm{pr}} \subset M$ be the open and dense subset of principal orbits, and let B_{pr} be a component of the fixed-point set of H in M_{pr}. Then a core domain is the closure of B_{pr} in M. Clearly $\bar{B}_{\mathrm{pr}}$ is invariant under the action of the identity component $N(\mathsf{H})_0$ of the normalizer of H.

THEOREM 6. *Let G be a connected compact Lie group acting smoothly on a simply connected manifold M with principal isotropy group H. Choose not necessarily different points $p_1, \ldots, p_f$ in a core domain $\bar{B}_{\mathrm{pr}}$ such that each of the f faces of M/G contains at least one of the orbits $\mathsf{G} \star p_1, \ldots, \mathsf{G} \star p_f$.*

If $\mathsf{K} \subset \mathsf{G}$ is a compact subgroup containing $N(\mathsf{H})_0$ as well as the isotropy groups of the points $p_1, \ldots, p_f$, then there is an equivariant smooth map $M \to \mathsf{G}/\mathsf{K}$.

Notice that if all faces of the orbit space intersect, one may choose $p_1 = \cdots = p_f$ as one point on the orbit of this intersection. If the orbit space has no boundary, one may choose $\mathsf{K} = N(\mathsf{H})_0$. The theorem should be useful in other contexts as well, as it is a simple statement that guarantees the failure of primitivity of an action. Recall that a smooth action of a Lie group G on a manifold M is called primitive if there is no smooth equivariant map $M \to \mathsf{G}/\mathsf{L}$ with $\mathsf{L} \subsetneq \mathsf{G}$.

As a consequence of Theorem 6 we show that the identity component of H decomposes in at most $2f$ factors, provided that we assume in addition that the action is primitive (Corollary 11.1) or that it leaves a positively curved complete metric invariant (Corollary 12.1). This way one gets restrictions on the principal isotropy group in terms of the geometry (number of faces) of the orbit space.

In order to control the latter one uses Alexandrov geometry. Recall that the orbit space $(M, g)/\mathsf{G}$ of an isometric group action on a positively curved manifold is positively curved in the Alexandrov sense. It is then easy to see that the distance function of a face F in M/G is strictly concave. This elementary observation can be utilized to give an optimal upper bound on the number of faces.

THEOREM 7. *Let G be a compact Lie group acting almost effectively and isometrically on a compact manifold (M, g) with a positively curved orbit space $(M, g)/\mathsf{G}$ of dimension k. Then*:

a) *The number of faces of the orbit space is bounded by $(k + 1)$. If equality holds then M/G is a stratified space homeomorphic to a k-simplex.*

b) *If the orbit space has $l + 1 < k + 1$ faces, then it is homeomorphic to the join of an l-simplex and the space that is given by the intersection of all faces.*

On positively curved orbit spaces there is also a nice duality between faces and points of maximal distance to a face. More precisely there is a unique point

$\mathsf{G} \star q \in M/\mathsf{G}$ of maximal distance to a face $F \subset M/\mathsf{G}$, and the normal bundle of the orbit $\mathsf{G} \star q \subset M$ is equivariantly diffeomorphic to the manifold that is obtained from M by removing all orbits belonging to F; see the soul orbit theorem (Theorem 4.1).

The previously mentioned tools are mainly used to control the principal isotropy group of an isometric group action on a positively curved manifold. The final tool we would like to mention assumes that one already has control on the principal isotropy group. To motivate this, consider the representation of $\mathsf{Sp}(d)$ which is given by h times the $4d$-dimensional standard representation. The principal isotropy group of this representation is given by a $(d-h)$ block. It is straightforward to check that the isometry type of the orbit space $\mathbb{R}^{4hd}/\mathsf{Sp}(d)$ is independent of d as long as $h < d$. It turns out that this stability phenomenon can be recovered in a far more general context.

Theorem 8 (Stability Theorem). *Let* (G_d, u) *be one of the pairs* $(\mathsf{Spin}(d), 1)$, $(\mathsf{SU}(d), 2)$ *or* $(\mathsf{Sp}(d), 4)$. *Suppose* G_d *acts nontrivially and isometrically on a simply connected Riemannian manifold* M^n (*no curvature assumptions*) *with principal isotropy group* H. *We assume that* H *contains a subgroup* H' *which up to conjugacy is a lower* $k \times k$ *block for some integer* $k \geq 2$ *and* $k \geq 3$ *if* $u = 1, 2$. *Assume also that* k *is maximal. Then the following are true*:

a) *There is a Riemannian manifold* M_1 *with an action of* G_{d+1}, *that contains* M *as a totally geodesic submanifold and* $\dim(M_1) - \dim(M) = u(d-k)$.

b) *The orbit spaces* M/G_d *and* M_1/G_{d+1} *are isometric and* $\mathrm{cohom}(M, g) = \mathrm{cohom}(M_1, g)$.

c) *If* $k \geq d/2$, *then the sectional curvature of* M_1 *attains its maximum and minimum in* M.

We emphasize that M_1 is not given as $M \times_{\mathsf{G}_d} \mathsf{G}_{d+1}$. Clearly one can iterate the theorem and get a chain of Riemannian manifolds

$$M =: M_0 \subset M_1 \subset \cdots,$$

where M_i admits an isometric action of G_{d+i} and all inclusions are totally geodesic.

If we assume in addition that the manifold M is compact and has an invariant positively curved metric, then we will see that M as well as M_i is tangentially homotopically equivalent to a rank one symmetric space, see Theorem 5.1. The combination of Theorem 5.1 and the isotropy lemma is also crucial for the proof of the following result.

Theorem 9. *Let* G *be a Lie group acting isometrically and with finite kernel on a positively curved simply connected Riemannian manifold* (M, g).

Suppose the principal isotropy group H *contains a simple subgroup* H' *of rank* ≥ 2*. If* $\dim(M) \geq 235$*, then* M *has the integral cohomology ring of a rank one symmetric space.*

Contents

The theorems are not proved in the order in which they are stated. In Section 1 we survey some of the results in the literature which are crucial for our paper.

Next we establish the stability theorem (Theorem 8) in Section 2. One of the main difficulties in the proof is to show that the constructed metrics are of class C^∞. We establish preliminary results in subsection 2.1 and subsection 2.3, and put the pieces together in subsection 2.4.

In Section 3 we will prove the isotropy lemma (Lemma 5) as well as several generalizations of it. The isotropy lemma guarantees that certain orbit spaces have codimension one strata (faces). In Section 4 we will show that to each face of a positively curved orbit space corresponds precisely to one soul orbit, the unique point of maximal distance to the face. Theorem 4.1 (soul orbit theorem) also summarizes some of the main properties of soul orbits. For us the main application is that the inclusion map of the soul orbit into the manifold is l-connected, provided that the inverse image of the face has codimension $l+1$ in the manifold. Theorem 4.1 is also important for the proof of Theorem 7 which is contained in Section 4, too.

Section 5 contains the first main application of the techniques established by then. Theorem 5.1 provides a sufficient criterion for a manifold to be tangentially homotopically equivalent to a compact rank one symmetric space

(CROSS). The hypothesis is the same as in the stability theorem except that we now assume an invariant metric of positive sectional curvature on the manifold. The main strategy for recovering the homotopy type of M is to consider the limit space $M_\infty = \bigcup M_i$ of the chain $M = M_0 \subset M_1 \subset \cdots$. As a consequence of the connectedness lemma (Theorem 1.2), we will show that M_∞ has a periodic cohomology ring. On the other hand, we will use the soul orbit theorem (Theorem 4.1) to show that M_∞ has the homotopy type of the classifying space of a compact Lie group. By combination of both statements it easily follows that M_∞ has the homotopy type of a point, $\mathbb{CP}^\infty$ or $\mathbb{HP}^\infty$. The connectedness lemma then allows us to recover the homotopy type of M.

It will turn out that the recovery of the tangential homotopy type is more or less equivalent to determining the isotropy representation at a soul orbit. For the latter task several theorems of Bredon on group actions on cohomology CROSS'es are very useful.

Section 6 contains another refinement of Theorem 5.1. We will show that under the hypothesis of Theorem 5.1 one can find a linear model for the simply connected manifold M. That is, M is tangentially homotopically equivalent to a rank one symmetric space S, and there is a linear action of the same group on S such that the isotropy groups of the two actions are in one to one correspondence.

In Section 7 we classify homogeneous spaces G/H, with H and G being compact and simple and with spherical isotropy representations, i.e., any nontrivial irreducible subrepresentation of the isotropy representation of G/H is transitive on the sphere. The only reason why we are interested in this problem is that, by Lemma 3.4, the identity component of the principal isotropy group of an isometric group action on a positively curved manifold has a spherical isotropy representation.

This in turn is used in Section 8, where we analyze the following situation. What pairs $(\mathsf{G}, \mathsf{H}')$ occur if we consider isometric group actions of a simple Lie group G on a positively curved manifold M whose principal isotropy group contains a simple normal subgroup H' of rank ≥ 2. It turns out that these are precisely the pairs occurring for linear actions on spheres. If we assume that the hypothesis of Theorem 5.1 is not satisfied for the action of G on M, then 14 pairs occur. This allows us to prove Theorem 9 in Section 9 and Theorem 1 in Section 10.

Section 11 might be of independent interest as it does not make use of any curvature assumptions. We prove Theorem 6 as well as applications.

In Section 12 we use these results in order to show that the principal isotropy group of an almost effective isometric group action on a positively curved compact manifold contains at most $2f$ factors, where f denotes the number of faces of the orbit space. This is essential for the proof of Theorem 2 in Section 13. We actually first prove a special case. In fact Proposition 13.1

says that the conclusion holds if $\mathrm{symrank}(M,g) > 9(\mathrm{cohom}(M,g)+1)$. The proof of this case is more straightforward and does not use the results of Section 8.

The proof of Theorem 2 can be briefly outlined as follows. We consider the cohomogeneity k action of $\mathsf{L} = \mathrm{Iso}(M,g)_0$ on the positively curved manifold (M,g). First, as has 4 observed by Püttmann [16], one can use an old lemma of Berger [4] to bound the corank of the principal isotropy group P from above by $(k+1)$,

$$\mathrm{rank}(\mathsf{P}) \geq \mathrm{rank}(\mathsf{L}) - k - 1 > 2(k+1);$$

see rank lemma (Proposition 1.4) for a slightly refined version. As mentioned above we show that P has at most $2f$ factors, where f denotes the number of faces of M/L; see Corollary 12.1. Because of $f \leq k+1$ (Theorem 7) these two statements yield the first crucial step in the proof of Theorem 2, namely the principal isotropy group P contains a simple normal subgroup of rank at least 2. It is then straightforward to show that this subgroup is contained in a normal simple subgroup G of L. Thereby we obtain an isometric action of a simple group G on M whose principal isotropy group H contains a simple normal subgroup H' of rank at least 2. Using Theorem 8.1 we are able to show that for a suitable choice of G the hypothesis of the stability theorem (Theorem 8) is satisfied, unless possibly M is fixed-point homogeneous with respect to a $\mathsf{Spin}(9)$-subaction. Thus we can either apply Theorem 5.1 or Grove and Searle's [11] classification of fixed-point homogeneous manifolds.

In Section 14 we prove Theorem 4 as well as Corollary 3. The proof also shows that for any n-manifold M satisfying the hypothesis of Corollary 3 there is a sequence of positively curved manifolds

$$M^n \subset M_1^{n+h} \subset M_2^{n+2h} \subset \cdots$$

all of which have cohomogeneity k. Furthermore all inclusions are totally geodesic embeddings and $h \leq 4k+4$. This might be useful for further applications, for example if one wants to recover more than just the tangential homotopy type.

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1. Preliminaries

According to Grove and Searle [11] a Riemannian manifold is called fixed-point homogeneous if there is an isometric nontrivial nontransitive action of a Lie group G such that $\dim(M/\mathsf{G}) - \mathrm{Fix}(\mathsf{G}) = 1$ or equivalently there is a component N of the fixed-point set $\mathrm{Fix}(\mathsf{G})$ such that G acts transitively on a normal sphere of N.

THEOREM 1.1 (Grove-Searle). *Let M be a compact simply connected manifold of positive sectional curvature. If M is fixed-point homogeneous, then M is equivariantly diffeomorphic to a rank one symmetric space endowed with a linear action.*

The following theorem (connectedness lemma) was proved by the author in [22].

THEOREM 1.2 (Connectedness Lemma). *Let M^n be a compact positively curved Riemannian manifold.*

a) *Suppose $N^{n-k} \subset M^n$ is a compact totally geodesic embedded submanifold of codimension k. Then the inclusion map $N^{n-k} \to M^n$ is $(n-2k+1)$-connected. If there is a Lie group G acting isometrically on M^n and fixing N^{n-k} pointwise, then the inclusion map is $\big(n-2k+1+\delta(\mathsf{G})\big)$-connected, where $\delta(\mathsf{G})$ is the dimension of the principal orbit.*

b) *Suppose $N_1^{n-k_1}, N_2^{n-k_2} \subset M^n$ are two compact totally geodesic embedded submanifolds, $k_1 \le k_2$, $k_1+k_2 \le n$. Then the intersection $N_1^{n-k_1} \cap N_2^{n-k_2}$ is a totally geodesic embedded submanifold as well, and the inclusion map*

$$N_1^{n-k_1} \cap N_2^{n-k_2} \longrightarrow N_2^{n-k_2}$$

is $(n-k_1-k_2)$-connected.

Recall that a map $f\colon N \to M$ between two manifolds is called h-connected, if the induced map $\pi_i(f)\colon \pi_i(N) \to \pi_i(M)$ is an isomorphism for $i < h$ and an epimorphism for $i = h$. If f is an embedding, this is equivalent to saying that up to homotopy M can be obtained from $f(N)$ by attaching cells of dimension $\ge h+1$.

Since fixed-point components of isometries are totally geodesic, Theorem 1.2 turns out to be a very powerful tool in analyzing positively curved manifolds with symmetry. In fact by combining the theorem with the following lemma (see [22]), one sees that a totally geodesic submanifold of low codimension in a positively curved manifold has immediate consequences for the cohomology ring.

LEMMA 1.3. *Let M^n be a closed differentiable oriented manifold, and let N^{n-k} be an embedded compact oriented submanifold without boundary. Suppose the inclusion $\iota\colon N^{n-k} \to M^n$ is $(n-k-l)$-connected and $n-k-2l > 0$. Let $[N] \in H_{n-k}(M,\mathbb{Z})$ be the image of the fundamental class of N in $H_*(M,\mathbb{Z})$, and let $e \in H^k(M,\mathbb{Z})$ be its Poincaré dual. Then the homomorphism*

$$\cup e\colon H^i(M,\mathbb{Z}) \to H^{i+k}(M,\mathbb{Z})$$

is surjective for $l \le i < n-k-l$ and injective for $l < i \le n-k-l$.

As mentioned before a crucial point in the proofs of the main results is gaining control on the principal isotropy group H of an isometric group action on a positively curved manifold. By making iterated use of an lemma of Berger [1961] on the vanishing of a Killing field on an even dimensional positively curved manifold one obtains

Proposition 1.4 (Rank Lemma). *Let G be a compact Lie group acting isometrically on a positively curved manifold with principal isotropy group H. There is a sequence of isotropy groups $\mathsf{K}_0 \supset \cdots \supset \mathsf{K}_h = \mathsf{H}$ such that* $\mathrm{rank}(\mathsf{K}_{i-1}) - \mathrm{rank}(\mathsf{K}_i) = 1$*. The orbit type of K_i is at least i-dimensional in M/G. Furthermore* $\mathrm{rank}(\mathsf{K}_0) = \mathrm{rank}(\mathsf{G})$ *if* $\dim(M)$ *is even and* $\mathrm{rank}(\mathsf{G}) - \mathrm{rank}(\mathsf{K}_0) \leq 1$ *if* $\dim(M)$ *is odd.*

In particular, $\mathrm{rank}(\mathsf{G}) - \mathrm{rank}(\mathsf{H}) \leq k + 1$ if k denotes the cohomogeneity of the action. The latter inequality has been previously observed by Püttmann [16].

2. Proof of the stability theorem

2.1. *Smoothness of metrics.* One of the technical difficulties in the proof of the stability theorem (Theorem 8) is to show that the constructed metrics are smooth. In this subsection we establish a few preliminary results in that direction. We start by observing that the problem can be reduced to polynomials.

Proposition 2.1. *Let V be a finite dimensional Euclidean vectorspace, $\rho\colon \mathsf{G} \to \mathsf{O}(V)$ an orthogonal representation, G' a subgroup of G, and let $V' \subset V$ be a $\rho(\mathsf{G}')$-invariant vector subspace. Suppose that for any continuous G'-invariant Riemannian metric g' on V' there is a unique continuous G-invariant Riemannian metric g on V for which $(V', g') \to (V, g)$ is an isometric embedding. Then the following statements are equivalent.*

a) *For all integers $k \geq 0$ the following holds. Consider the induced representations of G' and G in $S^2V' \otimes S^kV'$ and $S^2V \otimes S^kV$, respectively. Let $U'_k \subset S^2V' \otimes S^kV'$ and $U_k \subset S^2V \otimes S^kV$ be the vector subspaces that are fixed-pointwise by G' and G, respectively. Then the orthogonal projection* $\mathrm{pr}\colon S^2V \otimes S^kV \to S^2V' \otimes S^kV'$ *satisfies* $\mathrm{pr}(U_k) = U'_k$.

b) *For any G'-invariant C^∞ Riemannian metric g' on V' there is a G-invariant C^∞ Riemannian metric g on V such that the natural inclusion $(V', g') \to (V, g)$ is an isometric embedding.*

Proof. We first explain why b) implies a). We identify V' with $\mathbb{R}^n$. Notice that $p' \in S^2V' \otimes S^kV'$ may be viewed as a matrix valued function $\mathbb{R}^n \to$

$\mathrm{Sym}(n,\mathbb{R})$ such that the coefficients are homogeneous polynomials of degree k. Furthermore $p' \in U'_k$, if and only if the symmetric two form given by p' is G'-invariant.

Notice that for all $p' \in U'_k$ the corresponding two form occurs in the Taylor expansions of a suitable G'-invariant metric g' of V' at 0. By assumption g' is the restriction of a G-invariant metric g on V. Of course this implies that the polynomials in the Taylor expansion of g' are restrictions of the polynomials in the Taylor expansion of g. Hence a) holds.

Next we show that a) implies b). Suppose g' is a G'-invariant metric on $V' \cong \mathbb{R}^n$. If we think of g' as a matrix-valued function, we can approximate its coefficients by polynomials.

It follows that there is a sequence $p'_i \in \bigoplus_{k=1}^{\infty} S^2V' \otimes S^kV'$ such that the symmetric two form given by p'_i converges on compact sets in the C^∞-topology to g'. Since the metric g' is G'-invariant, we can choose p'_i to be G'-invariant as well.

By assumption p'_i is the restriction of a G-invariant element $p_i \in \bigoplus_{k=1}^{\infty} S^2V \otimes S^kV$. It suffices to prove that a subsequence of the two-forms given by p_i converges in the C^∞-topology.

We fix an integer $l > 0$. For all k we put $W'_k := \bigoplus_{i=1}^{k} U'_i$, and consider the map f_x assigning an element $p \in W'_\infty$ to the element $f_x(p) \in \bigoplus_{i=1}^{l} S^2V \otimes S^iV$ that is characterized by the fact that the two form given by $p - f_x(p)$ vanishes with degree l in x. For all positive integers h and k the set

$$L_{hk} := \big\{ x \in V \;\big|\; \dim(f_x(W_k)) \le h \big\}$$

is a variety. Furthermore $L_{hk} \supset L_{hk+1} \supset \cdots$. Therefore there exists a number $n(l)$ such that $L_{hk} = L_{hk'}$ for all $k, k' \ge n(l)$ and $h = 0, \ldots, \dim\big(\bigoplus_{i=1}^{l} S^2V \otimes S^iV\big)$.

This proves that given any element $q' \in W'_\infty$ and any point $x \in V'$ we can find $p' \in W'_{n(l)}$ such that $q' - p'$ vanishes in x up to degree l. Furthermore, for a given number $r > 0$ there is a compact subset $K' \subset W'_{n(l)}$ such that for all $i > 0$ and $x \in B_r(0) \subset V'$ there is a $p'_{ix} \in K'$ for which $p'_i - p'_{ix}$ vanishes with degree l in x.

Since there are unique elements $p_i \in W_\infty$ and $p_{ix} \in W_{n(l)}$ whose restrictions are given by p'_i and p'_{ix}, it follows that $p_i - p_{ix}$ vanishes up to degree l in x.

Consider the isomorphism $r\colon W_{n(l)} \to W'_{n(l)}$ obtained by restriction, and put $K := r^{-1}(K')$. There are *a priori* C^l-bounds on the ball $B_r(0) \subset V$ for all elements in the compact set K. Because of the above observations these bounds give *a priori* C^l-bounds for the sequence p_i on the ball $B_r(0)$. Since l, r are arbitrary, it follows that a subsequence of p_i converges in the C^∞-topology. $\square$

Definition 2.2. We say that a triple $\big(\rho\colon \mathsf{G}\to \mathsf{O}(V), \mathsf{G}', V'\big)$ has property (G) if and only if the following hold: V is a finite dimensional Euclidean vectorspace, G is a Lie group, ρ is an orthogonal representation in V, G' is a subgroup of G, V' is a $\rho(\mathsf{G}')$-invariant subspace of V and for all $a\in\mathsf{G}$ there is a $c\in\mathsf{G}'$ such that $\mathrm{pr}\circ\rho(ca)_{|V'}\colon V'\to V'$ is a self-adjoint positive semidefinite endomorphism, where $\mathrm{pr}\colon V\to V'$ denotes the orthogonal projection.

LEMMA 2.3. *Let $\mathsf{G}_n\in\big\{\mathsf{O}(n),\mathsf{U}(n),\mathsf{Sp}(n)\big\}$, and choose $\mathbb{K}\in\big\{\mathbb{R},\mathbb{C},\mathbb{H}\big\}$ correspondingly. Consider the standard representation $\rho\colon \mathsf{G}_n\to\mathsf{O}(\mathbb{K}^n)$. Let $\mathbb{K}^{n-k}\subset\mathbb{K}^n$ be the fixed-point set of the lower $k\times k$ block, and let $\mathsf{G}_{n-k}\subset\mathsf{G}_n$ be the upper $n-k$ block. Then the triple $\big(\rho\colon \mathsf{G}_n\to\mathsf{O}(\mathbb{K}^n),\mathsf{G}_{n-k},\mathbb{K}^{n-k}\big)$ has property* (G)*.*

The proof follows immediately from the Cartan decomposition. At first view property (G) does seem to be extremely restrictive, and the above lemma might not convince the reader that there are many examples. However, it turns out that property (G) is stable under various natural operations:

PROPOSITION 2.4. *Suppose $\big(\rho\colon \mathsf{G}\to \mathsf{O}(V), \mathsf{G}', V'\big)$ has property* (G) *and $k>0$.*

a) *The triple $\big(\otimes^k\rho\colon \mathsf{G}\to\mathsf{O}(\otimes^k V),\mathsf{G}',\otimes^k V'\big)$ has property* (G) *as well.*

b) *Let $h\colon \mathsf{G}\to\mathsf{O}(Z)$ denote the trivial representation in some Euclidean vectorspace Z. Then $\Big(h\oplus\bigoplus_{i=1}^k\rho\colon \mathsf{G}\to\mathsf{O}(Z\oplus\bigoplus_{i=1}^k V),\mathsf{G}',Z\oplus\bigoplus_{i=1}^k V'\Big)$ has property* (G) *as well.*

c) *Let $W\subset V$ be a G-invariant subspace of V, and suppose $W':=\mathrm{pr}(W)=V'\cap W$, where $\mathrm{pr}\colon V\to V'$ denotes the orthogonal projection. Then the triple $\big(\rho\colon \mathsf{G}\to\mathsf{O}(W),\mathsf{G}',W'\big)$ has property* (G) *as well.*

Proof. a) Let $a\in\mathsf{G}$. Because of property (G) we can choose $c\in\mathsf{G}'$ and an orthonormal basis $b_1,\dots,b_l$ of V' such that

$$\rho(ca)b_i=\lambda_i b_i+w_i \text{ with } \lambda_i\geq 0 \text{ and } w_i\perp V'.$$

It is straightforward to check that

$$\otimes^k\rho(ca)(b_{i_1}\otimes\cdots\otimes b_{i_k})=\lambda_{i_1}\cdots\lambda_{i_k}b_{i_1}\otimes\cdots\otimes b_{i_k}+w$$

with $w\perp\otimes^k V'\subset\otimes^k V$. Hence a) holds.

b) follows similarly.

c) Let $a\in\mathsf{G}$, and choose $c\in\mathsf{G}'$ such that $\mathrm{pr}\circ\rho(ca)_{|V'}$ is a selfadjoint positive semidefinite endomorphism of V'. Since W' is an invariant subspace of this endomorphism, it follows that $\mathrm{pr}\circ\rho(ca)_{|W'}\colon W'\to W'$ is a self-adjoint positive semidefinite endomorphism of W'. $\square$

Proposition 2.5. *Suppose* $\big(\rho\colon \mathsf{G}\to \mathsf{O}(V), \mathsf{G}', V'\big)$ *has property* (G). *Then statement* a) *of Proposition* 2.1 *holds for this triple.*

Proof. We view $W = S^2V\otimes S^kV$ as a G-invariant subspace in $\otimes^k V$. The orthogonal projection $\mathrm{pr}\colon \otimes^k V\to \otimes^k V'$ maps W to $S^2V'\otimes S^kV'$. Therefore we can employ a) and c) of Proposition 2.4 to see that the triple

$$\big(\rho\colon \mathsf{G}\to \mathsf{O}(S^2V\otimes S^kV), \mathsf{G}', S^2V'\otimes S^kV'\big)$$

has property (G) as well.

Clearly, $\mathrm{pr}(U_k)\subset U_k'$, where U_k, U_k' and pr are as defined in Proposition 2.1. Suppose we can find a vector $u\in U_k'\setminus\{0\}$ that is perpendicular to $\mathrm{pr}(U_k)$. This is equivalent to saying that u is perpendicular to U_k. Since $cu = u$ for all $c\in G'$, property (G) implies that $\langle gu, u\rangle \geq 0$ for all $g\in \mathsf{G}$. Hence the center of mass v of the orbit $\mathsf{G}\star u$ satisfies $\langle v, u\rangle > 0$. Because of $v\in U_k$ this is a contradiction. □

Corollary 2.6. *Suppose* $\big(\mathsf{G}_{n+1}, \mathsf{G}_n, u\big)$ *is one of the triples*

$$\big(\mathsf{O}(n+1), \mathsf{O}(n), 1\big),\quad \big(\mathsf{U}(n+1), \mathsf{U}(n), 2\big)\quad \text{or}\quad \big(\mathsf{Sp}(n+1), \mathsf{Sp}(n), 4\big).$$

Let $\rho\colon \mathsf{G}_n\to\mathsf{O}(p)$ *be a representation which decomposes as the sum of a trivial representation and* $(n-k)$ *pairwise equivalent* $u\cdot n$*-dimensional standard representations with* $k\geq 1$. *Put* $\bar p := p + u(n-k)$, *and consider the corresponding representation* $\bar\rho\colon \mathsf{G}_{n+1}\to \mathsf{O}(\bar p)$. *Then for any* (*not necessarily complete*) G_n*-invariant* C^∞*-Riemannian metric* g *on* $\mathbb{R}^p$ *there is unique metric* G_{n+1}*-invariant Riemannian metric* $\bar g$ *on* $\mathbb{R}^{\bar p}$ *for which* $(\mathbb{R}^p, g)\to(\mathbb{R}^{\bar p}, \bar g)$ *is an isometric embedding, and* $\bar g$ *is of class* C^∞ *as well.*

Proof. Let us first show that there is a unique continuous G_{n+1}-invariant metric $\bar g$ on $\mathbb{R}^{\bar p}$ for which $(\mathbb{R}^p, g)\to(\mathbb{R}^{\bar p}, \bar g)$ is an isometric embedding.

As any G_{n+1}-orbit in $\mathbb{R}^{\bar p}$ intersects $\mathbb{R}^p$, it suffices to show that there is a unique way of extending the given inner product on $T_x\mathbb{R}^p$ to $T_x\mathbb{R}^{\bar p}$ for $x\in\mathbb{R}^p$. Since the principal isotropy group of the representation ρ is conjugate to the lower by a $k\times k$ block, we also may assume that the lower $k\times k$ block B_k in G_n fixes x. Clearly this implies that the lower $(k+1)\times(k+1)$ block B_{k+1} of G_{n+1} fixes x as well.

The isotropy representation of B_k in $T_x\mathbb{R}^p$ consists of $(n-k)$ standard representations and a $(p-u(n-k)k)$-dimensional trivial representation. The isotropy representation of G_{k+1} in $T_x\mathbb{R}^{\bar p}$ consists of $(n-k)$ standard representations and a $(p-u(n-k)k)$-dimensional trivial representation. Thus we see that the moduli space of inner products of $\mathbb{R}^{\bar p}$ which are invariant under B_{k+1} is canonically isomorphic to the moduli space of inner products of $\mathbb{R}^p$ which are invariant under B_k. Clearly the result follows.

By Lemma 2.3 and Proposition 2.4 b) the triple $\big(\bar\rho\colon \mathsf{G}_{n+1} \to \mathsf{O}(\bar p), \mathsf{G}_n, \mathbb{R}^p\big)$ has property (G). Thus we can employ Proposition 2.5 and Proposition 2.1 to see that the metric $\bar g$ is smooth. □

2.2. *Extensions of Lie subgroups.*

LEMMA 2.7. *Let* $(\mathsf{G}_d, u) \in \big\{(\mathsf{SO}(d), 1), (\mathsf{SU}(d), 2), (\mathsf{Sp}(d), 4)\big\}$, *and let* $\mathsf{K} \subset \mathsf{G}$ *be a connected subgroup. Suppose that for some positive* k *there is a subgroup* $\mathsf{B}_k \subset \mathsf{K}$ *such that* B_k *is conjugated to the lower* $k \times k$ *block. Choose* k *maximal, and assume* $k \geq 3$ *if* $u < 4$. *Then* B_k *is a normal subgroup of* K.

Proof. We may assume B_k is the lower $k \times k$ block. Suppose, on the contrary, that we can find a subspace V of the Lie algebra of K which corresponds to a nontrivial irreducible subrepresentation of the isotropy representation of K/B_k. It is straightforward to show that up to conjugacy with an element in the upper $d-k$ block V is contained in the lower $(k+1)\times(k+1)$ block. But then K contains the lower $(k+1)\times(k+1)$ block — a contradiction. □

2.3. *Chains of homogeneous vectorbundles.* For a subgroup $\mathsf{H} \subset \mathsf{G}$, we let $N(\mathsf{H})$ denote the normalizer of H in G. In this subsection we prove the following local version of the stability theorem.

PROPOSITION 2.8. *Let* $\big(\mathsf{G}_{n+1}, \mathsf{G}_n, u\big)$ *be one of the triples*

$$\big(\mathsf{SO}(n+1), \mathsf{SO}(n), 1\big), \quad \big(\mathsf{SU}(n+1), \mathsf{SU}(n), 2\big) \quad or \quad \big(\mathsf{Sp}(n+1), \mathsf{Sp}(n), 4\big),$$

K *a closed subgroup of* G_n, *and let* $\rho\colon \mathsf{K} \to \mathsf{O}(p)$ *be a representation with principal isotropy group* $\mathsf{H} \subset \mathsf{K}$. *Suppose that* H *contains the lower* $k \times k$ *block* B_k *of* G_n *with* $k \geq 1$ *if* $u = 4$, *and* $k \geq 3$ *if* $u = 1, 2$. *Assume that* B_k *is normal in* $N(\mathsf{H}) \cap \mathsf{K}$.

a) *Then there is a unique normal subgroup* $\mathsf{B}_l \subset \mathsf{K}$ *with* $\mathsf{B}_k \subset \mathsf{B}_l$ *which is conjugate to the lower* $l \times l$ *block.*

b) *There is a natural choice for a subgroup* $\bar{\mathsf{K}} \subset \mathsf{G}_{n+1}$ *and for a representation* $\bar\rho\colon \bar{\mathsf{K}} \to \mathsf{O}(\bar p)$ *with* $\bar p - p = u(l-k)$ *such that*

   (i) *There is a natural inclusion* $\iota\colon \mathsf{G}_n \times_{\rho|\mathsf{K}} \mathbb{R}^p \to \mathsf{G}_{n+1} \times_{\bar\rho|\bar{\mathsf{K}}} \mathbb{R}^{\bar p}$ *between the two corresponding homogeneous vectorbundles.*

   (ii) *For any (not necessarily complete)* G_n*-invariant Riemannian metric (of class* C^∞*) on the vectorbundle* $\mathsf{G}_n \times_{\rho|\mathsf{K}} \mathbb{R}^p$ *there is a unique extension to a* G_{n+1}*-invariant Riemannian metric (of class* C^∞*) on the vectorbundle* $\mathsf{G}_{n+1} \times_{\bar\rho|\bar{\mathsf{K}}} \mathbb{R}^{\bar p}$.

Proof. a) Choose l maximal such that there is a subgroup $\mathsf{B}_l \subset \mathsf{K}$ which is conjugate to a lower $l \times l$ block containing B_k. Then B_l is normal in the identity component of K by Lemma 2.7. Since B_k is normal in the normalizer of H it is easy to deduce that B_l is a normal subgroup of K.

b) We may assume that B_l is given by the lower $l \times l$ block. There is a subgroup L of K such that $\mathsf{K} = \mathsf{L} \cdot \mathsf{B}_l$ and $\mathsf{L} \cap \mathsf{B}_l = 1$. We consider G_n as the upper $n \times n$ block of G_{n+1}. Let B_{l+1} be the lower $(l+1) \times (l+1)$ block in G_{n+1}, and put $\bar{\mathsf{K}} := \mathsf{L} \cdot \mathsf{B}_{l+1}$.

Next we want to 'extend' the representation $\rho\colon \mathsf{K} \to \mathsf{O}(p)$ to a representation $\bar\rho\colon \bar{\mathsf{K}} \to \mathsf{O}(\bar p)$ with $\bar p - p = u(l-k)$.

The fact that $\mathsf{B}_k \subset \mathsf{B}_l$ is contained in a principal isotropy group of the representation ρ implies that $\rho_{|\mathsf{B}_l}$ is the sum of a trivial representation and $(l-k)$ equivalent $u \cdot l$-dimensional standard representations. This in turn shows that ρ decomposes as follows:

$$\rho = \rho_1 \otimes_{\mathbb{K}} \rho_2 \oplus \rho',$$

where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ is determined by $\dim_{\mathbb{R}} \mathbb{K} = u$, ρ_1 is an $l-k$-dimensional representation of K over the field $\mathbb{K}$ with $\mathsf{B}_l \subset \mathrm{kernel}(\rho_1)$, ρ_2 is an irreducible $u \cdot l$-dimensional representation of K over the field $\mathbb{K}$ with $\rho_{2|\mathsf{B}_l}$ being the standard representation, and ρ' is another representation of K with $\mathsf{B}_l \subset \mathrm{kernel}(\rho')$.

Because of $\bar{\mathsf{K}}/\mathsf{B}_{l+1} \cong \mathsf{K}/\mathsf{B}_l$ we can extend ρ_1 and ρ' to representations of $\bar{\mathsf{K}}$. Furthermore it is obvious that we can 'extend' ρ_2 to a $u(l+1)$-dimensional representation $\bar\rho_2\colon \bar{\mathsf{K}} \to \mathsf{O}(u(l+1))$. Hence we may define $\bar\rho := \bar\rho_1 \otimes \bar\rho_2 \oplus \bar\rho'$.

Clearly, there is a natural inclusion

$$\iota\colon \mathsf{G}_n \times_{\rho_{|\mathsf{K}}} \mathbb{R}^p \to \mathsf{G}_{n+1} \times_{\bar\rho_{|\bar{\mathsf{K}}}} \mathbb{R}^{\bar p}$$

between the two corresponding homogeneous vectorbundles.

Next we want to check that for any continuous G_n-invariant Riemannian metric on $\mathsf{G}_n \times_{\rho_{|\mathsf{K}}} \mathbb{R}^p$ there is a unique continuous G_{n+1}-Riemannian metric on $\mathsf{G}_{n+1} \times_{\bar\rho_{|\bar{\mathsf{K}}}} \mathbb{R}^{\bar p}$ that extends the given metric.

Since any G_{n+1}-orbit in $\mathsf{G}_{n+1} \times_{\bar\rho_{|\bar{\mathsf{K}}}} \mathbb{R}^{\bar p}$ intersects $\mathsf{G}_n \times_{\rho_{|\mathsf{K}}} \mathbb{R}^p$, it suffices to check that at a point x in $\mathsf{G}_n \times_{\rho_{|\mathsf{K}}} \mathbb{R}^p$ there is a unique way of extending the given inner product of $T_x\mathsf{G}_n \times_{\rho_{|\mathsf{K}}} \mathbb{R}^p$ to an inner product of $T_x\mathsf{G}_{n+1} \times_{\bar\rho_{|\bar{\mathsf{K}}}} \mathbb{R}^{\bar p}$. Similarly we also may assume that the lower $(k+1) \times (k+1)$ block $\mathsf{B}_{k+1} \subset \mathsf{G}_{n+1}$ is contained in the isotropy group of x. The isotropy representation of B_{k+1} in $T_x\mathsf{G}_{n+1} \times_{\bar\rho_{|\bar{\mathsf{K}}}} \mathbb{R}^{\bar p}$ decomposes into $(n-k)$ pairwise equivalent $u(k+1)$-dimensional standard representations and an $(l-u(k+1))$-dimensional trivial representation, where $l = \dim(\mathsf{G}_{n+1} \times_{\bar\rho_{|\bar{\mathsf{K}}}} \mathbb{R}^{\bar p})$.

Put $\mathsf{B}_k := \mathsf{B}_{k+1} \cap \mathsf{G}_n$. Then the isotropy representation of B_k in $T_x\mathsf{G}_n \times_{\rho_{|\mathsf{K}}} \mathbb{R}^p$ decomposes into $(n-k)$ pairwise equivalent $u(k+1)$-dimensional standard representations and an $(l-u(k+1))$-dimensional trivial representation. Consequently, the moduli spaces of B_{k+1}-invariant inner products on $T_x\mathsf{G}_{n+1}\times_{\bar\rho_{|\bar{\mathsf{K}}}}\mathbb{R}^{\bar p}$ and B_k-invariant inner products on $T_x\mathsf{G}_n\times_{\rho_{|\mathsf{K}}}\mathbb{R}^p$ coincide.[1]

Notice that we can actually repeat this construction, i.e., we can extend any continuous metric of $\mathsf{G}_{n+1}\times_{\bar\rho_{|\bar{\mathsf{K}}}}\mathbb{R}^{\bar p}$ to a continuous metric of the corresponding vectorbundle of G_{n+2}. Clearly all elements of G_{n+2} leaving $\mathsf{G}_{n+1}\times_{\bar\rho_{|\bar{\mathsf{K}}}}\mathbb{R}^{\bar p}$ invariant are isometries of $\mathsf{G}_{n+1}\times_{\bar\rho_{|\bar{\mathsf{K}}}}\mathbb{R}^{\bar p}$. Similarly all isometries leaving $\mathsf{G}_n\times_{\rho_{|\mathsf{K}}}\mathbb{R}^p$ invariant are isometries of this bundle.

For $u=1$ this consideration shows that there are orbit equivalent isometric actions of $\mathsf{O}(n)$ and $\mathsf{O}(n+1)$ on the bundles $\mathsf{G}_n\times_{\rho_{|\mathsf{K}}}\mathbb{R}^p$ and $\mathsf{G}_{n+1}\times_{\bar\rho_{|\bar{\mathsf{K}}}}\mathbb{R}^{\bar p}$, respectively. For $u=2$ there are orbit equivalent isometric actions of $\mathsf{U}(n)$ and $\mathsf{U}(n+1)$ on these bundles.

For $u=1,2$ we change notation. Subsequently, if $u=1,2$, then $\big(\mathsf{G}_n,\mathsf{G}_{n+1},u\big)$ is one of the triples $\big(\mathsf{O}(n+1),\mathsf{O}(n),1\big)$ or $\big(\mathsf{U}(n+1),\mathsf{U}(n),2\big)$. The above argument shows that there is a canonical way to write the homogeneous vectorbundles as homogeneous vectorbundles of these larger groups. We change the groups K and $\bar{\mathsf{K}}$ consistently and continue to write ρ and $\bar\rho$ for the extended representations. For $u=4$ we leave everything as it was.

It remains to prove: for any smooth Riemannian metric g on $\mathsf{G}_n\times_{\rho_{|\mathsf{K}}}\mathbb{R}^p$ the unique extension of g to a continuous Riemannian metric on $\mathsf{G}_{n+1}\times_{\bar\rho_{|\bar{\mathsf{K}}}}\mathbb{R}^{\bar p}$ is smooth as well. Consider the principal K bundle $\pi\colon \mathsf{G}_n\times\mathbb{R}^p\to\mathsf{G}_n\times_{\rho_{|\mathsf{K}}}\mathbb{R}^p$, and choose a $\mathsf{G}_n\times\mathsf{K}$-invariant C^∞ Riemannian metric $\hat g$ on $\mathsf{G}_n\times\mathbb{R}^p$ that turns the projection into a Riemannian submersion. Similarly, we can show that there is a unique extension of $\hat g$ to a continuous $\mathsf{G}_{n+1}\times\bar{\mathsf{K}}$-invariant Riemannian metric $\hat g_{n+1}$ on $\mathsf{G}_{n+1}\times\mathbb{R}^{\bar p}$. Clearly it suffices to prove that $\hat g_{n+1}$ is smooth. Without loss of generality we may assume that $\mathsf{K}=\mathsf{B}_l$ is given by the lower $l\times l$ block. Then $\bar{\mathsf{K}}$ is given by the lower $(l+1)\times(l+1)$ block in G_{n+1}.

Let $\bar g_{n+1}$ denote the continuous Riemannian metric on $\mathbb{R}^{\bar p}$ that turns the projection $\mathrm{pr}\colon(\mathsf{G}_{n+1}\times\mathbb{R}^{\bar p},\hat g_{n+1})\to(\mathbb{R}^{\bar p},\bar g_{n+1})$ into a Riemannian submersion. By Corollary 2.6 $\bar g_{n+1}$ is smooth. It remains to check that the horizontal distribution is smooth and that the metric on the vertical distribution is smooth, which can be done similarly. We indicate the proof for the horizontal distribution.

The horizontal distribution of $\mathsf{G}_{n+1}\times\mathbb{R}^{\bar p}$ is given by a $\bar{\mathsf{K}}$-equivariant map

$$\omega_{n+1}\colon T\mathbb{R}^{\bar p}=\mathbb{R}^{\bar p}\times\mathbb{R}^{\bar p}\to\mathfrak{g}_{n+1}$$

which is linear in the second component and where the Lie algebra $\mathfrak{g}_{n+1}$ of G_{n+1} is endowed with the adjoint representation of $\bar{\mathsf{K}}$. Equivalently we can

[1] It is here where we need the assumption $k\geq 3$ for $u=1,2$ because otherwise the type of the representation could change from complex to real or from symplectic to complex which in turn would mean that the moduli space of $T_x\mathsf{G}_n\times_{\rho_{|\mathsf{K}}}\mathbb{R}^p$ is larger than the one of $T_x\mathsf{G}_{n+1}\times_{\bar\rho_{|\bar{\mathsf{K}}}}\mathbb{R}^{\bar p}$.

view ω_{n+1} as a map

$$\omega_{n+1}\colon \mathbb{R}^{\bar p} \to \mathbb{R}^{\bar p} \otimes \mathfrak{g}_{n+1}.$$

We have to show that this map is of class $\mathbb{C}^\infty$, provided that the K-equivariant corresponding map

$$\omega_n\colon \mathbb{R}^{p} \to \mathbb{R}^{p} \otimes \mathfrak{g}_{n}$$

is of class $\mathbb{C}^\infty$. Similarly to Proposition 2.1 it suffices to prove that the orthogonal projection

$$S^q\mathbb{R}^{\bar p} \otimes \mathbb{R}^{\bar p} \otimes \mathfrak{g}_{n+1} \longrightarrow S^q\mathbb{R}^{p} \otimes \mathbb{R}^{p} \otimes \mathfrak{g}_{n}$$

maps the subspace of the domain fixed by $\bar{\mathsf{K}}$ surjectively onto the subspace of the target fixed by K, where in order to define the orthogonal projection we may choose a fixed biinvariant metric on G_{n+1}.

By Proposition 2.5 for this in turn, it suffices to verify that the triple consisting of the natural representation of $\bar{\mathsf{K}} \cong \mathsf{O}(l+1)$ in $S^q\mathbb{R}^{\bar p} \otimes \mathbb{R}^{\bar p} \otimes \mathfrak{g}_{n+1}$, the subgroup $\mathsf{K} \cong \mathsf{O}(l)$ and the subspace $S^q\mathbb{R}^{p} \otimes \mathbb{R}^{p} \otimes \mathfrak{g}_{n}$ has property (G). But this can be deduced by making iterated use of Proposition 2.4. ☐

2.4. *Proof of Theorem* 8. We first want to explain why the local version of the stability theorem indeed follows from Proposition 2.8. Let $\mathsf{K} \subset \mathsf{G}_d$ be the isotropy group of $p \in M$ containing H as the principal isotropy group of its slice representation. By Lemma 11.8 below, B_k is normal in $N(\mathsf{H}) \cap \mathsf{K}$. This is the only time when we use that the underlying manifold is simply connected. Instead of $\pi_1(M) = 1$ one can require that B_k be normal in $N(\mathsf{H})$ — this will be used later on. Although Lemma 11.8 is proved very late in the paper, we remark that Section 11 can be read independently so that it cannot create logical problems. In either case the slice theorem allows one to apply Proposition 2.8 to a tubular neighborhood of the orbit $\mathsf{G} \star p$.

a) We identify G_d with the upper $d \times d$ block in G_{d+1}. Since M is a disjoint union of orbits, and each orbit $\mathsf{G}_d \star p$ can be identified with the homogeneous space $\mathsf{G}_d/\mathsf{H}_p$, we may think of M as disjoint union of the homogeneous spaces $\mathsf{G}_d/\mathsf{H}_p$, where p runs through a set representing each orbit precisely once.

We can now define the underlying set of the manifold M_1 we want to construct as follows: For each orbit we can choose a point $p \in M$ in that orbit whose isotropy group $\mathsf{H}_p \subset \mathsf{G}_d$ contains the lower $(d - r_p) \times (d - r_p)$ block as a normal subgroup, where $r_p \le r$ is an integer depending on p.

After choosing p the orbit is naturally diffeomorphic to $\mathsf{G}_d/\mathsf{H}_p$. We define $\hat{\mathsf{H}}_p \subset \mathsf{G}_{d+1}$ as the group generated by $\mathsf{H}_p \subset \mathsf{G}_d \subset \mathsf{G}_{d+1}$ and the lower $(d - r_p + 1) \times (d - r_p + 1)$ block in G_{d+1}.

We now define the underlying set of M_1 as the disjoint union of the homogeneous spaces $\mathsf{G}_{d+1}/\hat{\mathsf{H}}_p$, where p runs through a set representing each G_d-orbit in M precisely once.

Notice that the set M_1 comes with a natural G_{d+1}-action and that the natural inclusion $\mathsf{G}_d/\mathsf{H}_p \to \mathsf{G}_{d+1}/\hat{\mathsf{H}}_p$ induces a natural inclusion $M \to M_1$. Furthermore, the orbit spaces M_1/G_{d+1} and $A := M/\mathsf{G}_d$ are naturally isomorphic. Let $\mathrm{pr}_d\colon M \to A$ and $\mathrm{pr}_{d+1}\colon M_1 \to A$ denote the projection onto the orbit space. For each orbit $\mathsf{G}_d \star p$ in M there is, by the slice theorem, a small neighborhood U which is equivariantly diffeomorphic to a homogeneous vectorbundle. From Proposition 2.8 it follows that the set $U_1 = \mathrm{pr}_{d+1}^{-1}(\mathrm{pr}_d(U))$ may be identified with a homogeneous vectorbundle and that there is a unique G_{d+1}-invariant Riemannian metric of class C^∞ on U_1 that extends the given metric on U. This shows that M_1 admits a unique structure of a Riemannian manifold with a G_{d+1}-invariant metric that extends the given metric on M.

Clearly this finishes the proof of a). For b) it only remains to check that $\mathrm{cohom}(M, g) = \mathrm{cohom}(M_1, g)$. Since M is a fixed-point component of an isometry of M_1, we clearly have $\mathrm{cohom}(M, g) \le \mathrm{cohom}(M_1, g)$. If there is a group action of a Lie group L on M that commutes with the given G_d-action, then it is easy to see that one gets an isometric $\mathsf{L} \times \mathsf{G}_{d+1}$-action on M. This finishes the argument if $\mathsf{G}_d \subset \mathrm{Iso}(M, g)_0$ is normal. Finally one can show that the smallest normal subgroup N of $\mathrm{Iso}(M, g)_0$ containing $\mathsf{G}_d \subset \mathrm{Iso}(M, g)_0$ also satisfies the hypothesis of the stability theorem, and M_1 is one of the manifolds that occurs in the chain that one can construct with respect to the N-action.

c) Suppose that $v, w \in T_qM_1$ span a plane of minimal (maximal) sectional curvature in M_1. Because of $k \ge d/2$ it is straightforward to find an element $a \in \mathsf{G}_{d+1}$ with $a_*v \in TM$. In other words we may assume $v \in T_qM$. Since M is totally geodesic, the tangent space T_qM is an invariant subspace of the curvature operator $R(\cdot, v)v$. Hence we may choose either $w \in T_qM$ or $w \in \nu_q(M)$. In the former case we are done. In the latter it is straightforward to check that w is fixed by a $k \times k$ block H'. By switching the roles of w and v we can show similarly that we may assume that up to conjugacy there is a $k \times k$ block which keeps v fixed. But now it is straightforward to check that there is an element $g \in \mathsf{G}_{d+1}$ with $L_{g*}(v) = v$ and $L_{g*}(w) \in T_qM$. But this proves that the minimal (maximal) sectional curvature is attained in M.

3. Isotropy lemmas

LEMMA 3.1. *Let G be a Lie group acting on a positively curved manifold M. Suppose that K is an isotropy group, whose orbit type has dimension k in M/G. Choose an irreducible subrepresentation U of the isotropy representation of G/K, and define $u \in \{1, 2, 4\}$ depending on whether the representation is of real $(u = 1)$ complex $(u = 2)$ or symplectic $(u = 4)$ type. Let l be the maximal number such that there are l linear independent subrepresentations of the slice representation of K which are equivalent to U. Suppose that $k - ul > 0$.*

Then there is an isotropy group $\bar{\mathsf{K}}$ *in the closure of the orbit type of* K*, such that* U *is equivalent to a subrepresentation of the isotropy representation of* $\bar{\mathsf{K}}/\mathsf{K}$*. Furthermore the orbit type of* $\bar{\mathsf{K}}$ *has dimension at least* $(k-1-u\cdot l)$.

If K is the principal isotropy group, then the slice representation is trivial, and the isotropy lemma (Lemma 5) follows immediately. There are also some other useful consequences:

LEMMA 3.2. *Let* G *be a Lie group acting on a positively curved manifold with cohomogeneity* k*. Suppose that* H *is the principal isotropy group. Given up to* k *nontrivial irreducible subrepresentation of the isotropy representation of* G/H *which are pairwise nonequivalent, one can find an isotropy group* $\bar{\mathsf{K}}$ *such that each of the* k *representations is equivalent to a subrepresentation of* $\bar{\mathsf{K}}/\mathsf{H}$.

Definition 3.3. Let G be a Lie group and H a connected compact subgroup. We call the isotropy representation of G/H *spherical* if any nontrivial irreducible subrepresentation of the isotropy representation of G/H is transitive on the sphere.

LEMMA 3.4. *Suppose* G *acts not transitively on a positively curved manifold with principal isotropy group* H*. Then the isotropy representation of* G/H_0 *is spherical.*

In fact, each irreducible subrepresentation of H_0 is equivalent to a subrepresentation of the isotropy representation of $\mathsf{K}_0/\mathsf{H}_0$, where K is an isotropy group. Furthermore we may assume K corresponds to a codimension 1 orbit type. Thus $\mathsf{K}_0/\mathsf{H}_0$ is a sphere and its isotropy representation is spherical as can be easily deduced from the classification of transitive actions on spheres.

Proof of Lemma 3.1. Let $p\in M$ be a point with isotropy group K, and let M' be the fixed-point component of K with $p\in M'$. In the Lie algebra $\mathfrak{g}$ of G we consider the orthogonal complement $\mathfrak{m}$ of the subalgebra $\mathfrak{k}$ of K with respect to a biinvariant metric on G. Let V be the maximal K-invariant subspace of $\mathfrak{m}$ for which the following holds. Any irreducible subrepresentation of V is equivalent to U. We endow V with the induced invariant metric.

For $u\in V$ let J_u denote the Killing field corresponding to u. Assume, on the contrary, that $J_{u|q}\neq 0$ for all $p\in M'$ and all $u\in V\setminus\{0\}$. Choose a $p\in M'$, and a unit vector $v\in V$ with

$$\|J_{v|p}\| = \min\bigl\{\|J_{w|q}\| \bigm| q\in M', w\in V, \|w\|=1\bigr\}.$$

If we let $H\subset T_pM'$ denote the vectors perpendicular to $\mathsf{G}\star p$, then the map

$$T\colon H\otimes V\to T_pM,\ x\otimes u\mapsto \nabla_x J_u$$

is K-equivariant. We put $Y := J_v$. By the equivariance of T it is easy to see that for each $x \in H$ the orthogonal projection of $\nabla_x Y$ onto the normal space of $\mathsf{G}\star p$ is contained in a $u\cdot l$-dimensional subspace. Because of $\dim(H) \geq k > u\cdot l$, we can find an $x \in H \setminus \{0\}$ such that $\nabla_x Y$ is tangential to the orbit.

Thus there is an element $u \in V$ with $\nabla_x Y = J_{u|p}$. The particular choice of (v, p) gives $u \perp v$ and $L_p(u) \perp L_p(v)$. Put $Z := J_u$, $c(t) := \exp(tx)$, and consider the vectorfield

$$A(t) = \frac{Y_{\circ c} - tZ_{\circ c}}{(1 + t^2\|u\|^2)^{1/2}} = \frac{J_{(v+tu)|c(t)}}{\|v + tu\|}.$$

From the choice of (v, p) it is clear that $\|A(t)\|$ attains a minimum at $t = 0$. Clearly the same holds for the norm of the vectorfield $B(t) = Y_{\circ c} - tZ_{\circ c}$. On the other hand $B'(0) = 0$, and

$$\begin{aligned}
\tfrac{d^2}{dt^2}{}_{t=0}\|B(t)\|^2 &= 2\langle B''(0), B(0)\rangle \\
&= 2\langle Y''_{\circ c}(0), Y_{\circ c}(0)\rangle - 4\langle Z'_{\circ c}(0), Y_{\circ c}(0)\rangle \\
&= -2\langle R(x, Y_{|p})Y_{|p}, x\rangle - 4\langle Z_{\circ c}(0), Y'_{\circ c}(0)\rangle \\
&< -4\|Z_{|p}\|^2 \leq 0
\end{aligned}$$

— a contradiction. In the calculation above we used the fact that Y and Z are Jacobifields along c satisfying $\langle Y_{\circ c}, Z'_{\circ c}\rangle = \langle Y'_{\circ c}, Z_{\circ c}\rangle$.

Thus there is an isotropy group $\bar{\mathsf{K}}$ such that $\bar{\mathsf{K}}/\mathsf{K}$ contains a subrepresentation which is equivalent to U. Let h be the dimension of the orbit type of $\bar{\mathsf{K}}$ in the orbit space. The slice representation of $\bar{\mathsf{K}}$ decomposes as an h-dimensional trivial representation and a nontrivial representation. If $h < k - 1 - ul$, then the nontrivial part of the slice representation induces an action on the sphere such that the orbit type of K has dimension $k - 1 - h > ul$. We could repeat the argument and find a group $\bar{\mathsf{K}}'$ between K and $\bar{\mathsf{K}}$ such that $\bar{\mathsf{K}}'/\mathsf{K}$ contains a subrepresentation which is equivalent to U. In other words, if we choose $\bar{\mathsf{K}}$ minimal, then $h \geq k - 1 - u \cdot l$. □

4. Soul orbits

In this section we will establish the estimate on the number of faces of a positively curved orbit space (Theorem 7) which in turn relies on

THEOREM 4.1 (Soul Orbit Theorem). *Let G be a Lie group acting isometrically on a Riemannian manifold with principal isotropy group H. Suppose that the orbit space M/G has positive curvature in the Alexandrov sense. Let* $\mathrm{pr}\colon M \to M/\mathsf{G}$ *denote the projection. Suppose there is an isotropy group K corresponding to a face F of the orbit space M/G. If there is more than one face with isotropy group K, F is allowed to be the union of these faces. Then*

a) *There is a unique point $\mathsf{G}\star q \in M/\mathsf{G}$ of maximal distance to F.*

b) *$M \setminus \mathrm{pr}^{-1}(F)$ is equivariantly diffeomorphic to the normal bundle of the orbit $\mathsf{G} \star q$.*

c) *The inclusion map $\mathsf{G} \star q \to M$ is $\dim(\mathsf{K}/\mathsf{H})$-connected.*

For later applications it is important to note that we only assumed positive curvature for the orbit space. We will often refer to the orbit $\mathsf{G} \star q$ as a soul orbit.

Proof. It is straightforward to check that the distance function $d(F, \cdot)$ of the face F defines a strictly concave function on the Alexandrov space M/G. Thus there is a unique point of maximal distance $\mathsf{G} \star q$. Next consider the Lipschitz function $r\colon M \to \mathbb{R}$ given by $r(p) = d(F, \mathrm{pr}(p)) = d(\mathrm{pr}^{-1}(F), p)$. Since $d(F, \cdot)$ is strictly concave in $M/\mathsf{G} \setminus F$ it is easy to see that the distance function r has no critical points (in the sense of Grove-Shiohama) on $M' := M \setminus \big(\mathrm{pr}^{-1}(F) \cup \mathsf{G} \star q\big)$. Thus we can construct a gradient-like vectorfield X on M' with respect to r. A simple averaging argument shows that we may choose a G-invariant vectorfield X. Consequently $M \setminus \mathrm{pr}^{-1}(F)$ is diffeomorphic to the normal bundle of $\mathsf{G} \star q$. Notice that the set $\mathrm{pr}^{-1}(F)$ has codimension $\geq \dim(\mathsf{K}/\mathsf{H}) + 1$. Consequently the inclusion map $M \setminus \mathrm{pr}^{-1}(F) \to M$ is $\dim(\mathsf{K}/\mathsf{H})$-connected. □

Proof of Theorem 7. a) We argue by induction on k. Suppose there are at least $k+1$ faces, $F_0, \ldots, F_k$ in the orbit space M/G. Let $\bar{p}_i \in M/\mathsf{G}$ be the point of maximal distance to F_i. By Theorem 4.1, $M/\mathsf{G} \setminus F_i$ is isomorphic as a stratified space to the tangent cone $C_{\bar{p}_i} M/\mathsf{G}$. In particular, the tangent cone $C_{\bar{p}_i} M/\mathsf{G}$ has at least $k-1$ faces, and the same holds for the space of directions $\Sigma_{\bar{p}_0} M/\mathsf{G}$ at $\bar{p}_0$. Since $\Sigma_{\bar{p}_0} M/\mathsf{G}$ is a positively curved $(k-1)$-dimensional orbit space, our induction hypothesis implies that $\Sigma_{\bar{p}_0} M/\mathsf{G}$ is, as a stratified space, isomorphic to a $(k-1)$-simplex. Thus $M/G \setminus F_i$ is, as a stratified space, isomorphic to the cone over a $(k-1)$-simplex, and it easily follows that M/G is isomorphic to a k-simplex.

b) Let $F_0, \ldots, F_l$ denote the faces of M/G, $l < k$. Let $\bar{p}_0$ be again the soul point corresponding to F_0. Then $M/\mathsf{G} \setminus F_0$ is isomorphic to the tangent cone $C_{\bar{p}_0} M/\mathsf{G}$. This implies that M/G is homeomorphic to the natural compactification of $C_{\bar{p}_0} M/\mathsf{G}$, i.e., to the subspace of all vectors of norm (distance to the origin) ≤ 1. By the induction hypothesis the space of directions $\Sigma_{\bar{p}_0} M/\mathsf{G}$ is homeomorphic to the join of an $(l-1)$ simplex and the space $F_1 \cap \cdots \cap F_l \cap \Sigma_{\bar{p}_0} M/\mathsf{G}$. The latter is homeomorphic to $F_0 \cap F_1 \cap \cdots \cap F_l$ and hence the result follows. □

Proposition 4.2. *Let G be a connected Lie group acting isometrically on a simply connected positively curved manifold M with principal isotropy* *group* H. *Let* $\mathsf{F} := \pi_1(\mathsf{G}/\mathsf{H})$ *be the fundamental group of the principal orbit and* $C(\mathsf{F})$ *the center of* F. *Then* $\mathsf{F}/C(\mathsf{F})$ *is isomorphic to* $\mathbb{Z}_2^d$ *for some* $d \geq 0$.

Proof. We argue by induction on the dimension of M. Consider first a special case. Suppose that any irreducible subrepresentation of the isotropy representation of G/H is one dimensional. Without loss of generality we can assume that G acts effectively and then $\mathsf{H} \cong (\mathbb{Z}_2)^d$. Since the abelian fundamental group of G is mapped onto the center of the fundamental group of G/H, the result follows.

Suppose next that there is an irreducible subrepresentation of the isotropy representation of G/H of dimension at least 2. By the isotropy lemma there is an isotropy group K corresponding to a face F of the orbit space such that $\mathsf{K}/\mathsf{H} \cong \mathbb{S}^h$ with $h \geq 2$. Let $\mathsf{G} \star p \in M/\mathsf{G}$ be the orbit of maximal distance to F. By Theorem 4.1 the inclusion map $\mathsf{G} \star p \to M$ is 2-connected. Let K denote the isotropy group of p. Since G/K is simply connected, it follows that the natural map $\pi_1(\mathsf{K}/\mathsf{H}) \to \pi_1(\mathsf{G}/\mathsf{H})$ is surjective. Therefore the statement follows from the induction hypothesis applied to the slice representation of K.

□

5. Recovery of the tangential homotopy type of a chain

THEOREM 5.1. *Let* (G_d, u) *be one of the pairs* $(\mathsf{SO}(d), 1)$, $(\mathsf{SU}(d), 2)$ *or* $(\mathsf{Sp}(d), u)$. *Suppose* G_d *acts isometrically and nontrivially on a positively curved compact manifold* M. *Suppose also that the principal isotropy group* H *of the action contains up to conjugacy a lower* $k \times k$ *block* B_k *with* $k \geq 2$ *if* $u = 4$ *and* $k \geq 3$ *if* $u = 1, 2$. *Choose* k *maximal.*

If M *is simply connected, then* M *is tangentially homotopically equivalent to a rank one symmetric space. If* M *is not simply connected, then* $\pi_1(M)$ *is isomorphic to the fundamental group of a space form of dimension* $u(d-k)-1$.

For the remainder of the section the assumption of Theorem 5.1 is assumed to be valid. Since the isotropy representation of G/H is spherical, B_k is normal in the normalizer of H. Thus we can apply the stability theorem (Theorem 8) even if M is not simply connected; see the beginning of subsection 2.4. Consequently there is a Riemannian manifold M_i with an isometric action of G_{d+i} such that M_i/G_{d+i} is isometric to M/G_d. Furthermore there are natural totally geodesic inclusions $M = M_0 \subset M_1 \subset \cdots$. We do not know in this general situation whether M_i has positive sectional curvature for $i > 0$. We will establish the theorem in four steps.

Step 1. The union $\bigcup_{i=1}^{\infty} M_i$ has the homotopy type of a classifying space of a compact Lie group L. If M_0 is simply connected, then L is connected.

Step 2. The classifying space $B\mathsf{L}$ of the group L has periodic cohomology. If M_0 is simply connected, then $\mathsf{L} \cong \{e\}, \mathsf{S}^1, \mathsf{S}^3$. Furthermore M_i is homotopic to a rank one symmetric space.

Step 3. If M_0 is simply connected, then M_i is tangentially homotopically equivalent to a rank one symmetric space.

Step 4. If M_0 is not simply connected, then $\pi_1(M_0)$ is isomorphic to the fundamental group of a space form of dimension $u(d-k)-1$.

5.1. *Proof of Step* 1. We let H_{d+i} denote the principal isotropy group of the action of G_{d+i} on M_i. Notice that H_{d+i} contains the lower $(k+i)\times(k+i)$ block B_{k+i} of G_{d+i}. Consequently H_d has a $u\cdot k$-dimensional irreducible subrepresentation. By Lemma 3.1 there is an isotropy group K corresponding to a face such that the isotropy representation of K/H_d contains this representation. Notice that up to conjugacy K contains the lower $(k+1)\times(k+1)$ and $\mathsf{K}/\mathsf{H} \cong \mathbb{S}^{uk+u-1}$. Let F be the union of all faces with isotropy group K in the orbit space.

By Theorem 4.1 there is a soul orbit $\mathsf{G}_d \star y_0$ whose inclusion map is $(uk+u-1)$-connected. Notice that the isotropy group of y_0 does not contain a lower $(k+1)\times(k+1)$ block, because otherwise $\mathsf{G}_d \star y_0$ would be contained in F.

Thus we may assume that the isotropy group $\mathsf{G}_d^{y_0}$ of y_0 is given by $\mathsf{L}\cdot\mathsf{B}_k$, where L a subgroup with $\mathsf{L}\cap\mathsf{B}_k = 1$.

It is clear from the construction of the chain that the isotropy group of $y_0 \in M_0 \subset M_i$ with respect to the G_{d+i}-action on M_i is then given by $\mathsf{G}_{d+i}^{y_0} = \mathsf{L}\cdot\mathsf{B}_{k+i}$.

Recall that the orbit spaces M_i/G_{d+i} and M/G_d are canonically isometric. Furthermore the face F corresponds in M_i/G_{d+i} to an isotropy group K_{d+i} containing the lower $(k+1+i)\times(k+1+i)$ block of G_{d+i} and hence $\mathsf{K}_{d+i}/\mathsf{H}_{d+i} \cong \mathbb{S}^{u(k+1+i)-1}$. By Theorem 4.1 the inclusion map $\mathsf{G}_{d+i}\star y_0 \to M_i$ is $(u(k+1+i)-1)$-connected.

The natural inclusion

$$\mathsf{G}_\infty \star y_0 := \bigcup_{i=0}^{\infty} \mathsf{G}_{d+i}\star y_0 \to \bigcup_{i=0}^{\infty} M_i =: M_\infty$$

is a weak homotopy equivalence, and by Whitehead it is a homotopy equivalence. We may identify the orbit $\mathsf{G}_{d+i}\star y_0$ with the homogeneous space $\mathsf{G}_{d+i}/\mathsf{L}\cdot\mathsf{B}_{k+i}$. Since L is in the normalizer of B_{k+i}, we may think of $\mathsf{G}_{d+i}/\mathsf{L}\cdot\mathsf{B}_{k+i}$ as the quotient of $\mathsf{G}_{d+i}/\mathsf{B}_{k+i}$ by a free L-action. Given that $\mathsf{G}_{d+i}/\mathsf{B}_{k+i}$ is $(k+i)$-connected, we deduce that $\mathsf{G}_\infty\star y_0$ is homotopically equivalent to the classifying space $B\mathsf{L}$. If M_0 is simply connected, then L is connected as the inclusion map $\mathsf{G}_d\star y_0 \to M_0$ is 3-connected.

5.2. *Proof of Step* 2. The construction of the chain $M_0 \subset M_1 \subset \cdots$ implies that M_{i-1} is given as the intersection of two totally geodesic copies of M_i in M_{i+1}. If M_{i+1} had positive sectional curvature we could employ the connectedness lemma to see that the inclusion map $M_{i-1} \to M_i$ is $\dim(M_{i-1})$-connected. Recall that the proof of the connectedness lemma only needs the fact that one has positive curvature along any geodesic emanating perpendicularly to M_i in M_{i+1}; see [22]. Given a unit vector v in the normal bundle of $M_i \subset M_{i+1}$ one can find an isometry $\iota \in \mathsf{G}_{d+i}$ such that $\iota_*(v) \in TM_{-1}$, where M_{-1} is the fixed-point set of the matrix $\mathrm{diag}(1, \dots, 1, -1, -1) \in \mathsf{G}_{d+1}$ in M_1. In other words we just have to check that $R(\cdot, \dot{c})\dot{c}$ is a positive definite endomorphism of $T_{c(t)}M_{i+1}$ for any geodesic in c in M_{-1}. In order to check this, notice that $R(\cdot, \dot{c})\dot{c}$ is equivariant with respect the isotropy representation of the lower $(i+1)$ block of G_{d+i} in $T_{c(t)}M_{i+1}$. Since any irreducible subspace of this representation has nontrivial intersection with $T_{c(t)}M_0$, we see that $R(\cdot, \dot{c})\dot{c}$ is indeed positive definite.

Consequently the inclusion $M_{i-1} \to M_i$ is $\dim(M_{i-1})$-connected. Therefore M_{i-1} has a periodic cohomology ring with period $h = \dim(M_i) - \dim(M_{i-1}) = u(d-k)$; see Lemma 1.3. This shows that the classifying space $B\mathsf{L} \cong \bigcup_{i=1}^{\infty} M_i$ has periodic cohomology. If M_0 is simply connected, then L is connected. Since $\{e\}, \mathsf{S}^1$, and S^3 are the only connected Lie groups whose classifying spaces have periodic cohomology, it follows that $\bigcup_{i=1}^{\infty} M_i$ has the homotopy type of a point, of $\mathbb{CP}^{\infty}$, or of $\mathbb{HP}^{\infty}$. Next recall that the inclusion map $M_i \to \bigcup_{i=1}^{\infty} M_i$ is n_i-connected with $n_i = \dim(M_i)$. If $\mathsf{L} = \{e\}$, this clearly implies that M is a homology sphere and hence a topological sphere by Smale's solution of the generalized Poincaré conjecture.

If $B\mathsf{L} \cong \mathbb{CP}^{\infty}$, then the map $M_i \to \mathbb{CP}^{\infty}$ may be viewed as a map between M_i and the n_i-skeleton $\mathbb{CP}^{n_i/2}$ of $\mathbb{CP}^{\infty}$. Since this map induces an isomorphism on cohomology, it is a homotopy equivalence by Whitehead. Similarly if $\mathsf{L} \cong \mathsf{S}^3$, then the map induces a homotopy equivalence between M_i and the n_i-skeleton $\mathbb{HP}^{n_i/4}$ of $\mathbb{HP}^{\infty}$.

5.3. *Proof of Step* 3.

5.3.1. *Recovery of the stable tangent bundle of* $\mathbb{HP}^n$. We want to recover the stable tangent bundle in the case that M is homotopically equivalent to $\mathbb{HP}^n$.

We consider again the chain of manifolds

$$M = M_0 \subset M_1 \subset \cdots .$$

First we want to show that the homotopy equivalence $M_i \to \mathbb{HP}^{n_i/4}$ from Step 2 pulls back the normal bundle of $\mathbb{HP}^{n_i/4}$ in $\mathbb{HP}^{n_{i+1}/4}$ to the normal bundle of M_i in M_{i+1}. Clearly this can be equivalently restated as follows. Let $\mathsf{S}^3 \to \Sigma^{n_i+3} \to M_i$ be the unique S^3-principal bundle whose Euler class is a generator of $H^4(M_i,\mathbb{Z})$. Then we claim that the normal bundle of M_i in M_{i+1} is isomorphic to the vectorbundle associated to the representation $\mathsf{S}^3 \to \mathsf{O}(n_{i+1}-n_i)$ given as the sum of pairwise equivalent irreducible four dimensional representations. For the proof we call the latter vector bundle V_i. Furthermore we let W_i denote the normal bundle of M_i in M_{i+1}. Clearly the restriction of V_i to M_{i-1} is isomorphic to V_{i-1}. Recall that M_{i-1} can be realized as the intersection of two copies of M_i in M_{i+1}. Thus the restriction of W_i to M_{i-1} is isomorphic to W_{i-1}.

Recall also that the isotropy group of the orbit $\mathsf{G}_{d+i}\star y_0$ is given by $\mathsf{S}^3\cdot\mathsf{B}_{k+i}$, and we can think of S^3 as being contained in the upper $(d-k)\times(d-k)$ block of G_{d+i}. The restriction of W_i to the orbit $\mathsf{G}_{d+i}\star y_0$ is given by the homogeneous vectorbundle

$$\big(\mathsf{G}_{d+i}/\mathsf{B}_{d+i}\big)\times_{\rho|\mathsf{S}^3}\mathbb{R}^{u(d-k)},$$

where ρ is the representation of S^3 in $\mathbb{R}^{u(d-k)}$ induced by the embedding of S^3 in the upper $(d-k)\times(d-k)$ block of G_{d+i}.

Since the inclusion map $M_i \to M_{i+1}$ is n_i-connected, the Euler class of the normal bundle W_i is a generator of its homology group. In fact this follows from Lemma 1.3 as the Euler class is given by the pull back of $e \in H^*(M_{i+1},\mathbb{Z})$ to $H^*(M_i,\mathbb{Z})$, where e is as defined in Lemma 1.3. By Lemma 5.2 from below, ρ splits into pairwise equivalent four dimensional irreducible representations.

Clearly we can find a number l such that the inclusion map $\mathsf{G}_{n+l}\star y_0 \to M_l$ is (n_i+1)-connected. We have seen that the restrictions of the bundles V_l and W_l to the orbit $\mathsf{G}_{n+l}\star y_0 \to M$ are isomorphic. Since the inclusion map $M_i \to M_l$ is (n_i+1)-connected as well, M_i is in M_l homotopic to the n_i-skeleton of $\mathsf{G}_{n+l}\star y_0$. Thus the restrictions of V_l and W_l to M_i are isomorphic, too. On the other hand we have seen that these bundles are isomorphic to V_i and W_i, respectively.

Notice that it now suffices to prove that for some large i the map $M_i \to \mathbb{HP}^{n_i/4}$ is a tangential homotopy equivalence.

Next we will establish the result for $\mathsf{G}_{d+i} = \mathsf{Sp}(d+i)$. Recall that the representation ρ viewed as a real representation splits into pairwise equivalent four dimensional irreducible subrepresentations. This actually determines the embedding $\mathsf{S}^3 \subset \mathsf{Sp}(d-k)$ up to conjugacy. Namely we can assume that

$$\mathsf{S}^3 = \big\{\mathrm{diag}(g,\ldots,g) \bigm| g \in \mathsf{S}^3\big\} \subset \mathsf{Sp}(d-k) =: \mathsf{Sp}(r).$$

Our next goal is to determine the isotropy representation of $\mathsf{S}^3 \subset \mathsf{Sp}(r)$ at y_0. Since we have determined the embedding of $\mathsf{S}^3 \subset \mathsf{Sp}(r)$, this amounts to determining the slice representation of S^3. For some positive integer q put

$$\varphi := 2\pi/q \text{ and } A := \mathrm{diag}(e^{i\varphi},\ldots,e^{i\varphi},1\ldots,1),$$

where the entry $e^{i\varphi}$ occurs precisely $2r^2$ times. The matrix A is contained in a principal isotropy group for $i > 2r^2$, and one component F_{iA} of $\mathrm{Fix}(A)$

is isometric to M_{i-2r^2}. Notice that this component is the unique component realizing the maximal dimension for $i > 4r^2$. We can estimate the dimension of the component N_{iy_0} of $\mathrm{Fix}(A)$ of A containing the point y_0 by

$$\dim(N_{iy_0}) \geq (r^2-1) + (2r^2-r)2r = 4r^3 - r^2 - 1. \tag{1}$$

In fact the right-hand side of the inequality is the dimension of $N_{iy_0} \cap \mathsf{G} \star y_0$, provided that the integer q is larger than 2. Notice that N_{iy_0} is contained in M_{2r^2}. In particular, $N_{iy_0} \neq F_{iA}$.

We claim that there is no component other than N_{iy_0} and F_{iA}. Suppose, on the contrary, there would be a third component F_i'. For $q > 2$ this component would be invariant under the action of the group $\mathsf{SU}(2r^2) \cdot \mathsf{Sp}(d+i-2r^2)$. If this group fixed F_i' pointwise, then G_{d+i} would fix F_i' pointwise, too; indeed this follows from the fact that the principal isotropy group of G_{d+i} contains $\mathsf{Sp}(k+i) = \mathsf{Sp}(d+i-r)$. But then clearly F_i' would intersect F_{iA} — a contradiction. Thus $\mathsf{SU}(2r^2) \cdot \mathsf{Sp}(d+i-2r^2)$ acts nontrivially on F_i' and hence $\dim(F_i') \geq 4r^2 - 2$.

Notice that this estimate also holds if we choose for q an irrational number. If q is irrational, then the fixed-point set of A equals the fixed-point set of the circle generated by A. We can then use [5, Ch. VII, Th. 5.2] to see that all components of $\mathrm{Fix}(A)$ other than F_{iA} have the rational cohomology ring of complex projective spaces. Furthermore the Euler characteristics of the fixed-point set equal $\chi(M)$, and by the dimension estimate, $\chi(N_{iy_0}) \geq 2r^3 - \frac{1}{2}r^2$ and $\chi(F_{iA}) = \frac{1}{4}(n_i - 8r^3) + 1$. Finally a third component would have Euler characteristic at least $2r^2$. But this is clearly not possible. Thus $\mathrm{Fix}(A)$ has only two components if q is irrational. This in turn implies that $\mathrm{Fix}(A)$ has precisely two components namely F_{iA} and N_{iy_0} for any integer $q > 1$. Furthermore N_{iy_0} is independent of the integer q and $\dim(N_{iy_0}) = 4r^3 - 2$ as long as $q > 2$. Since equation (1) determines the dimension of $N_{iy_0} \cap \mathsf{Sp}(d+i) \star y_0$, we deduce

$$\dim(TN_{iy_0} \cap \nu(\mathsf{Sp}(d+i) \star y_0)) = r^2 - 1 \tag{2}$$

for all odd primes q and for $q = 4$. If $\bar{\rho}\colon \mathsf{S}^3 \to O\big(\nu(\mathsf{Sp}(d+i) \star y_0)\big)$ denotes the slice representation, we obtain that for any element a of odd prime order q in S^3, the eigenspace of $\bar{\rho}(a)$ corresponding to the eigenvalue 1 has dimension $r^2 - 1$. The same holds for any element of order 4.

This shows that the slice representation has only weights 0, 1, and 2. Equivalently, the irreducible subspaces of the slice representation are of real dimension 1, 3 or 4.

Finally we consider A for $q = 2$. Since $\mathrm{Fix}(A)$ has two components, we can employ [5, Ch. VII, Th. 3.2] to see that their dimensions add up to $n_i - 4$. Thus $\dim(N_{iy_0}) = 8r^3 - 4$ if $q = 2$. This implies for the element $-1 \in \mathsf{S}^3$ that $\bar{\rho}(-1)$ has precisely $(2r^2 - r - 1)$ times the eigenvalue 1. Combining this with equation (2) we find that $\bar{\rho}_{|\mathsf{S}^3}$ decomposes as follows:

$$\bar{\rho}_{|\mathsf{S}^3} = \left(\tfrac{r(r+1)}{2} - 1\right) \cdot [1] + \tfrac{1}{2} r(r-1) \cdot [3] + h \cdot [4],$$

where [1], [3], and [4] denote the trivial, the three dimensional, and the four dimensional representation of S^3, respectively and where $4h = n_i - r^2 + 2 - 2r(d+i+1)$.

Because of $h \geq 0$ this actually gives the optimal lower bound on the dimension of M_i. In other words there is a linear action of $\mathsf{Sp}(d+i)$ on $\mathbb{HP}^{n_i/4}$ whose principal isotropy group contains $\mathsf{Sp}(d+i-r)$ as a normal subgroup. Of course our arguments apply to this linear action as well. Since we have determined the normal bundle as well as the isotropy group of the orbit, the pull back of the tangent bundle of M_i to $\mathsf{Sp}(d+i) \star y_0$ is determined.

As before we choose a number l such that the inclusion map $\mathsf{Sp}(d+l) \star y_0 \to M_l$ is (n_i+1)-connected. Since M_i is in M_l homotopic to the n_i-skeleton of $\mathsf{Sp}(d+l) \star y_0$, we see that the homotopy equivalence $h\colon M_i \to \mathbb{HP}^{n_i/4}$ pulls back the restriction of the tangentbundle of $\mathbb{HP}^{n_l/4}$ to the restriction of the tangentbundle of M_l to M_i. Since we have already established that the pull back of the normal bundle of $\mathbb{HP}^{n_i/4}$ in $\mathbb{HP}^{n_l/4}$ is the normal bundle of M_i in M_l, it follows that h is tangential.

One could argue similarly for the groups $\mathsf{SU}(d+i)$ and $\mathsf{SO}(d+i)$. However, one can actually reduce these cases to the previous one as follows.

Recall that the first part of the argument was carried out for all groups, and thereby it suffices to prove that for some large i the homotopy equivalence $M_i \to \mathbb{HP}^{n_i/4}$ is tangential. Clearly for some large i we can assume that $d+i$ is divisible by 4 and that $\frac{7}{8}(d+i) < (d+i-k)$. Consider now the subaction of $\mathsf{Sp}((d+i)\cdot u/4) \subset \mathsf{G}_{d+i}$. This subaction satisfies again the hypothesis of the stability theorem. It is actually easy to see that manifolds in the new chain form just a subsequence of the old ones. Since the original chain had an orbit whose inclusion map is h_i-connected (with $h_i \to \infty$), it is easy to see that the new one also has an orbit of this type. In other words the previous proof goes through.

LEMMA 5.2. *Let $\rho\colon \mathsf{S}^3 \to \mathsf{O}(4r)$ be a representation. Consider the homogeneous vectorbundle $E\mathsf{S}^3 \times_{\rho|\mathsf{S}^3} \mathbb{R}^{4r}$ over $B\mathsf{S}^3 = \mathbb{HP}^\infty$. If the Euler class of that vectorbundle is a generator of $H^{4r}(\mathbb{HP}^\infty, \mathbb{Z})$, then ρ is the direct sum of pairwise equivalent four dimensional irreducible subrepresentations.*

Proof. Consider the natural map $\mathrm{pr}\colon \mathbb{CP}^\infty \to \mathbb{HP}^\infty$. Recall that pr induces an isomorphism on cohomology in dimensions divisible by four. Thus if we pull back the homogeneous vectorbundle $E\mathsf{S}^3 \times_{\rho|\mathsf{S}^3} \mathbb{R}^{4r}$ to $\mathbb{CP}^\infty$, then the Euler class of the bundle is a generator of $H^{4r}(\mathbb{CP}^\infty, \mathbb{Z})$. On the other hand the pull back bundle is the homogeneous vectorbundle $E\mathsf{S}^1 \times_{\rho|\mathsf{S}^1} \mathbb{R}^{4r}$ which is the sum of two dimensional subbundles corresponding to the two dimensional invariant subspaces of $\rho_{|\mathsf{S}^1}$. Since the Euler class of the bundle is the product of the Euler classes of the subbundles, it follows that the Euler class of each of the two dimensional subbundles is a generator of $H^2(\mathbb{CP}^\infty, \mathbb{Z})$. This in turn implies that the weights of the representation ρ are all equal to 1. Clearly the result follows. ☐

5.3.2. *Recovery of the stable tangent bundle of* $\mathbb{CP}^n$. Analogously to the previous case one can show that the homotopy equivalence $M_i \to \mathbb{CP}^{n_i/2}$ pulls back the normal bundle of $\mathbb{CP}^{n_i/2}$ in $\mathbb{CP}^{n_{i+1}/2}$ to the normal bundle of M_i in M_{i+1}. As before the problem can then be reduced to the case $\mathsf{G}_{d+i} = \mathsf{Sp}(d+i)$.

Consider again the orbit $\mathsf{Sp}(d+i) \star y_0$ whose inclusion map is $4(d+i-k)$-connected. The isotropy group of y_0 is $\mathsf{L} \cdot \mathsf{B}_{k+i}$, where we can think of $\mathsf{L} \cong \mathsf{S}^1$ as being contained in the upper $(d-k) = r$ block. The embedding of $\mathsf{S}^1 \subset \mathsf{Sp}(r)$ is again determined by the fact that the Euler class of the normal bundle of M_i in M_{i+1} generates its cohomology group. In this case it follows that up to conjugacy

$$\mathsf{L} = \mathsf{S}^1 = \left\{ \operatorname{diag}(z, \ldots, z) \mid z \in \mathsf{S}^1 \right\} \subset \mathsf{Sp}(r).$$

For a positive integer q put $\varphi := 2\pi/q$, and consider the matrix

$$A_{w,q} = \operatorname{diag}(e^{i\varphi}, \ldots, e^{i\varphi}, 1, \ldots, 1)$$

which has precisely w entries equal to $e^{i\varphi}$. In the present situation we need to establish that $\operatorname{Fix}(A_{w,q})$ has precisely three components for sufficiently large i and $q > 2$. At all times we assume we assume that $i > 4w > 3r$.

Since $A_{w,q}$ is a contained in a principal isotropy group, one component F_w of $\operatorname{Fix}(A_{w,q})$ is isometric to M_{i-w}. Clearly F_w does not depend on the value of q and

$$\dim(F_w) = n_i - 4wr.$$

For $q = 2$ it is known that there are only two components and their dimensions add up to $\dim(M) - 2$, see [5, Ch VII, Th. 3.2]. Consequently

$$\dim(N_{y_0,2}) = 4wr - 2. \tag{3}$$

For $q > 2$ the component $N_{y_0,q}$ of $\operatorname{Fix}(A_{w,q})$ containing the point y_0 satisfies

$$2wr - r^2 - 1 \leq \dim(N_{y_0}) \leq 2wr - r^2 - 1 + c,$$

where c is the cohomogeneity of the action of $\mathsf{Sp}(d+i)$ — a constant that does not depend on i. We will subsequently always assume $w > c$. Next we want to show that $N_{y_0,q}$ does not depend on $q > 2$ either.

Given an odd prime q consider the sequence $N_{y_0,q} \supset N_{y_0,q^2} \supset \cdots$. Using [5, Ch. VII, Th. 3.1] we see that the components of $\operatorname{Fix}(A_{w,q^{i+1}}) \cap N_{y_0,q^i}$ have the cohomology of complex projective spaces with respect to $\mathbb{Z}_q$-coefficients.

One of these components is $N_{y_0,q^{i+1}}$. By the above dimension estimate it has codimension at most c. If there were another component, then it is easy to verify, as in the previous case, that it would have dimension at least w. But this is a contradiction as the Euler characteristic of $\mathrm{Fix}(A_{w,q^{i+1}}) \cap N_{y_0,q^i}$ is equal to the Euler characteristic of N_{y_0,q^i}.

This proves $N_{y_0,q} = N_{y_0,q^i}$ for all odd primes q, $i > 0$. Similarly one proves $N_{y_0,4} = N_{y_0,2^i}$ for all $i > 2$. Moreover it is clear that the set $\bigcap_{i>0} N_{y_0,q^i}$ is independent of the integer $q \geq 2$. In summary we can say that $N_{y_0,q}$ is independent of $q > 2$.

This is equivalent to saying that the slice representation of S^1 has only weights 0, 1, and 2. If we can determine the number of each of the weights, then we can proceed as in the previous case and we are done. For that in turn it suffices to determine the dimensions of $N_{y_0,2}$ and $N_{y_0,3}$. The former is given by equation 3. For the latter we claim that

$$\dim(N_{y_0,3}) = 2wr - 2.$$

It is known that $\mathrm{Fix}(A_{w,3})$ has at most three components and if there are three components, then their dimension adds up to $n_i - 4$; see Bredon. One of the components, F_w, has dimension $n_i - 4wr$. Thus we have to prove that there is a third component which has the same dimension as $N_{y_0,3}$. Consider the matrix $J := \mathrm{diag}(j, \ldots, j, 1 \ldots, 1)$ for which the entry $j \in \mathbb{H}$ occurs precisely w times. Notice that $JA_{w,q}J^{-1} = A_{w,q}^{-1}$. Thus J leaves the fixed-point set of $A_{w,q}$ invariant. We claim that $JN_{y_0,q}$ is a different component of $\mathrm{Fix}(A_{w,q})$ for $q > 2$.

Consider the action of $\mathsf{U}(w)$ on $N_{y_0,q}$. If $JN_{y_0,q} = N_{y_0,q}$, then the group $\bar{\mathsf{U}}(w)$ generated by $\mathsf{U}(w)$ and J would leave $N_{y_0,q}$ invariant. Notice that the orbit $\bar{\mathsf{U}}(w) \star y_0$ is disconnected. Since $N_{y_0,q}$ is connected, there must be an isotropy group K of a point $z_0 \in N_{y_0,q}$ corresponding to a face of the orbit space $N_{y_0,q}/\bar{\mathsf{U}}(w)$ such that $\bar{\mathsf{U}}(w)/\mathsf{K}$ is connected. Then K necessarily has precisely two components, and the identity component is given by the principal isotropy group $\bar{\mathsf{H}}$ of the action of $\bar{\mathsf{U}}(w)$ on $N_{y_0,q}$. Since $N_{y_0,q}$ is fixed by $A_{w,q}$ for all q, we have $\bar{\mathsf{H}} = \mathsf{U}(w-k) \cdot \mathsf{C}$, where C is the center of $\mathsf{U}(w)$. This in turn implies that the isotropy group of z_0 with respect to the G_{d+i}-action has two components as well, and the identity component is $\mathsf{S}^1 \cdot \mathsf{B}_{k+i}$. In particular, the orbit $\mathsf{G}_{n+i} \star z_0$ is not in the face F from Step 1, i.e., in the collection of all G_{n+i}-orbits whose isotropy groups contain a $k+i+1$ block. By the soul orbit theorem (Theorem 4.1) the manifold without these orbits is equivariantly diffeomorphic to the normal bundle of the soul orbit $\mathsf{G}_{n+i} \star y_0$ — a contradiction.

5.4. *Proof of Step* 4. There is nothing to prove in the even dimensional case. Thus we may assume that $\dim(M_i) = n_i$ is odd. We consider again the soul orbit $\mathsf{G}_{n+i} \star y_0$. The isotropy group is given by $\mathsf{L} \cdot \mathsf{B}_{k+i}$, where L is in the normalizer of the $(k+i) \times (k+i)$ block B_{k+i}. In the cases $\mathsf{G}_{n+i} = \mathsf{SO}(n+i)$,

$\mathsf{SU}(n+i)$, it is actually useful to consider the corresponding orbit-equivalent isometric actions of $\mathsf{O}(n+i), \mathsf{U}(n+i)$, which can be obtained as subactions from G_{n+i+1} on M_{i+1}. So we change notation and put $\mathsf{G}_{n+i} = \mathsf{O}(n+i), \mathsf{U}(n+i)$ if $u = 1, 2$. We change B_{k+i} consistently and think of it as the lower $k \times k$ block in G_n. After this change of notation we may assume that L is contained in the upper $(d-k)$ block. Since the inclusion map $\mathsf{G}_{n+i} \star y_0 \to M_i$ is $u(k+i)$-connected, it follows that L is isomorphic to $\pi_1(M)$. The embedding of L in the upper $(d-k)$ block induces a faithful representation

$$\rho\colon \mathsf{L} \to \mathbb{R}^{u(d-k)}.$$

It suffices to prove that ρ induces a free action on the sphere $\mathbb{S}^{u(d-k)-1}$.

We argue by contradiction. Suppose there is a finite cyclic subgroup $\mathbb{Z}_p \subset \mathsf{L}$ of prime order p such that $\rho(\mathbb{Z}_p)$ does act freely on the sphere. We now consider the manifold M_i' that is obtained from the universal cover $\tilde{M}_i$ of M_i by dividing the cyclic group $\mathbb{Z}_p$. Clearly the manifold M_i' also admits an action G_{n+i}, and the inverse image of the soul orbit in M_i' is again an orbit whose inclusion map is $u(k+i)$-connected. In other words we have reduced the problem to the case of a cyclic group of prime order.

Now, suppose that L has prime order and that the representation ρ does not induce a free action. Then there is a nontrivial subspace of $\mathbb{R}^{u(d-k)}$ that is fixed by ρ. This in turn implies that the homogeneous vectorbundle

$$\mathsf{G}_{n+i}/\mathsf{B}_{k+i} \times_{\rho|\mathsf{L}} \mathbb{R}^{u(d-k)}$$

over the soul orbit $\mathsf{G}_{n+i} \star y_0$ has a section. In particular, its Euler class is zero. On the other hand we have seen that this bundle is the pull back of the normal bundle of M_i in M_{i+1}. Since the inclusion map $M_i \to M_{i+1}$ is $\dim(M_i)$-connected, the Euler class is a generator of its cohomology group, a contradiction as the even dimensional cohomology groups are equal to $\mathbb{Z}_p$.

6. The linear model of a chain

THEOREM 6.1. *Suppose that G_d acts isometrically on a simply connected positively curved manifold M and that the hypothesis of Theorem* 5.1 *is satisfied. Then M is homotopically equivalent to a compact rank one symmetric space S* (*the model*). *There is a linear action of G_d on S such that the isotropy groups of the two actions are in one-to-one correspondence.*

In the case that M is homeomorphic to a sphere Σ^n, the theorem is actually a consequence of a theorem of Hsiang and Davis [7], and Hsiang, Davis and Morgan [8]. In fact they also show that Σ is then necessarily given by a Brieskorn variety. A theorem of Hsiang and Straume [15] asserts that an action of $\mathsf{SU}(d)$ on a simply connected manifold with the cohomology ring of $\mathbb{S}^n$, $\mathbb{CP}^n$ or $\mathbb{HP}^n$ has a linear model, provided that the number of orbit types is at most

$p(d)$, the number of partitions of d. Using the soul orbit theorem (Theorem 4.1) it is easy to verify the hypothesis. Hence one can use this result to prove Theorem 6.1 for $\mathsf{G}_d = \mathsf{SU}(d)$. By their theorem all slice representations coincide with the corresponding slice representations of the linear model as well. The authors also indicate that similar theorems should hold for $\mathsf{G}_d = \mathsf{SO}(d), \mathsf{Sp}(d)$, but this has not been carried out yet.

For us the main use of Theorem 6.1 is that it guarantees large isotropy groups. This will be particularly useful in Sections 8 and 14

Proof. By assumption the principal isotropy group H_{d+i} of the G_{d+i}-action on M_i contains a $k+i \times k+i$ block B_{k+i} as a normal subgroup. Furthermore, if we remove all orbits whose isotropy groups contain a $(k+i+1)\times(k+i+1)$ block, then the resulting manifold is equivariantly diffeomorphic to the normal bundle of the soul orbit $\mathsf{G}_{d+i} \star y_0$. As we have seen in the proof of Theorem 5.1, the isotropy group of y_0 is given by $\mathsf{L} \cdot \mathsf{B}_{k+i}$ with L being in the normalizer of B_{k+i} and $\mathsf{L} = \{1\}, \mathsf{S}^1$ or S^3.

Consider first the case that L is trivial or equivalently that M_i is homeomorphic to a sphere. By the previous argument any nonprincipal isotropy group K contains a $(k+i+1) \times (k+i+1)$ block. Next we show that K is a block. Consider the slice representation of K as an action on the normal sphere. The principal isotropy group of that action is H_{d+i}. Since K contains a $(k+i+1) \times (k+i+1)$ block, the soul orbit theorem (Theorem 4.1) tells us that if we remove all orbits from the normal sphere whose isotropy group contains a $(k+i+1)\times(k+1)$ block, then the resulting manifold is equivariantly diffeomorphic to a the normal bundle of a soul orbit. Since the corresponding isotropy group contains no $(k+i+1) \times (k+i+1)$ block, it must be principal by the previous argument. Thus the principal orbit is $(k+i-1)$-connected, as its inclusion map is $(k+i)$-connected. Since H_{d+i} is a block, this implies that K is a block as well for sufficiently large i and hence for all i.

If G_d has a fixed-point in M, we are done. Thus we may assume G_d has no fixed-point. We plan to prove next that $\dim(M_{d+i}) = u(d+i)(d-k) - 1$. In case of $u = 1, 2$ it is useful to replace G_d by the orbit equivalent actions of $\mathsf{O}(d)$ or $\mathsf{U}(d)$, respectively. Since the isotropy groups of the G_{d+i}-actions are blocks, the nonexistence of a fixed-point of G_{d+i} implies that $-\mathrm{id} \in \mathsf{G}_d$ has no fixed-points either. The quotient $N_i := M_i/\pm\mathrm{id}$ is a cohomology $\mathbb{RP}^{d_i}$.

For the induced action of G_{d+i} on N_i we consider the element

$$\iota = \mathrm{diag}(-1, \ldots, -1, 1 \ldots 1)$$

with precisely $d-k$ entries being equal to -1 for $i > d-k$. By Bredon, the fixed-point set of ι has two components, and their dimensions add up to $n_i - 1$. Clearly one of the components is isometric to M_{i-d+k}, and its codimension is given by $u(d-k)^2$. Notice that $-\iota$ is contained in the lower $(k+i)\times(k+i)$ block.

Thus the other component has codimension $u(d-k)(k+i)$. Consequently, $n_i = u(d-k)(d+i)-1$.

This in turn shows that there is a linear action of G_d on the sphere $\mathbb{S}^{n_i}$ with principal isotropy group H_{n+i}. It only remains to check that G_{d+i-1} occurs as an isotropy group of the G_{d+i}-action on M_i. Suppose not. Then we could pass to the fixed-point free G_{n+i-1}-subaction on M_i and prove as above that $n_i = u(d-k)(d+i-1)-1$, which is nonsense.

In the case that M is homotopically equivalent to $\mathbb{CP}^{n/2}$ or $\mathbb{HP}^{n/4}$ the theorem follows by a careful analysis of the proof that the homotopy equivalence is tangential combined with arguments that are similar to the above case. □

7. Homogeneous spaces with spherical isotropy representations

We call the isotropy representation of a homogeneous space G/H spherical if any nontrivial irreducible subrepresentation is transitive on the sphere; cf. Definition 3.3. The reason why we are interested in these homogeneous spaces is Lemma 3.4. We are actually mainly interested in how the normal simple subgroups of H can sit in G. Notice that if H' is a normal subgroup of H, then G/H' has a spherical isotropy representation. We begin by the following observation:

LEMMA 7.1. *If* G/H *has a spherical isotropy representation, and* H *is a simple connected group of rank* ≥ 2, *then* H *is contained in a simple normal subgroup of* G.

Notice that this is actually obvious because if H were to lie diagonally in G, then the adjoint representation of H would be a subrepresentation of the isotropy representation of G/H. If H is a simple group of rank 1, one can easily conclude that for any projection $\bar{\mathsf{H}}$ to a simple factor G' of G the space $\mathsf{G}'/\bar{\mathsf{H}}$ also has a spherical isotropy representation. This way we have effectively limited our problem to the case of a simple Lie group G.

LEMMA 7.2. *Let* $\mathsf{G} = \mathsf{SO}(n), \mathsf{SU}(n), \mathsf{Sp}(n)$ *and* $\mathsf{H} \cong \mathsf{S}^3, \mathsf{SO}(3)$. *Suppose that* G/H *has a spherical isotropy representation. Then one of the following holds*:

a) H *is embedded as a* 3×3 *block of* $\mathsf{SO}(n)$.

b) $\mathsf{G} = \mathsf{SO}(n)$ *and the embedding* $\mathsf{H} \to \mathsf{SO}(n)$ *is a real representation that decomposes into a trivial representation and irreducible subrepresentations of real dimension* 4.

c) $\mathsf{G} = \mathsf{SU}(n)$ *and the embedding* $\mathsf{H} \to \mathsf{SU}(n)$ *is a complex representation that decomposes as a trivial representation and representations of complex dimension* 2.

d) $\mathsf{G} = \mathsf{Sp}(n)$ *and up to conjugacy* $\mathsf{H} = \{\mathrm{diag}(q, q \ldots, q, 1, \ldots, 1) \mid q \in \mathsf{S}^3\}$.

Proposition 7.3. *Let* $\mathsf{G} = \mathsf{SO}(n), \mathsf{SU}(n), \mathsf{Sp}(n)$, *and let* H *be a simple, connected proper subgroup of rank* ≥ 2 *such that* G/H *has a spherical isotropy representation. Then one of the following holds*:

a) *Up to conjugacy* H *is given as a* $k \times k$ *block in* G.

b) $\mathsf{G} = \mathsf{SO}(n)$ *and*

  (i) $\mathsf{H} = \mathsf{SU}(3) \subset \mathsf{SO}(6) \subset \mathsf{SO}(n)$,

  (ii) $\mathsf{H} = \mathsf{G}_2 \subset \mathsf{SO}(7) \subset \mathsf{SO}(n)$,

  (iii) $\mathsf{H} = \mathsf{Sp}(2) \subset \mathsf{SU}(4) \subset \mathsf{SO}(8) \subset \mathsf{SO}(n)$,

  (iv) $\mathsf{H} = \mathsf{SU}(4) \subset \mathsf{SO}(8) \subset \mathsf{SO}(n)$ *or*

  (v) $\mathsf{H} = \mathsf{Spin}(7) \subset \mathsf{SO}(8) \subset \mathsf{SO}(n)$.

c) $\mathsf{G} = \mathsf{SU}(n)$ *and* $\mathsf{H} = \mathsf{Sp}(2) \subset \mathsf{SU}(4) \subset \mathsf{SU}(n)$.

Proposition 7.4. *Let* G *be one of the exceptional compact simple Lie groups, and let* H *be a connected simple proper subgroup of rank* ≥ 2 *such that* G/H *has a spherical isotropy representation. Then one of the following holds*:

a) $(\mathsf{G}, \mathsf{H}) = (\mathsf{G}_2, \mathsf{SU}(3))$.

b) G *is one of the other exceptional groups and* $\mathsf{H} \subset \mathsf{Spin}(9) \subset \mathsf{F}_4 \subset \mathsf{E}_6 \subset \mathsf{E}_7 \subset \mathsf{E}_8$. *Furthermore* $\mathsf{H} \subset \mathsf{Spin}(9)$ *is given by a* $k \times k$ *block with* $k = 5, 6, 7, 8, 9$, *unless* $\mathsf{H} = \mathsf{SU}(3)$ *or* G_2.

By a $k \times k$ block of $\mathsf{Spin}(9)$ we mean that the epimorphism $\mathsf{Spin}(9) \to \mathsf{SO}(9)$ maps the group H up to conjugacy to a $k \times k$ block of $\mathsf{SO}(9)$.

The proofs of the above results are straightforward and we omit them.

8. Exceptional actions with large principal isotropy groups

Theorem 8.1. *Let* G *be a simple, simply connected Lie group acting nontrivially, isometrically and nontransitively on a positively curved manifold* M. *Suppose that the principal isotropy group of that action has a simple, connected normal subgroup* H' *of rank* ≥ 2. *Suppose also that the hypothesis of Theorem* 5.1 *is not satisfied. Then* $(\mathsf{G}, \mathsf{H}')$ *is one of the following pairs*:

- $(\mathsf{E}_7, \mathsf{Spin}(8))$, $(\mathsf{E}_6, \mathsf{Spin}(8))$,
- $(\mathsf{F}_4, \mathsf{Spin}(8))$, $(\mathsf{F}_4, \mathsf{SU}(3))$,
- $(\mathsf{Spin}(10), \mathsf{SU}(4))$, $(\mathsf{Spin}(10), \mathsf{SU}(3))$,
- $(\mathsf{Spin}(9), \mathsf{Spin}(7))$, $(\mathsf{Spin}(9), \mathsf{G}_2)$, $(\mathsf{Spin}(9), \mathsf{SU}(3))$,
- $(\mathsf{Spin}(8), \mathsf{G}_2)$, $(\mathsf{Spin}(8), \mathsf{SU}(3))$,
- $(\mathsf{Spin}(7), \mathsf{G}_2)$, $(\mathsf{Spin}(7), \mathsf{SU}(3))$ *or* $(\mathsf{G}_2, \mathsf{SU}(3))$.

These are precisely the pairs that occur for linear actions on spheres. Of course, these pairs can occur in arbitrarily high dimensions if one stabilizes the representation by a trivial one. The embedding $\mathsf{H}' \subset \mathsf{G}$ is uniquely determined by the previous section. In the case of $\mathsf{G} = \mathsf{Spin}(8)$ we assumed that the hypothesis of the stability theorem is not satisfied even up to an outer automorphism. Notice that the theorem implies in particular that E_8 cannot act on a positively curved manifold with a large principal isotropy group; see also Proposition 8.9 and Corollary 12.2.

Propositions 7.3 and 7.4 reduce the problem of proving Theorem 8.1 considerably. We treat the remaining cases in the following seven lemmas. The hypothesis of Theorem 8.1 is assumed to be valid for the remainder of this section. We may also assume that M is simply connected.

LEMMA 8.2. *Suppose* H *contains* $\mathsf{SU}(3)$ *as a normal subgroup. Then* G *is one of the groups* $\mathsf{F}_4, \mathsf{Spin}(7), \mathsf{Spin}(8), \mathsf{Spin}(9), \mathsf{Spin}(10)$, *or* G_2.

Proof. We argue by contradiction. By the previous section $\mathsf{G} = \mathsf{Spin}(l)$ or E_k with $l \geq 11$ or $k = 6, 7, 8$. Consider first the case of $\mathsf{G} = \mathsf{Spin}(l)$. The first step is to show that there is an isotropy group containing $\mathsf{Spin}(h) \supset \mathsf{SU}(3)$ embedded as an $h \times h$ block. For that we consider the possibility that there is an isotropy group K containing $\mathsf{K}' \cong \mathsf{SU}(4)$ or $\mathsf{Spin}(7)$ as a normal subgroup. We may assume that K' is not embedded as a block. Consider the central involution ι in K'. Then $\mathrm{Fix}(\iota)$ has a component M' with an action of $\mathsf{Spin}(l-8) \cdot \mathsf{Spin}(8)$ whose principal isotropy group contains $\mathsf{K}' \subset \mathsf{Spin}(8)$. It is easy to see that this action is not transitive. This implies that the subaction of $\mathsf{Spin}(8)$ has cohomogeneity ≥ 3 on M'. Notice that the embedding of $\mathsf{K}' \subset \mathsf{Spin}(8)$ is up to an outer automorphism of $\mathsf{Spin}(8)$ given by a block. Theorem 6.1 tells us that $\mathsf{Spin}(8)$ has a fixed-point.

To establish the first step it only remains to rule out the possibility that any isotropy group has either $\mathsf{SU}(3)$ or G_2 as a normal subgroup. By the rank lemma (Proposition 1.4) there is an isotropy group K of rank ≥ 3 whose orbit type is not isolated. If we choose K minimal, it is easy to see that $\mathsf{K} = \mathsf{S}^1 \cdot \mathsf{SU}(3)$ or $\mathsf{K} = \mathsf{S}^3 \cdot \mathsf{SU}(3)$. The isotropy representation of $\mathsf{Spin}(l)/\mathsf{K}$ contains a $6k$-dimensional irreducible representation with finite kernel. Clearly such a representation cannot degenerate in an isotropy group containing G_2 as a normal subgroup.

Thus there is an isotropy group K containing $\mathsf{Spin}(h) \supset \mathsf{SU}(3) \subset \mathsf{H}$ embedded as an $h \times h$ block. We may assume that $\mathsf{Spin}(h)$ is a normal subgroup of K; see Lemma 2.7. We choose h minimal. Consider the central involution $\iota \in \mathsf{Spin}(h)$ that is also central in $\mathsf{Spin}(l)$. Then $\mathrm{Fix}(\iota)$ has a component M' with an action of $\mathsf{Spin}(l)$ whose principal isotropy group contains $\mathsf{Spin}(h)$. By Theorem 6.1 $\mathsf{Spin}(l-1)$ occurs as the identity component of an isotropy group. For $l \geq 12$ this is a contradiction as there is no representation of

$\mathsf{Spin}(l-1)$ with principal isotropy group being $\mathsf{SU}(3)$. For $l=11$ the slice representation of $\mathsf{Spin}(11)/\mathsf{Spin}(10)$ necessarily consists of the 32-dimensional spin representation, the 10-dimensional standard representation and a trivial representation. If the trivial representation were larger than one dimensional, then we could employ Theorem 6.1, to guarantee a fixed-point of $\mathsf{Spin}(11)$. Thus $\dim(M)=52,53$.

Consider the subgroups $\{e\}\subset \mathsf{SU}(2)\subset \mathsf{SU}(3)\subset \mathsf{G}_2\subset \mathsf{Spin}(7)_{ns}\subset \mathsf{Spin}(8)$ of $\mathsf{Spin}(10)$, where $\mathsf{Spin}(7)_{ns}$ is embedded by the 8-dimensional representation. We denote the fixed-point components containing a component of $\mathrm{Fix}(\mathsf{Spin}(10))$ of these groups by

$$M^{53-\varepsilon}\supset N^{29-\varepsilon}_{\mathsf{SU}(2)}\supset N^{17-\varepsilon}_{\mathsf{SU}(3)}\supset N^{12-\varepsilon}_{\mathsf{G}_2}\supset N^{7-\varepsilon}_{ns}\supset N^{5-\varepsilon}_{\mathsf{Spin}(8)},$$

where the upper index denotes the dimension and $\varepsilon\in\{1,0\}$. The connectedness lemma implies that all manifolds are simply connected. If $\varepsilon=0$, we can use the totally geodesic embedding $N^5_{\mathsf{Spin}(8)}\subset N^7_{ns}$ to see that both manifolds are homotopy spheres; see [22, Prop. 7.3].

Suppose for a moment we can show this no matter whether $\varepsilon=0$ or 1. Then we can argue as follows. Notice that $N^{7-\varepsilon}_{ns}$ is invariant under an action of $\mathsf{Spin}(3)$ embedded as a normal subgroup of $\mathsf{Spin}(3)\cdot\mathsf{Spin}(7)_{ns}\subset\mathsf{Spin}(11)$. The central involution $\iota\in\mathsf{Spin}(3)$ coincides with the central involution in $\mathsf{Spin}(10)$, and $N^{5-\varepsilon}_{\mathsf{Spin}(8)}$ is a component of the fixed-point set of ι restricted to $N^{7-\varepsilon}_{ns}$. Since the manifolds are spheres, we can use Smith theory to see that ι has no other components. Thus $\mathsf{Spin}(3)$ has no singular orbits in $B:=N^{7-\varepsilon}_{ns}\setminus N^{5-\varepsilon}_{\mathsf{Spin}(8)}$. This implies that the rational $\mathsf{Spin}(3)$-equivariant cohomology of B is given by the cohomology of the orbit space B/S^3. On the other hand B is a cohomology $\mathbb{S}^1$, and a simple computation shows that the cohomology of $B\times_{\mathsf{S}^3}E\mathsf{S}^3$ is given by the cohomology of $\mathbb{S}^1\times\mathbb{HP}^\infty$.

In the case of $\varepsilon=1$ it remains to verify that $N^4_{\mathsf{Spin}(8)}\subset N^6_{ns}$ are homotopy spheres. Suppose not. Since $N^4_{\mathsf{Spin}(8)}$ is invariant under a nontrivial $\mathsf{Spin}(3)$-action, we can employ a result of Hsiang and Kleiner [14] to see that $N^4_{\mathsf{Spin}(8)}$ is homeomorphic to $\mathbb{CP}^2$. Then N^6_{ns} is a homotopy $\mathbb{CP}^3$. Furthermore the involution $\iota\in\mathsf{Spin}(3)$ has in N^4_{ns} precisely one isolated fixed-point outside $N_{\mathsf{Spin}(8)}$. Using the fact that ι is central, we see that $\mathsf{Spin}(3)$ fixes this point as well. Hence there is an isotropy group K containing $\mathsf{Spin}(3)\cdot\mathsf{Spin}(7)_{ns}$. But then it is easy to see that there is also an isotropy group K_2 containing $\mathsf{Spin}(3)\cdot\mathsf{SU}(4)$ with the additional property that its orbit type is not isolated. The irreducible 24-dimensional subrepresentation of $\mathsf{Spin}(11)/\mathsf{K}_2$ is not equivalent to a subrepresentation of the slice. Thus it has to degenerate by Lemma 3.1 which can only happen in a fixed-point of $\mathsf{Spin}(11)$ — a contradiction.

Thus $\mathsf{G}=\mathsf{E}_6,\mathsf{E}_7,\mathsf{E}_8$. Essentially as in the spin case one can show that there is an isotropy group K which contains $\mathsf{Spin}(h)\supset\mathsf{SU}(3)$. The embedding is unique in this case, and again we may assume that $\mathsf{Spin}(h)$ is normal in K. If we choose h minimal, then it is easy to see that $h \leq 7$.

Let ι be the central involution in $\mathsf{Spin}(h)$. Then a fixed-point component M' of ι admits an isometric action of the normalizer $N(\iota)$ whose principal isotropy group contains $\mathsf{Spin}(h)$. If $\mathsf{G} = \mathsf{E}_7$ or E_8, then $N(\iota) = \mathsf{S}^3 \cdot \mathsf{Spin}(12)$ or $\mathsf{Spin}(16)$. It then follows from Theorem 6.1 that there is an isotropy group containing $\mathsf{Spin}(11)$. But this is clearly impossible as there is no representation of $\mathsf{Spin}(11)$ or of a bigger group whose principal isotropy group contains $\mathsf{SU}(3)$ as a normal subgroup. Thus we may assume $\mathsf{G} = \mathsf{E}_6$. Then M' comes with an action of $\mathsf{S}^1 \times \mathsf{Spin}(10)$. If $h < 7$, then there is an isotropy group containing $\mathsf{Spin}(9)$ as a normal subgroup whose slice representation contains 3 times the 9-dimensional representation. This is clearly impossible, since the principal isotropy group contains $\mathsf{SU}(3)$ as a normal subgroup. Thus $h = 7$. If $\dim(M') \geq 30$, then we can employ Theorem 6.1 to show that $\mathsf{Spin}(10)$ is contained in an isotropy group. Furthermore the slice representation of $\mathsf{Spin}(10)$ contains three 10-dimensional subrepresentations — a contradiction.

Thus we may assume $\dim(M') < 30$. By Theorem 6.1 $\dim(M') = 29$ and thereby $\dim(M) = 77$. As above we can show that $\mathsf{Spin}(9)$ is contained in an isotropy group such that the slice representation contains two 9-dimensional subrepresentations of $\mathsf{Spin}(9)$. It is then easy to see that $\mathrm{Fix}(\mathsf{Spin}(9))$ has a 2-dimensional component which is invariant under an S^1-action. Thus there is a larger isotropy group K with identity component $\mathsf{S}^1 \cdot \mathsf{Spin}(9)$. The slice representation of K restricted to $\mathsf{Spin}(9)$ contains a unique 16-dimensional subrepresentation (of real type). Thus the action of the circle $\mathsf{S}^1 \subset \mathsf{K}$ must fix the 16-dimensional subrepresentation pointwise. On the other hand the order 2 element in $\mathsf{S}^1 \subset \mathsf{S}^1 \cdot \mathsf{Spin}(9) \subset \mathsf{E}_6$ coincides with the central element of $\mathsf{Spin}(9)$ — a contradiction. □

LEMMA 8.3. *Suppose that* H *contains the exceptional Lie group* G_2 *as a normal subgroup. Then* $\mathsf{G} = \mathsf{Spin}(9)$ *or* $\mathsf{Spin}(8)$.

Proof. Since G/H_0 has a spherical isotropy representation, $\mathsf{H}_0 = \mathsf{G}_2$. Notice that there is an isotropy group K corresponding to a face with $\mathsf{K}_0 \cong \mathsf{Spin}(7)$.

Let us first rule out $\mathsf{G} \cong \mathsf{E}_7, \mathsf{E}_8$. Choose the central element $\iota \in \mathsf{Spin}(7)$. Then a component of $\mathrm{Fix}(\iota)$ admits an action of $\mathsf{S}^3 \cdot \mathsf{Spin}(12)$ or $\mathsf{Spin}(16)$ whose principal isotropy group contains a 7×7 block. By Theorem 6.1, it follows that there is an isotropy group $\bar{\mathsf{K}}$ containing $\mathsf{Spin}(11)$. Since there is no representation of $\mathsf{Spin}(11)$ whose principal isotropy group is G_2 this is a contradiction.

Next we consider $\mathsf{G} = \mathsf{E}_6$. We can use a similar argument to show that $\mathsf{Spin}(8)$ occurs as a normal subgroup of an isotropy group. Given that $\mathsf{Spin}(8)$ is a regular subgroup, we can find an isotropy group $\bar{\mathsf{K}}$ of rank ≥ 5 containing $\mathsf{Spin}(8)$. Since $\mathsf{Spin}(10)$ has no representation with principal isotropy group being G_2, it follows that $\bar{\mathsf{K}}_0 = \mathsf{Spin}(9) \cdot \mathsf{K}'$ or $\mathsf{Spin}(8) \cdot \mathsf{K}'$, where $\mathsf{K}' \subset \bar{\mathsf{K}}_0$ is a normal subgroup of rank at least 1. Furthermore it is easy to reduce the former case to the latter. Thus we may assume $\bar{\mathsf{K}}_0 = \mathsf{Spin}(8)\cdot\mathsf{K}'$. The slice representation contains two nonequivalent representations of $\mathsf{Spin}(8)$, and it is clear that these two 8-dimensional subspaces are necessarily fixed by K'. Therefore there is also an isotropy group $\tilde{\mathsf{K}}$ of the form $\tilde{\mathsf{K}}_0 = \mathsf{Spin}(7)\cdot\mathsf{K}'$ with the additional property that $\tilde{\mathsf{K}}$ does not correspond to an isolated orbit type. It is now straightforward to check that $\mathsf{E}_6/\mathsf{Spin}(7)\cdot\mathsf{K}'$ contains an irreducible subrepresentation of dimension 16 or of dimension 32. This representation is necessarily not equivalent to a subrepresentation of the slice. Hence it has to degenerate by Lemma 3.1. Thus there is a larger isotropy group $\hat{\mathsf{K}}$ such that $\hat{\mathsf{K}}/\mathsf{Spin}(7)\cdot\mathsf{K}'$ contains a 16- or 32-dimensional isotropy representation. This implies $\hat{\mathsf{K}}_0 = \mathsf{Spin}(10)\cdot\mathsf{S}^1$ or E_6 — a contradiction as these groups have no representations whose principal isotropy groups contain G_2 as a normal subgroup.

Next we consider $\mathsf{G} = \mathsf{Spin}(l)$ for $l \geq 10$. As before, let K be an isotropy group corresponding to a face with $\mathsf{K}_0 \cong \mathsf{Spin}(7)$. If $\mathsf{Spin}(7)$ is embedded by the 8-dimensional representation, then we can argue as follows. Let ι be the central involution of $\mathsf{Spin}(7)$. Then $\mathrm{Fix}(\iota)$ has a component M' with an isometric action of $\mathsf{Spin}(8)\cdot\mathsf{Spin}(l-8)$, whose principal isotropy group has $\mathsf{Spin}(7)$ as its identity component. Clearly $\mathsf{Spin}(8)$ has a fixed-point in M' and M' is fixed-point homogeneous. By Grove-Searle [12] $l \leq 11$ and $\mathsf{Spin}(l-8)$ has a fixed-point as well. Therefore an isotropy group of $\bar{\mathsf{K}}$ contains $\mathsf{Spin}(l-8)\cdot\mathsf{Spin}(7)$. It is then easy to see that there also must be an isotropy group containing $\mathsf{Spin}(l-8)\cdot\mathsf{G}_2$ with the additional property that it is not isolated. As before Lemma 3.1 yields the desired contradiction.

Thus we may assume that $\mathsf{Spin}(7)$ is embedded by the 7-dimensional representation. Consider again the central involution ι of $\mathsf{Spin}(7)$. Then ι is central in $\mathsf{Spin}(l)$, and $\mathrm{Fix}(\iota)$ has a component M' with an isometric action of $\mathsf{Spin}(l)$, whose principal isotropy group has $\mathsf{Spin}(7)$ as its identity component. This implies that $\mathsf{Spin}(8)$ occurs as an isotropy group. Notice that the slice representation of $\mathsf{Spin}(8)$ consists of 2 nonequivalent 8-dimensional representations. It is then easy to see that there is an isotropy group K' whose identity component is $\mathsf{Spin}(7)$ embedded by the 8-dimensional representation.

We are left with the case of $\mathsf{G} = \mathsf{F}_4$. Again $\mathsf{Spin}(7)$ occurs as the identity component of an isotropy group corresponding to a face. Let ι be the central involution in $\mathsf{Spin}(7)$. Then $\mathrm{Fix}(\iota)$ has a component M' that is invariant under an action of $\mathsf{Spin}(9)$ whose principal isotropy group has a $\mathsf{Spin}(7)$ block as identity component embedded as a 7×7 block. Clearly the action cannot be transitive. Thus $\dim(M) \geq 40$. If $\dim(M) \geq 42$, then $\dim(M') \geq 18$ and it follows that $\mathsf{Spin}(9)$ has a fixed-point in M' by Theorem 6.1. This implies that the slice representation of $\mathsf{F}_4/\mathsf{Spin}(9)$ has two 9-dimensional subrepresentations which is nonsense as its principal isotropy group is G_2. Hence $\dim(M) = 40, 41$.

We consider a fixed-point component N of G_2 of codimension 35. The normalizer of G_2 in F_4 is connected and isomorphic to $\mathsf{SO}(3)\times\mathsf{G}_2$. The orbit spaces $N/\mathsf{SO}(3)$ and M/F_4 are isometric, and the $\mathsf{Spin}(7)$-face corresponds in $N/\mathsf{SO}(3)$ to a face with finite isotropy group. Since this isotropy group cannot preserve an orientation of N, it follows that N is not orientable. By Synge's theorem the dimension of N is even and $\pi_1(N)\cong\mathbb{Z}_2$. Thus $\dim(M)=41$ and $\dim(N)=6$.

Let N_i be the fixed-point component of $\mathsf{SU}(i)\subset\mathsf{SU}(3)$ containing N $(i=2,3)$. It is easy to see that $\dim(N_2)=21$ and $\dim(N_3)=11$. Applying the connectedness lemma repeatedly to the chain $N^6\subset N_2^{11}\subset N_3^{21}\subset M^{41}$ we see that the inclusion map $N\to M$ is 2-connected — a contradiction. □

LEMMA 8.4. *The isotropy representation of* G/H *cannot contain both a* 5- *as well an* 8-*dimensional irreducible subrepresentation.*

Proof. We argue by contradiction. By the previous section $\mathsf{G}=\mathsf{Spin}(n)$, $\mathsf{SU}(n),\mathsf{F}_4,\mathsf{E}_6$, E_7, E_8, and H contains $\mathsf{Sp}(2)$ as a normal subgroup. If the cohomogeneity is larger than 1, then there is an isotropy group K such that the isotropy representation of K/H contains both an 8- and a 5-dimensional representation. The slice representation induces a cohomogeneity-one action on the normal sphere and the principal isotropy group is H. No such representation exists.

If the cohomogeneity equals 1, then the 8-dimensional representation has to degenerate. It follows that $\mathsf{K}/\mathsf{H}\cong\mathbb{S}^{11}$, and the corank of K is at most one. In summary we can say $\mathsf{K}\subset\mathsf{G}$ has $\mathsf{Sp}(3)$ as a normal subgroup of and $\mathrm{corank}(\mathsf{K})\le 1$. This actually rules out all groups G except for $\mathsf{G}=\mathsf{F}_4$. In this remaining case one can argue as follows. Consider the central element $\iota\in\mathsf{Sp}(3)$, which acts necessarily as $-\,\mathrm{id}$ on the slice. There is also a 28-dimensional subspace $U\subset T\mathsf{F}_4/\mathsf{K}$ on which ι acts as $-\,\mathrm{id}$. This subspace is totally geodesic. On the other hand this subspace has intrinsic zero curvatures — a contradiction. □

LEMMA 8.5. *If the isotropy representation of* G/H *contains both a* 7-*dimensional as well as an* 8-*dimensional irreducible subspace*, *then* $\mathsf{G}\cong\mathsf{Spin}(9)$.

Proof. From the previous subsection it follows that $\mathsf{Spin}(7)$ is a normal subgroup of G. Furthermore the irreducible subrepresentations of $\mathsf{G}/\mathsf{Spin}(7)$ are of real type. Since G/H_0 has a spherical isotropy representation, it follows that $\mathsf{H}_0\cong\mathsf{Spin}(7)$. Let us first consider the case of cohomogeneity >1. By the isotropy lemma there is an isotropy group K corresponding to a face of the orbit space, such that the isotropy representation of K/H contains the 8-dimensional representation of H. Since K/H is a sphere, it follows that K has a normal subgroup isomorphic to $\mathsf{Spin}(9)$, and the slice representation of K is given by the 16-dimensional spin representation of $\mathsf{Spin}(9)$. Suppose for a moment that the isotropy representation of G/K has an irreducible subrepresentation whose kernel does not contain $\mathsf{Spin}(9)$ and that is not equivalent to the slice representation of K. Then there is an isotropy group $\bar{\mathsf{K}}$ corresponding to a codimension 2 stratum such that the isotropy representation of $\bar{\mathsf{K}}/\mathsf{K}$ contains this representation. In particular, $\mathsf{Spin}(9)$ is not a normal subgroup of $\bar{\mathsf{K}}$. On the other hand, the slice representation of $\bar{\mathsf{K}}$ induces a cohomogeneity-one action on the sphere with principal isotropy group H — a contradiction, as such a representation does not exist.

Thus we now can assume that the isotropy representation of $\mathsf{G}/\mathsf{Spin}(9)$ decomposes as a trivial representation and a 16-dimensional irreducible subrepresentation. Consider the central involution ι in $\mathsf{Spin}(9)$. The centralizer $N(\iota)$ must contain $\mathsf{Spin}(9)$ as a normal subgroup. Using the fact that $\mathsf{G}/N(\iota)$ is a symmetric space we deduce $\mathsf{G} = \mathsf{F}_4$, $\mathsf{K} = \mathsf{Spin}(9)$, and $\mathsf{H} = \mathsf{Spin}(7)$.

If the cohomogeneity is larger than 2 we can employ Lemma 3.1 to see that the 8-dimensional spin representation of $\mathsf{Spin}(7)$ has to degenerate twice. This would imply that F_4 has a fixed-point which is nonsense as there is no representation of F_4 with principal isotropy group $\mathsf{Spin}(7)$.

If the cohomogeneity is 2, then $\dim(M) = 33$. It is easy to see that there are only two orbit types with isotropy groups $\mathsf{Spin}(7)$ and $\mathsf{Spin}(9)$, respectively. In particular, it follows from the soul orbit theorem that M is 7-connected, as this is true for $\mathsf{F}_4/\mathsf{Spin}(7)$. Consider next the fixed-point set of $\mathsf{SU}(2) \subset \mathsf{Spin}(7)$ embedded as a normal subgroup of a 4×4 block. It is easy to see that there is a component $N \subset \mathrm{Fix}(\mathsf{SU}(2))$ of dimension 17. Using the connectedness lemma (Theorem 1.2) we see that the inclusion map $N \to M$ is 5-connected and hence N is 4-connected. The manifold N is invariant under an action of $\mathsf{Sp}(3)$. There are only two orbit types for this action as well, corresponding to the isotropy groups $\mathsf{Sp}(2) \cdot \mathsf{S}^3$ and $\mathsf{Sp}(1)^2$. Using the soul orbit theorem again, we see that the inclusion map $\mathsf{Sp}(3)/\mathsf{Sp}(1)^2 \to N$ of a principal orbit is 7-connected — a contradiction as N is 4-connected.

So we are left with the case of cohomogeneity 1. Combining the previous subsection, the rank lemma and $\mathsf{H}_0 \cong \mathsf{Spin}(7)$, we see that G is one of the groups $\mathsf{F}_4, \mathsf{Spin}(10)$ or $\mathsf{Spin}(11)$, and it is not hard to rule out each of them separately. □

LEMMA 8.6. *Suppose* G/H *has a* 6-*dimensional as well as an* 8-*dimensional irreducible subrepresentation. Then* $\mathsf{G} = \mathsf{Spin}(10)$.

Proof. We argue by contradiction. Notice that our assumptions imply by the previous subsection that $\mathsf{SU}(4) \cong \mathsf{Spin}(6)$ is a normal subgroup of H. We first consider the case of cohomogeneity ≥ 3. By Lemma 3.2 there is an isotropy group $\bar{\mathsf{K}}$ corresponding to a stratum of codimension ≤ 2 such that K/H contains both the 6- and the 8-dimensional representation of H.

It is easy to see that $\mathsf{Spin}(10)$ is the only simple group which has a representation satisfying the hypothesis of the lemma. Since the orbit type of $\mathsf{K} \supset \mathsf{Spin}(10)$ is not isolated, we can use Lemma 3.1 to see that any nontrivial subrepresentation of $\mathsf{G}/\mathsf{Spin}(10)$ is isomorphic to the 32-dimensional subrepresentation of the slice representation. This implies that $\mathsf{Spin}(10)$ is a normal subgroup of the normalizer $N(\iota)$ of ι, where ι is the central involution in $\mathsf{Spin}(10)$. Using the fact that $\mathsf{G}/N(\iota)$ is a symmetric space one easily deduces $\mathsf{G} = \mathsf{E}_6$. Furthermore $\bar{\mathsf{K}}_0 = \mathsf{Spin}(10)$ or $\bar{\mathsf{K}}_0 = \mathsf{S}^1 \cdot \mathsf{Spin}(10)$. The slice representation of $\bar{\mathsf{K}}$ consists of a trivial representation and a 32-dimensional irreducible representation. Let $U \subset T_p\mathsf{E}_6/\bar{\mathsf{K}}$ be the unique 32-dimensional invariant subspace. Consider the equivariant tensor $B\colon S^2U \to \nu_p(T_p\mathsf{E}_6/\bar{\mathsf{K}})$. The unique central order 2 element $\iota \in \mathsf{Spin}(10)$ acts as $-\,\mathrm{id}$ on U. Thus it acts trivially on S^2U. Consequently the image of B lies in the subspace corresponding to the trivial representation. The trivial subrepresentations of S^2U correspond to the space of equivariant selfadjoint endomorphisms of U. Since the representation of $\mathsf{Spin}(10)$ in U is irreducible, it follows that S^2U contains only one trivial representation. Thus the image of $B\colon S^2U \to \nu_p\big(T_p\mathsf{E}_6/\bar{\mathsf{K}}\big)$ is at most one dimensional. Let $N \in \nu_p\big(T_p\mathsf{E}_6/\bar{\mathsf{K}}\big)$ be a unit vector such that $\mathbb{R}\cdot N$ contains $B(S^2U)$. Since we may represent the bilinear form $\langle B(u,v), N\rangle$ of U by a self-adjoint equivariant endomorphism of U, we deduce that $\langle B(u,v), N\rangle$ is semidefinite. The 32-dimensional subspace U can be canonically identified with the tangent space of the rank-two symmetric space $\mathsf{E}_6/\mathsf{S}^1\cdot\mathsf{Spin}(10)$. In particular, we can find linear independent vectors $u, v \in U$ that generate a plane of intrinsic curvature zero in $\mathsf{E}_6/\bar{\mathsf{K}}$. The previous argument shows that we can employ the Gauss equations to see that the actual curvature of the plane $\mathrm{span}\{u, v\}$ is smaller or equal to the intrinsic curvature — a contradiction.

If the cohomogeneity of the action is one, we can use more direct methods to show that this cannot happen for $\mathsf{G} \neq \mathsf{Spin}(10)$. Thus we may assume that the cohomogeneity is 2. As before, there is an isotropy group K containing $\mathsf{Spin}(10)$ as a normal subgroup. Since K acts with cohomogeneity 2 on the slice, the orbit type of G/K is necessarily isolated, and the slice representation restricts to the 32-dimensional representation of $\mathsf{Spin}(10)$. This also implies that $\mathsf{K}_0/\mathsf{Spin}(10) \cong \mathsf{H}/\mathsf{SU}(4)$ is abelian.

Let $N(\iota) \subset \mathsf{G}$ be the normalizer of the central element ι of order 2 in $\mathsf{Spin}(10)$. Then ι acts as $-\,\mathrm{id}$ on the slice. Thus the homogeneous space $N(\iota)/\mathsf{K}$ is a fixed-point component of ι. Since $(\mathsf{G}, N(\iota))$ is a symmetric pair and $N(\iota)/\mathsf{K}$ is a homogeneous space of positive sectional curvature, it follows that $\mathsf{G} = \mathsf{E}_6, \mathsf{Spin}(11)$. As before we can rule out E_6. Thus we are left with $\mathsf{G} = \mathsf{Spin}(11)$ and $\dim(M) = 42$. Suppose first that the two dimensional orbit space has only two faces. Then it is a biangle with isotropy groups $\mathsf{Spin}(7)$ and

$\mathsf{SU}(5)$ for the two faces and two vertices with isotropy group $\mathsf{Spin}(10)$. Notice that the fixed-point set of $\mathsf{Spin}(7)$ is a 4-dimensional manifold which is invariant under an isometric action of $\mathsf{Spin}(3)$. In fact using the above description of the orbit structure it easily follows that $\mathrm{Fix}(\mathsf{Spin}(7))$ is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$ contradicting [14].

Thus we may assume that the orbit space has three faces and is a triangle, and the $\mathsf{Spin}(10)$-vertex has an angle $\pi/4$. By Gauss Bonnet the sum of all three angles is larger than π. Thus at least one of the other angles is $\pi/2$, and the corresponding slice representation is reducible. From this it easily follows that the other face has the isotropy group $\mathsf{S}^1 \cdot \mathsf{SU}(4)$, and the other vertices have the isotropy groups $\mathsf{S}^1 \cdot \mathsf{SU}(5)$ and $\mathsf{S}^1 \cdot \mathsf{Spin}(7)$.

The isotropy representation of $\mathsf{Spin}(11)/\mathsf{SU}(4) \cdot \mathsf{S}^1$ cannot contain a 12-dimensional irreducible representation, by Lemma 3.1. This actually determines the embedding of the circle in $\mathsf{Spin}(11)$, namely it has to be embedded as a $\mathsf{Spin}(2)$ block. The fixed-point set of $\mathsf{Spin}(2)$ is a cohomogeneity-one manifold with an action of $\mathsf{Spin}(9)$ whose principal isotropy group $\mathsf{SU}(4)$ is embedded by the 8-dimensional representation. But the 8-dimensional representation of $\mathsf{Spin}(9)/\mathsf{SU}(4)$ cannot possibly degenerate. □

LEMMA 8.7. H *does not contain* $\mathsf{Spin}(9)$.

Proof. Otherwise the 16-dimensional representation of $\mathsf{Spin}(9)$ would occur as a subrepresentation of the isotropy representation of G/H. But it cannot degenerate. □

LEMMA 8.8. *If* E_8 *acts isometrically and nontrivially on a positively curved manifold, then the principal isotropy group has no simple subgroup of rank* ≥ 2.

Proof. Using the previous cases and Proposition 7.4, we only have to rule out the possibility that H contains $\mathsf{Spin}(8)$ as a normal subgroup. The normalizer $N(\iota)$ of a central involution ι in $\mathsf{Spin}(8)$ is given by $\mathsf{Spin}(16)$. By applying Theorem 6.1 to the action of $\mathsf{Spin}(16)$ on a fixed-point component we deduce that $\mathsf{Spin}(15)$ is contained in an isotropy group. Furthermore it is also clear that there is an isotropy group K containing $\mathsf{Spin}(14)$ such that its orbit type is not isolated. The isotropy representation of E_8/K contains a 128-dimensional irreducible subrepresentation. By Lemma 3.1 this representation has to degenerate which could only happen in a fixed-point. □

PROPOSITION 8.9. *Suppose the Lie group* E_8 *acts isometrically and nontrivially on a manifold* M *of positive sectional curvature. Then the identity component* H_0 *of the principal isotropy group* H *is abelian.*

Proof. We argue by contradiction and assume that H_0 contains a simple normal subgroup H'. Because of Lemma 8.8 H_0 does not contain any simple subgroups of rank ≥ 2. If H contains a subgroup which is isomorphic to a 3×3 block of $\mathsf{Spin}(16)/\mathbb{Z}_2 \subset \mathsf{E}_8$, then one can argue as in the proof of Lemma 8.8.

Since the isotropy representation of E_8/H is spherical, this only leaves one possibility namely H' is given as a normal subgroup of $\mathsf{S}^3\cdot\mathsf{E}_7 \subset \mathsf{E}_8$. In that case the fixed-point set of $\mathrm{Fix}(\mathsf{H}')$ has codimension $112 < \dim(M)/2$. Thus it follows from Frankel's theorem that for all $a \in \mathsf{E}_8$ the group K generated by H' and $a\mathsf{H}'a^{-1}$ has a nontrivial fixed-point set, namely $\mathrm{Fix}(\mathsf{H}')\cap a\star\mathrm{Fix}(\mathsf{H}')$. For a suitable choice we can arrange that K be given by an 8×8 block of $\mathsf{Spin}(16)$. It is then easy to deduce that there is an isotropy group $\bar{\mathsf{K}}$ of a point p containing a $k\times k$ block $\mathsf{Spin}(k)$ of $\mathsf{Spin}(16)/\mathbb{Z}_2 \subset \mathsf{E}_8$ as a normal subgroup, $k \geq 8$.

Notice that the central involution ι of $\mathsf{Spin}(16)/\mathbb{Z}_2$ is contained in K'. We consider the component N of $\mathrm{Fix}(\iota)$ containing p. Since the slice representation of $\mathsf{Spin}(k)$ contains H' as subgroup of the principal isotropy group, we deduce that the slice representation of $\mathsf{Spin}(k)$ decomposes into spin and standard representations. Thus the principal isotropy group of the $\mathsf{Spin}(16)$-action on N contains an $l\times l$ block with $l \geq 3$. The hypothesis of Theorem 6.1 is satisfied for the action of $\mathsf{Spin}(16)$ on N, except that N might not be simply connected. By Theorem 6.1 $\mathsf{Spin}(15)$ is contained in an isotropy group. It is then easy to see that there is an isotropy group $\bar{\mathsf{K}} \subsetneq \mathsf{E}_8$ whose orbit type is not isolated and which contains $\mathsf{Spin}(14)$. The isotropy representation $\mathsf{E}_8/\bar{\mathsf{K}}$ contains a 128-dimensional irreducible subrepresentation which is not equivalent to any subrepresentation of the slice representation. By Lemma 3.1 this representation has to degenerate which can only happen in a fixed-point. Since there is no representation of E_8 whose principal isotropy group is not abelian, this is a contradiction. □

9. Proof of Theorem 9

Without loss of generality we may assume that H' is normal in H_0 and the hypothesis of Theorem 5.1 is not satisfied. By Lemma 7.1 we may assume that G is simple. Hence (G,H') is one of the pairs that are listed in Theorem 8.1. If (G,H') is one of the pairs

$$(\mathsf{E}_6,\mathsf{Spin}(8)),\ (\mathsf{F}_4,\mathsf{Spin}(8)),\ (\mathsf{Spin}(10),\mathsf{SU}(4)),\ (\mathsf{Spin}(9),\mathsf{G}_2),\ \text{or}\ (\mathsf{Spin}(8),\mathsf{G}_2)$$

we pass to the subactions of $\mathsf{F}_4 \subset \mathsf{E}_6$, $\mathsf{Spin}(8) \subset \mathsf{F}_4$, $\mathsf{Spin}(9) \subset \mathsf{Spin}(10)$, $\mathsf{Spin}(8) \subset \mathsf{Spin}(9)$ or $\mathsf{Spin}(7) \subset \mathsf{Spin}(8)$, respectively, whose corresponding principal isotropy groups contain $\mathsf{SU}(3)$. For each of the remaining 9 pairs we can make iterated use of Lemma 3.1 to see that G has a fixed-point p.

For a subgroup $\mathsf{K} \subset \mathsf{G}$ we let $\mathrm{Fix}(\mathsf{K})_0$ denote the fixed-point component of K containing p. By the connectedness lemma (Theorem 1.2), the inclusion map $\mathrm{Fix}(\mathsf{G})_0 \to M$ is h-connected with $h = 2\,\mathrm{Fix}(\mathsf{G})_0 - n + \dim(\mathsf{G}/\mathsf{H}) + 1$.

Using $n \geq 235$, we see that $h \geq \left[\frac{n}{2}\right]$. Therefore it is sufficient to show that $\mathrm{Fix}(\mathsf{G})_0$ is a cohomology CROSS.

If $(\mathsf{G}, \mathsf{H}')$ is one of the pairs $(\mathsf{Spin}(9), \mathsf{Spin}(7))$,$(\mathsf{Spin}(7), \mathsf{G}_2)$, or $(\mathsf{G}_2, \mathsf{SU}(3))$, then M is fixed-point homogeneous and we are done.

If $(\mathsf{G}, \mathsf{H}')$ is one of the pairs $(\mathsf{Spin}(10), \mathsf{SU}(3))$, or $(\mathsf{Spin}(8), \mathsf{SU}(3))$, we choose a central involution $\iota \in \mathsf{G}$, which is contained in the kernel of a nontrivial subrepresentation of the the fixed-point representation. It is straightforward to check that the hypothesis of Theorem 5.1 is satisfied for the induced action of G on $\mathrm{Fix}(\iota)_0$. Thus $\mathrm{Fix}(\iota)_0$ is a homotopy CROSS and by the connectedness lemma (Theorem 1.2) the same holds for $\mathrm{Fix}(\mathsf{G})_0$.

The same kind of reasoning works if $(\mathsf{G}, \mathsf{H}')$ is one of the pairs $(\mathsf{Spin}(9), \mathsf{SU}(3))$ or $(\mathsf{Spin}(7), \mathsf{SU}(3))$, unless the fixed-point representation is given as the sum of a trivial representation and two equivalent spin representations. In the latter case we argue as follows. For $\mathsf{G} = \mathsf{Spin}(9)$ we consider a nonstandard $\mathsf{Spin}(7) \subset \mathsf{Spin}(9)$, and notice that $\mathrm{Fix}(\mathsf{Spin}(9))_0$ is given by the intersection of two copies of $\mathrm{Fix}(\mathsf{Spin}(7))_0$ inside $\mathrm{Fix}(\mathsf{G}_2)_0$. Since $\mathrm{Fix}(\mathsf{Spin}(7))_0$ has codimension 2 in $\mathrm{Fix}(\mathsf{G}_2)_0$ we can apply [22, Prop. 7.3] to see that $\mathrm{Fix}(\mathsf{Spin}(9))$ is homotopically equivalent to a sphere or a complex projective space. The argument for $\mathsf{G} = \mathsf{Spin}(7)$ is similar.

For $(\mathsf{G}, \mathsf{H}') = (\mathsf{F}_4, \mathsf{SU}(3))$ the nontrivial part of the fixed-point representation is necessarily given by two equivalent 26-dimensional representations. We look at the groups $\mathsf{Spin}(8) \subset \mathsf{Spin}(9) \subset \mathsf{F}_4$. Then $\mathrm{Fix}(\mathsf{F}_4)_0$ is given by the intersection of two isometric copies of $\mathrm{Fix}(\mathsf{Spin}(9))_0$ in $\mathrm{Fix}(\mathsf{Spin}(8))_0$. Since $\mathrm{Fix}(\mathsf{Spin}(9))_0$ has codimension 2 in $\mathrm{Fix}(\mathsf{Spin}(8))_0$, we can conclude as before.

In the remaining case of $(\mathsf{G}, \mathsf{H}') = (\mathsf{E}_7, \mathsf{Spin}(8))$ the nontrivial part of the fixed-point representation is the irreducible 112-dimensional representation of E_7. We consider the subgroup $\mathsf{Spin}(12) \subset \mathsf{E}_7$. There is a central involution $\iota \in \mathsf{Spin}(12)$ such that $\mathrm{Fix}(\iota)_0$ has codimension 64. The totally geodesic submanifold $\mathrm{Fix}(\iota)_0$ is invariant under a $\mathsf{Spin}(12)$-action, and the principal isotropy group of that action is a $\mathsf{Spin}(8)$ block. By Theorem 5.1 $\mathrm{Fix}(\iota)_0$ is homotopically equivalent to a CROSS, and, by the connectedness lemma, the same holds for $\mathrm{Fix}(\mathsf{E}_7)$.

10. Positively curved manifolds with large symmetry degree

Theorem 1 will be proved as a consequence of the following theorem.

THEOREM 10.1. *Let (M, g) be a simply connected Riemannian manifold of positive sectional curvature. Let G be a connected Lie group acting almost effectively and isometrically on (M^n, g). Suppose $\dim(\mathsf{H}) \geq \max\left\{\frac{\dim(\mathsf{G})-4}{2}, 2\right\}$ holds for a principal isotropy group H. Then one of the following holds*:

a) *M is tangentially homotopically equivalent to $\mathbb{S}^n$, $\mathbb{CP}^{n/2}$ or $\mathbb{HP}^{n/4}$.*

b) *M is diffeomorphic to the Cayley plane* $\mathrm{Ca}\mathbb{P}^2$.

c) *M is isometric to a homogeneous space of positive sectional curvature.*

As mentioned before a compact manifold is tangentially homotopically equivalent to $\mathbb{S}^n$ or $\mathbb{HP}^{n/4}$ if and only if it is homeomorphic to $\mathbb{S}^n$ or $\mathbb{HP}^{n/4}$, $n \neq 3$.

Proof. We argue by induction on $\dim(\mathsf{G})$. We may assume that G does not act transitively. After replacing G by a finite cover we also may assume that its semisimple part is simply connected and that G is the product of a semisimple group and a torus. The proof is divided into six cases. In each case we will assume that the assumptions of the preceding cases are not satisfied.

Case 1. There is a decomposition $\mathsf{G} = \mathsf{L}_1 \times \mathsf{L}_2$ with $\dim(\mathsf{L}_i) > 0$, and the image of the projection $\mathrm{pr}_1\colon \mathsf{H} \to \mathsf{L}_1$ has dimension at most $\frac{\dim(\mathsf{L}_1)-4}{2}$.

Put $\mathsf{H}_2 := \mathsf{H} \cap \mathsf{L}_2$. Notice that $\dim(\mathsf{H}_2) \geq \frac{\dim(\mathsf{L}_2)}{2}$. This actually implies $\dim(\mathsf{H}_2) \geq 2$ and the theorem follows by applying the induction hypothesis to the subaction of L_2 on M.

Case 2. H contains a simple subgroup H' of rank ≥ 2.

We may assume that H' is normal in H_0. By Lemma 7.1, H' is contained in a simple normal subgroup L_1 of G. We write $\mathsf{G} = \mathsf{L}_1 \times \mathsf{L}_2$ and let H_1 denote the projection of H onto the first factor. Clearly we may assume that the hypothesis of Theorem 5.1 is not satisfied for the subaction of L_1 on M. By Theorem 8.1, $(\mathsf{L}_1, \mathsf{H}')$ is one of the pairs listed there. Combining $\dim(\mathsf{H}_1) \geq \frac{\dim(\mathsf{L}_1)-4}{2}$ with the fact that, by Lemma 3.4, the isotropy representation of $\mathsf{L}_1/\mathsf{H}_1$ is spherical, gives that $(\mathsf{L}_1, \mathsf{H}')$ is one of the pairs $(\mathsf{F}_4, \mathsf{Spin}(8))$, $(\mathsf{Spin}(8), \mathsf{G}_2)$, $(\mathsf{Spin}(7), \mathsf{SU}(3))$ $(\mathsf{Spin}(9), \mathsf{Spin}(7))$, $(\mathsf{Spin}(7), \mathsf{G}_2)$ or $(\mathsf{G}_2, \mathsf{SU}(3))$. In the last three cases the manifold is fixed-point homogeneous. The first two cases are dealt with in Lemma 10.2 below. Thus we may assume $(\mathsf{L}_1, \mathsf{H}') = (\mathsf{Spin}(7), \mathsf{SU}(3))$. By Lemma 3.1 there is an isotropy group $\mathsf{K} \subset \mathsf{G}$ corresponding to face such that $\mathsf{L}_1 \cap \mathsf{K}$ contains the exceptional Lie group G_2. It is then straightforward to check that the projection of K_0 to L_1 equals G_2. Because of $\mathsf{K}/\mathsf{H} \cong \mathbb{S}^6$ the identity component of H_1 equals $\mathsf{H}' \cong \mathsf{SU}(3)$. But then $\dim(\mathsf{H}_1) = 8 < \frac{\dim(\mathsf{L}_1)-4}{2}$ — a contradiction, as we are not in Case 1.

Case 3. G has a simple factor L_1 which is not isomorphic to $\mathsf{Sp}(2)$, $\mathsf{SU}(3)$ or S^3.

We write $\mathsf{G} = \mathsf{L}_1 \times \mathsf{L}_2$. Because of Case 1 the projection H_1 of H_0 to the first factor has dimension $\geq \frac{\dim(\mathsf{L}_1)-4}{2}$. By Case 2, H contains no simple subgroups of rank ≥ 2. Combining this with the fact that the isotropy representation of $\mathsf{L}_1/\mathsf{H}_1$ is spherical, we see that $\mathsf{L}_1/\mathsf{H}_1$ is given by $\mathsf{Sp}(3)/\mathsf{Sp}(1)^3$ or $\mathsf{SU}(4)/\mathsf{SU}(2)^2 \cong \mathsf{SO}(6)/\mathsf{SO}(4)$. In particular, $\dim(\mathsf{H}_1) = \frac{\dim(\mathsf{L}_1)-3}{2}$ and $\dim(\mathsf{H} \cap \mathsf{L}_2) \geq \frac{\dim(\mathsf{L}_2)-1}{2}$.

If $\dim(\mathsf{L}_2) > 3$, then the theorem follows by application of the induction hypothesis to the subaction of L_2. If $\dim(\mathsf{L}_2) = 3$, then M is fixed-point homogeneous with respect to $\mathsf{H} \cap \mathsf{L}_2$. Thus we may assume that L_2 is abelian, and consequently $\mathsf{H}_1 \subset \mathsf{H}_0$. Because of $\mathsf{L}_1/\mathsf{H}_1 \cong \mathsf{Sp}(3)/\mathsf{Sp}(1)^3$ or $\mathsf{SO}(6)/\mathsf{SO}(4)$ we can either apply Lemma 10.2 from below or Theorem 5.1.

Case 4. The isotropy representation of G/H contains a four-dimensional irreducible subrepresentation.

Because of the previous case any simple normal subgroup of G is isomorphic to $\mathsf{SU}(3)$, $\mathsf{Sp}(2)$ or S^3. By the isotropy lemma (Lemma 5), there is an isotropy group K corresponding to a face such that $\mathsf{K}/\mathsf{H} \cong \mathbb{S}^4, \mathbb{S}^5, \mathbb{S}^7$. Hence K contains a unique simple normal subgroup K' of rank ≥ 2. We may assume that K' is not normal in G because otherwise M is fixed-point homogeneous. Hence we can find a decomposition $\mathsf{G} = \mathsf{L}_1 \times \mathsf{L}_2 \times \mathsf{L}_3$ with L_i being simple such that K' projects nontrivially to L_i, $i = 1, 2$. Clearly $\mathsf{L}_1 \cong \mathsf{L}_2 \cong \mathsf{K}'$. Since K' and K_0 project to the same subgroup of $\mathsf{L}_1 \times \mathsf{L}_2$, we see that the projection of H to $\mathsf{L}_1 \times \mathsf{L}_2$ has dimension at most $\dim(\mathsf{K}') - 4 = \frac{\dim(\mathsf{L}_1 \times \mathsf{L}_2)-8}{2}$ — a contradiction, as we are not in Case 1.

Case 5. G has a simple normal subgroup L_1 of rank ≥ 2.

We write $\mathsf{G} = \mathsf{L}_1 \times \mathsf{L}_2$, and let H_1 denote the projection of H to L_1. Then $\mathsf{L}_1 \cong \mathsf{SU}(3)$ or $\mathsf{Sp}(2)$, and the isotropy representation of $\mathsf{L}_1/\mathsf{H}_1$ decomposes into 1-, 2- and 3-dimensional subrepresentations. From this it is easy to deduce $\dim(\mathsf{H}_1) \leq \frac{\dim(\mathsf{L}_1)-4}{2}$. Since we are not in Case 1, $\mathsf{L}_2 = 1$. But then $\mathsf{G}/\mathsf{H} \cong \mathsf{SO}(5)/\mathsf{SO}(3)$ or $\mathsf{SU}(3)/\mathsf{T}^2$ and we can either apply Theorem 5.1 or Lemma 10.2.

Case 6. G has no simple normal subgroup L_1 of rank ≥ 2.

Here we treat a special case first. Suppose H_0 contains a nontrivial connected simple normal subgroup H'. Let L_1 be the smallest normal subgroup of G containing H'. We write $\mathsf{G} = \mathsf{L}_1 \times \mathsf{L}_2$. Clearly $\mathsf{L}_1 \cong (\mathsf{S}^3)^i$ for some $i \geq 2$, and H' is given by the diagonal $\Delta\mathsf{S}^3$ in L_1. Furthermore the projection of H_0 to L_1 is also given by H'. Because of Case 1 we may assume $i \leq 3$. If we apply the isotropy lemma to the action of L_1, we see that there must be an isotropy group $\mathsf{K} \subset \mathsf{L}_1$ corresponding to a face of the orbit space such that K/H is a sphere of dimension ≥ 3. This implies $\mathsf{K}_0 \cong \left(\mathsf{S}^3\right)^2$, K contains a normal subgroup of G, and M is fixed-point homogeneous with respect to the subaction of this subgroup.

Hence we may assume that H has an abelian identity component. Let G_s denote the semisimple part of G. It is easy to see that $\dim(\mathsf{H}) \geq \mathrm{rank}(\mathsf{G}_s)$. Since H projects with finite kernel to a subgroup of G_s, it follows that M is fixed-point homogeneous with respect to the subaction of a suitable $\mathsf{S}^1 \subset \mathsf{H}_0$. □

LEMMA 10.2. *Suppose the simple Lie group* G *acts isometrically and not transitively on a simply connected positively curved manifold* M *with principal isotropy group* H. *If* $(\mathsf{G},\mathsf{H}_0) \in \big\{(\mathsf{Spin}(8),\mathsf{G}_2), (\mathsf{F}_4,\mathsf{Spin}(8)), (\mathsf{Sp}(3),\mathsf{Sp}(1)^3), (\mathsf{SU}(3),\mathsf{T}^2)\big\}$, *then* M *is diffeomorphic to* $\mathbb{S}^n$ *or* $\mathrm{Ca}\mathbb{P}^2$.

Proof. First we consider the case of cohomogeneity one. Then there are two singular orbits. In the cases of $\mathsf{G} = \mathsf{F}_4$, $\mathsf{SU}(3)$, $\mathsf{Sp}(3)$ we can use the isotropy lemma for the horizontal geodesic to see that at the two singular orbits two different subrepresentation of the isotropy representation of G/H have to degenerate. This in fact determines the picture of the sphere. In the case of $\mathsf{G} = \mathsf{Spin}(8)$ there are two possibilities left. Either the manifold is $\mathbb{S}^{15}$ or $\mathbb{S}^7 \times \mathbb{S}^8$ endowed with a cohomogeneity one action of $\mathsf{Spin}(8)$. In the latter case the fixed-point set of $\mathsf{SU}(3) \subset \mathsf{G}_2$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$, which is clearly impossible.

Suppose now that the cohomogeneity is ≥ 2. Then we can employ Lemma 3.1 twice in order to guarantee a fixed-point of G. Our assumptions on the principal isotropy group determines the isotropy representation of G, and it is straightforward to check that $\dim(M/\mathsf{G}) - \dim(\mathrm{Fix}(\mathsf{G})) = 2$. In other words, M has fixed-point cohomogeneity one in the sense of Grove and Kim. By [10], M is diffeomorphic to a rank one symmetric space. □

Proof of Theorem 1. Put $\mathsf{G} = \mathrm{Iso}(M,g)_0$, and let H be the principal isotropy group of the action. We may assume $\dim(M/\mathsf{G}) > 0$. Since $\dim(\mathsf{G}) \geq 2n-6$ it follows that $\dim(\mathsf{H}) \geq \dim(\mathsf{G}) - n + 1 \geq \frac{\dim(\mathsf{G})-4}{2}$. Thus we can apply Theorem 10.1, unless possibly $n \leq 6$. There is nothing to prove for $n \leq 3$, and we can apply [14] for $n = 4$. Thus we may assume $n = 5, 6$. If $\dim(M/\mathsf{L}) > 1$, then $\dim(\mathsf{H}) \geq n - 4$, and it is easy to see that M is fixed-point homogeneous. If $\dim(M/\mathsf{L}) = 1$, then we can apply [17]. □

11. Group actions with nontrivial principal isotropy groups

In this section no curvature assumptions are required. It will also be useful to allow M to be a noncompact manifold. The main purpose of this section is to prove Theorem 6. Before we start to establish the necessary lemmas for the proof, we draw some consequences. For a connected compact Lie group H_0 we put

$$\mathrm{nof}(\mathsf{H}_0) := \dim(\mathsf{C}) + \text{number of factors of } \mathsf{H}_0/\mathsf{C},$$

where C denotes the center of H_0.

Corollary 11.1. *Suppose that a connected compact Lie group* G *acts smoothly on a simply connected manifold* M *with principal isotropy group* H. *Let* f *denote the number of faces of* M/G.

a) *If* $2f < \mathrm{nof}(\mathsf{H}_0)$, *then there is a connected normal subgroup* $\mathsf{H}' \subset \mathsf{H}$ *with* $\mathrm{nof}(\mathsf{H}') \geq \mathrm{nof}(\mathsf{H}) - 2f$ *and an equivariant smooth map* $M \to \mathsf{G}/N(\mathsf{H}')_0$.

b) *If* $f < \dim(\mathsf{C})$, *where* C *denotes the center of* H_0, *then there is a subgroup* C' *of dimension* $\dim(\mathsf{C})' \geq \dim(\mathsf{C}) - 1$ *and an equivariant smooth map* $M \to \mathsf{G}/N(\mathsf{C}')_0$.

Proof of Corollary 11.1. We use the notation of Theorem 6. We can choose the points $p_1, \dots, p_f$ such that they correspond to generic points on the different faces. If K_i denotes the isotropy group of p_i, then $\mathsf{K}_i/\mathsf{H} \cong \mathbb{S}^{n_i}$. Furthermore $n_i > 0$ by Lemma 11.4 from below. a) The kernel of the slice representation of K_1 has at least $\mathrm{nof}(\mathsf{H}) - 2$ factors. Let H' be the identity component of the intersection of all the kernels of the slice representations of $\mathsf{K}_1, \dots, \mathsf{K}_f$. Then H' is a normal subgroup of H and $\mathrm{nof}(\mathsf{H}') \geq \mathrm{nof}(\mathsf{H}) - 2f$. Furthermore using the fact that K_i/H is connected it is easy to deduce that K_i is contained in the normalizer of H'. By Theorem 6 there is an equivariant map $M \to \mathsf{G}/N(\mathsf{H}')_0$. b) Since the center of the identity component of the kernel of the slice representation of K_i has dimension at least $\dim(\mathsf{C}) - 1$, one can prove b) analogously to a). □

The following result can be found in [12] up to small modifications. In fact Theorem 6 may be viewed as its generalization.

Proposition 11.2. *Let* G *be a connected compact Lie group acting smoothly on a manifold* M *with principal isotropy group* H. *Let* B *be a component of* $\mathrm{Fix}(\mathsf{H})$ *projecting surjectively onto the orbit space. Consider the subgroup* $N(\mathsf{H})'$ *of the normalizer leaving* B *invariant. Suppose* K *is a compact subgroup of* G *containing* $N(\mathsf{H})'$ *as well as all isotropy groups of points in* B. *Then there is an equivariant map* $M \to \mathsf{G}/\mathsf{K}$.

Since the proof is essentially contained in [12], we only sketch it here. We first show that $N := \mathsf{K}\star B$ is a smooth submanifold of M. In fact we show that for $p \in B$ a neighborhood of the orbit $\mathsf{K}\star p$ in $\mathsf{K}\star B$ is via the exponential map (of an invariant metric) diffeomorphic to the vectorbundle over $\mathsf{K}\star p$ given by the restriction of the normal bundle of $\mathsf{G}\star p$ in M. Once one knows that N is a submanifold one can consider the smooth map $f\colon \mathsf{G} \times N \to M$, $(g,p) \mapsto g \star p$. It is easy to see that f has full rank and that the fibers of f are given by the orbits of the diagonal action of K on $\mathsf{G} \times N$. Thus one gets a diffeomorphism of G-manifolds $\bar{f}\colon \mathsf{G} \times_{\mathsf{K}} N \to M$ and hence the above result.

PROPOSITION 11.3. *Suppose a compact Lie group* G *acts smoothly and effectively on a manifold* M *with nontrivial principal isotropy group* H. *If the orbit space has no boundary, then there is a smooth* G*-equivariant map* $M \to \mathsf{G}/N(\mathsf{H})$, *where* $N(\mathsf{H})$ *denotes the normalizer of* H.

Proof. By Proposition 11.2 it suffices to prove that for each isotropy group K of a point $p \in M$ the kernel of the slice representation of K is isomorphic to the principal isotropy group. Suppose K is a counterexample. Then the isotropy representation of K/H is nontrivial. As usual the slice representation induces an action on the normal sphere of the orbit $\mathsf{G}\star p$. Application of the isotropy lemma to this linear action gives that the orbit space $\nu_p(\mathsf{G}\star p)/\mathsf{K}$ has boundary. Hence the orbit $\mathsf{G}\star p$ corresponds to a boundary point of the orbit space M/G — a contradiction. □

LEMMA 11.4. *Suppose* G *is a connected Lie group acting on a simply connected manifold* M *with principal isotropy group* H.

a) *Suppose that* K *is an isotropy group such that the identity component* H_0 *of* H *lies in the kernel of the slice representation of* K. *Then* K *is contained in the identity component* $N(\mathsf{H}_0)_0$ *of the normalizer* $N(\mathsf{H}_0)$ *of* H_0.

b) *There is no isotropy group* K *with* $\dim(\mathsf{K}) = \dim(\mathsf{H})$ *corresponding to a face of the orbit space.*

Notice that the lemma says in particular that $\mathsf{H} \subset N(\mathsf{H}_0)_0$.

Proof. We first want to reduce the problem to the special case that all isotropy groups are of the form described in a). Suppose that K is an isotropy group whose slice representation has not a kernel of dimension $\dim(\mathsf{H}_0)$. Clearly $\dim(\mathsf{K}) - \dim(\mathsf{H}) \geq 2$ and thus the set of orbits whose isotropy groups contain K has codimension at least 3 in M. Thus we may remove all of these orbits from M without changing the fact that M is simply connected.

a) Let M' be a component of $\mathrm{Fix}(\mathsf{H}_0)$ containing a point with isotropy group H. By the previous reduction step we may assume that H_0 is a normal subgroup of any isotropy group of $p \in M'$. By Proposition 11.2 there is an equivariant smooth map $M \to \mathsf{G}/N(\mathsf{H}_0)$. This map is necessarily a fiber bundle and since M is simply connected, there is an equivariant smooth map $f\colon M \to \mathsf{G}/N(\mathsf{H}_0)_0$, too. Since M' is a fiber of f, all isotropy groups of points in M' are contained in $N(\mathsf{H}_0)_0$.

b) We keep the notation. Notice that the principal isotropy group of the action of $\mathsf{G}' := N(\mathsf{H}_0)_0/\mathsf{H}_0$ on M' has a trivial identity component. We do not know whether M' is simply connected. But since the normal bundle of the fiber M' is trivial, M' is orientable. Suppose now that there is a finite isotropy group $\mathsf{K}' \subset \mathsf{G}'$ corresponding to a face of the orbit space. Let $\iota \in \mathsf{K}'$ be an element that is not in the kernel of the slice representation ρ. Notice that $\rho(\iota)$ has determinant -1. Using the fact that K' is finite and that Ad_ι has determinant one we see that ι switches the orientation of M'. On the other hand ι is contained in the connected group G' — a contradiction. □

LEMMA 11.5. *Suppose that a connected compact Lie group* G *acts smoothly on a simply connected manifold* M. *Let* M' *denote the* G*-manifold that is obtained from* M *by removing all orbits corresponding to a collection of faces of* M/G. *Let* $p \in M'$. *Then there is no normal subgroup* $\Pi \subsetneq \pi_1(M')$ *containing the image of the natural map* $\pi_1(\mathsf{G} \star p) \to \pi_1(M')$.

Proof. We argue by induction on the number of faces which have been removed. Let $N \subset M$ denote the G-manifold that is obtained from M by removing all orbits from M which correspond to orbit types on the boundary of M/G but not to faces of M/G. By Lemma 11.4 b) $M \setminus N$ has codimension 3 in M and thus N is simply connected. We may assume that $N = M$. Then each face F of M/G is a manifold, and we can find in each face a subset F' of codimension 1 such that $F \setminus F'$ is contractible. Clearly we can remove all orbits in F' from M and obtain a simply connected G-manifold. In other words we may assume that all faces of M/G are contractible manifolds. Let $\Pi \subset \pi_1(M')$ be a normal subgroup containing the image of $\pi_1(\mathsf{G} \star p) \to \pi_1(M')$. Clearly we may assume that $\mathsf{G} \star p$ is a principal orbit. Choose one face $F \subset M/\mathsf{G}$ which has been removed, and consider a tubular neighborhood U of the inverse image of this face. Notice that the fundamental groups of U and $U \cap M'$ are contained in the image of the fundamental group of an orbit. Using van Kampen's theorem it is easy to see that the epimorphism $\pi_1(M') \to \pi_1(M')/\Pi$ induces an epimorphism $\pi_1(M' \cup U) \to \pi_1(M')/\Pi$. By the induction hypothesis $\Pi = \pi_1(M')$. □

LEMMA 11.6. *Suppose that a connected compact Lie group* G *acts smoothly on a simply connected manifold* M. *Let* K *be an isotropy group contained in the normalizer* $N(\mathsf{H})$ *of* H. *Then* $\mathsf{K} \subset N(\mathsf{H})_0 \cdot \mathsf{H}$.

Proof. Let M' be the manifold that is obtained from M by removing all orbits with isotropy group K such that K is not contained in $N(\mathsf{H})$. Since the isotropy representation of K/H is nontrivial, the isotropy lemma implies that all these orbits correspond to boundary orbits of M/G. Thus the hypothesis of Lemma 11.5 is satisfied for M'. Applying Proposition 11.2 gives an equivariant map $f\colon M' \to \mathsf{G}/\mathsf{N}(\mathsf{H})$. Notice that the image F of the natural map $\pi_1\big(\mathsf{G}/\mathsf{N}(\mathsf{H})_0 \cdot \mathsf{H}\big) \to \pi_1\big(\mathsf{G}/\mathsf{N}(\mathsf{H})\big)$ is normal in $\pi_1\big(\mathsf{G}/\mathsf{N}(\mathsf{H})\big)$. Furthermore F contains the image of $\pi_1\big(f_{|\mathsf{G}\star p}\big)\colon \pi_1\big(\mathsf{G} \star p\big) \to \pi_1\big(\mathsf{G}/\mathsf{N}(\mathsf{H})\big)$ for any point $p \in M'$. Thus we can apply Lemma 11.5 to see that the image of

$\pi_1(f)\colon \pi_1(M') \to \pi_1(\mathsf{G}/\mathsf{N}(\mathsf{H}))$ is contained in F as well. Consequently we can lift f to an equivariant map $\tilde{f}\colon M' \to \mathsf{G}/\mathsf{N}(\mathsf{H})_0 \cdot \mathsf{H}$. Thus the isotropy group K is contained in $\mathsf{N}(\mathsf{H})_0 \cdot \mathsf{H}$. □

We are now ready to introduce some type of Weyl group W for orbit spaces with boundary.

PROPOSITION 11.7. *Suppose that a connected compact Lie group* G *acts smoothly on a simply connected manifold* M *with principal isotropy group* H. *Let* M_{pr} *denote the set of all principal orbits,* $B_{\mathrm{pr}} \subset M_{\mathrm{pr}}$ *be a fixed-point component of* H, *and let* $B \subset M$ *be the component of* $\mathrm{Fix}(\mathsf{H}) \subset M$ *containing* B_{pr}.

a) *The group* $\mathsf{L}_0 = \{ g \in N(\mathsf{H})/\mathsf{H} \mid g \star b \in B_{\mathrm{pr}} \text{ for } b \in B_{\mathrm{pr}} \}$ *is connected.*

b) *The component group* $\mathsf{W} = \mathsf{L}/\mathsf{L}_0$ *of*

$$\mathsf{L} := \{ g \in N(\mathsf{H})/\mathsf{H} \mid g \star b \in B \text{ for } b \in B \}$$

is generated by at most k *involutions, where* k *is the number of faces of* M/G *whose inverse images in* B *have codimension* 1. *The orbit spaces* B/L *and* M/G *are isometric.*

Proof. a) Let $\tilde{M}$ be the manifold that is obtained from M by removing all boundary orbits, and let $\tilde{B}$ be the fixed-point component of H containing B_{pr}. Clearly B_{pr} is dense in $\tilde{B}$. Thus $\mathsf{L}_0 = \{ g \in N(\mathsf{H})/\mathsf{H} \mid g \star b \in \tilde{B} \text{ for } b \in \tilde{B} \}$. By Proposition 11.3, there is an equivariant map $f\colon \tilde{M} \to \mathsf{G}/N(\mathsf{H})$. As before we can employ Lemma 11.5 to guarantee an equivariant lift $\tilde{f}\colon \tilde{M} \to \mathsf{G}/N(\mathsf{H})_0 \cdot \mathsf{H}$. Since $\tilde{B}$ is a fiber of $\tilde{f}$, the subgroup of G which leaves $\tilde{B}$ invariant is given by $N(\mathsf{H})_0 \cdot \mathsf{H}$. In other words L_0 is connected. b) Recall that B/L is isometric to M/G; see [12]. Let M' be the manifold that is obtained from M by removing all orbits for which the inverse image of the corresponding orbit type in B has codimension at least two. If we put $B' := B \cap M'$, then

$$\mathsf{L} = \{ g \in N(\mathsf{H})/\mathsf{H} \mid g \star b \in B' \text{ for all } b \in B' \}.$$

Notice that $\mathsf{W} := \mathsf{L}/\mathsf{L}_0$ acts simply transitively on the components of $B' \cap M_{\mathrm{pr}}$. On the other hand $B' \cap M_{\mathrm{pr}}$ consists of isometric copies of B_{pr} which are separated by orbit types of codimension 1 in B' (and in M/G). For each such face there is an involution which switches two components of $B' \cap M_{\mathrm{pr}}$. Clearly the subgroup of L generated by these involutions also acts transitively on the components of $B' \cap M_{\mathrm{pr}}$ and hence the groups coincide. Furthermore one can generate the group W by the involutions corresponding to the k boundary components of $\bar{B}_{\mathrm{pr}} \cap B'$. □

Proof of Theorem 6. If the orbit space has no boundary, the theorem follows from Proposition 11.3. Let $F_1, \ldots, F_f$ denote the faces of M/G. We may assume $\mathsf{G}\star p_i \in F_i$ and also that $\mathsf{G}\star p_i$ corresponds to a generic point of F_i. In fact otherwise we just replace p_i by another point with smaller isotropy group. Let K_i be the isotropy group of p_i. Let B be the component of $\mathrm{Fix}(\mathsf{H})$ containing B_{pr}. The component group of

$$\mathsf{L} := \big\{ g \in N(\mathsf{H})/\mathsf{H} \;\big|\; g\star b \in B \text{ for all } b \in B \big\}$$

is generated by involutions contained in $N(\mathsf{H}) \cap \mathsf{K}_i/\mathsf{H}$, $i = 1,\ldots,f$. Therefore K contains the group

$$N(\mathsf{H})' = \big\{ g \in N(\mathsf{H}) \;\big|\; g\star b \in B \text{ for all } b \in B \big\}.$$

By Proposition 11.2, it remains to verify that K contains all isotropy groups of points in B. Because of $N(\mathsf{H}') \subset \mathsf{K}$ it suffices to prove that K contains one isotropy group of each orbit type. By assumption this holds for the isotropy groups corresponding to faces.

Let P be an isotropy group of a point $p \in B$. We have to show $\mathsf{P} \subset \mathsf{K}$. We may assume that P does not correspond to a face. If P is contained in the normalizer $N(\mathsf{H})$, then Lemma 11.6 gives $\mathsf{P} \subset \mathsf{N}(\mathsf{H})_0 \cdot \mathsf{H} \subset \mathsf{K}$. So we may assume that P is not contained in $N(\mathsf{H})$. Suppose, on the contrary, that P is not contained in K either. We may assume that P is a minimal counterexample. Consider the nontrivial part of the slice representation, i.e., the action on the normal sphere $\mathbb{S}^{n'}$ at p of the orbit type of P in M. As P is a minimal counterexample, all isotropy groups in $\mathbb{S}^{n'}$ which contain the principal isotropy group H are contained in K. Furthermore K also contains the normalizer of H in P, as this group is a subgroup of $N(\mathsf{H})'$. By Proposition 11.2 there is an equivariant map $\mathbb{S}^{n'} \to \mathsf{P}_0/\mathsf{K}\cap\mathsf{P}_0$. Since the action of P_0 on $\mathbb{S}^{n'}$ is not transitive, a simple topological results yields $\mathsf{K}\cap\mathsf{P}_0 = \mathsf{P}_0$. Because of $\mathsf{P} \not\subset N(\mathsf{H})$ the orbit $\mathsf{G}\star p$ corresponds to a boundary point of the orbit space and $\dim(\mathsf{P}) > \dim(\mathsf{H})$. Applying the soul orbit theorem to the slice representation of P yields the existence of an isotropy group $\mathsf{P}' \subsetneq \mathsf{P}$ such that P/P' is connected. Using $\mathsf{P}_0 \subset \mathsf{K}$, we see that P' is not contained in K either — a contradiction as P is a minimal counterexample. □

Proposition 11.7 also allows us to prove the lemma already used in the proof of the stability theorem.

LEMMA 11.8. *Let* (G_d, u) *be one of the pairs* $(\mathsf{SO}(d), 1)$, $(\mathsf{SU}(d), 2)$ *or* $(\mathsf{Sp}(d), 4)$. *Assume that* G_d *acts on a simply connected manifold* M. *Suppose also the principal isotropy group* H *of the action contains a* $k\times k$ *block* B_k *with* $k \geq 2$, *and* $k \geq 3$ *if* $u = 1, 2$. *Choose* k *maximal. Let* K *be an isotropy group such that* H *is a principal isotropy group of the slice representation of* K. *Then* B_k *is normal in* $N(\mathsf{H}) \cap \mathsf{K}$.

Proof. Consider a point with isotropy group H and the fixed-point component B of H containing that point. We consider again the subgroup $N(\mathsf{H})'$

of $N(\mathsf{H})$ which leaves this component invariant. It suffices to prove that B_k is normal in $N(\mathsf{H})'$.

By Lemma 2.7, B_k is a normal subgroup of H_0. Because of $\mathsf{H} \subset N(\mathsf{H}_0)_0$ (see Lemma 11.4) we deduce B_k is normal in $N(\mathsf{H}_0)_0 \supset \mathsf{H}$.

The action of $\mathsf{L} := N(\mathsf{H})'/\mathsf{H}$ on B has a trivial principal isotropy group. By Proposition 11.7, the component group L/L_0 is generated by involutions that correspond to faces of $B/\mathsf{L} = M/\mathsf{G}_d$ whose inverse image in B have codimension 1. We choose a generic point on the inverse images of such a face. Then the isotropy group K' of that point with respect to the L-action is isomorphic to $\mathbb{Z}_2$. We also consider the isotropy group $\bar{\mathsf{K}}$ with respect to the G_d-action. Since $\bar{\mathsf{K}}/\mathsf{H}$ is a sphere of positive dimension, it is easy to see that B_k is normal in $N(\mathsf{H}) \cap \mathsf{K}'$. Thus it follows that the component of L represented by K' normalizes B_k. From Proposition 11.7 we deduce that B_k is normal in $N(\mathsf{H})'$. □

12. On the number of factors of principal isotropy groups

First we we have the following corollary from the previous section.

Corollary 12.1. *Let* G *be a compact connected Lie group acting with finite kernel on a simply connected compact positively curved manifold, and let* H *denote its principal isotropy group.*

a) *Then* $\mathrm{nof}(\mathrm{pr}_1(\mathsf{H}_0)) \le 2f \le 2k+2$, *where* f *denotes the number of faces of the orbit space* M/G.

b) *If* C *denotes the center of* H_0, *then* $\dim(\mathsf{C}) \le f \le k+1$.

In fact if this were false we could use Corollary 11.1 to see that there are a nontrivial connected subgroup $\mathsf{H}' \subset \mathsf{H}$ and an equivariant map $f\colon M \to \mathsf{G}/N(\mathsf{H}')$. By Lemma 5 this implies that any isotropy representation of G/H is equivalent to a subrepresentation of the isotropy representation of $N(\mathsf{H}')/\mathsf{H}$. But this would clearly imply that H' is in the kernel of the action.

It is easy to see that both estimates are sharp.

Corollary 12.2. *If* E_8 *acts nontrivially and isometrically on a positively curved manifold* M, *then* $\dim(M) \ge 247$.

Notice that the lower bound is optimal as there is an action of E_8 on $\mathbb{S}^{247}$ induced by the adjoint representation.

Proof of Corollary 12.2. By Proposition 8.9 the principal isotropy group of the action has an abelian identity component H_0. Combining this with Corollary 12.1 b) gives the fact that the cohomogeneity k of the action is bounded below by $\dim(\mathsf{H}_0) - 1$. Hence $\dim(M) \ge \dim(\mathsf{E}_8) - 1 = 247$. □

Recall that by Theorem 7 all faces of a positively curved compact orbit space intersect, unless the orbit space is given by a simplex. In view of this the following corollary is very useful.

Corollary 12.3. *Let* G *be a connected Lie group acting on a compact positively curved manifold. Suppose there is a point* $p \in M$ *whose orbit is contained in the intersection of all faces of* M/G*. Let* K *be the isotropy group of* p*, and let* $\mathsf{H} \subset \mathsf{K}$ *be a principal isotropy group. Then any nontrivial irreducible subrepresentation of the isotropy representation of* G/H *is isomorphic to a subrepresentation of the isotropy representation of* K/H*.*

Proof. Let B be a component of $\mathrm{Fix}(\mathsf{H})$ containing p. Let $N(\mathsf{H})'$ be the subgroup of $N(\mathsf{H})$ leaving B invariant. Using Proposition 11.7 we see that $N(\mathsf{H})' \cap \mathsf{K}$ intersects each connected component of $N(\mathsf{H})'$. Let $c(t)$ be a generic horizontal geodesic in B; i.e., $c(t)$ is either a generic point or a point corresponding to a codimension 1 orbit type. Given a nontrivial irreducible subrepresentation of the isotropy representation of G/H we can find a t such that this representation is equivalent to a subrepresentation the isotropy representation of $\mathsf{K}_{c(t)}/\mathsf{H}$.

Since $B/N(\mathsf{H})'$ is isometric to M/G and $\mathsf{G}\star p$ is contained in all faces, we can find an $h \in N(\mathsf{H})'$ with $h\mathsf{K}_{c(t)}h^{-1} \subset \mathsf{K}$. We may assume $h \in N(\mathsf{H})_0$ as K intersects all components of $N(\mathsf{H})'$. But then the isotropy representations of $\mathsf{K}_{c(t)}/\mathsf{H}$ and $h\mathsf{K}_{c(t)}h^{-1}/\mathsf{H}$ are isomorphic. □

Lemma 12.4. *Let* G *be a connected Lie group containing* E_8 *as a normal subgroup. Suppose that* G *acts faithfully and isometrically on a positively curved manifold* M*. Then* $\dim(M/\mathsf{G}) \geq 3$ *and strict inequality holds if* $\dim(M) \neq 247$ *or* $\mathsf{G} \neq \mathsf{E}_8$*.*

Proof. Put $k := \dim(M/\mathsf{G})$. We argue by induction on $\dim(M)$. If the subaction of E_8 has a fixed-point p, then the induction hypothesis implies that the slice representation of the isotropy group of $\mathsf{K} \subset \mathsf{G}$ at p induces an action on the sphere of cohomogeneity ≥ 3. Thus we have that $\dim(M/\mathsf{G}) \geq 4$.

We can write $\mathsf{G} = \mathsf{E}_8 \times \mathsf{G}'$. Consider next the case of $\mathsf{G}' \neq \{1\}$. If G' acts with only one orbit type, then M/G' is a positively curved manifold endowed with an isometric action of E_8. Furthermore as a consequence of the rank lemma (Proposition 1.4) combined with the isotropy lemma (Lemma 5) G' necessarily has rank 1 and $\dim(M/\mathsf{G}')$ is even. Thus the statement follows from the induction hypothesis.

If G' does not act with one orbit type, we can argue as follows. Choose an isotropy group $\mathsf{K}' \subset \mathsf{G}'$ that is not principal. Let $N(\mathsf{K}')$ be the normalizer of K' in G'. On a component M' of $\mathrm{Fix}(\mathsf{K}')$ the group $\mathsf{E}_8 \times N(\mathsf{K}')_0/\mathsf{K}'$ acts, and the cohomogeneity of the action is strictly smaller than $\dim(M/\mathsf{G})$. Again the statement follows from the induction hypothesis.

In other words it suffices to prove the lemma for $\mathsf{G} = \mathsf{E}_8$. By Corollary 12.2 $\dim(M) \geq 247$. Suppose that $\dim(M/\mathsf{G}) \leq 3$. Then $\dim(\mathsf{H}) = 248 + \dim(M/\mathsf{G}) - \dim(M) \leq 4$, and strict inequality holds if $\dim(M) > 247$ or $\dim(M/\mathsf{G}) \leq 2$. The statement now follows from the rank lemma (Proposition 1.4). □

13. Proof of Theorem 2

We first want to prove a special case for those readers who only want to get a rough idea of the line of arguments and are not interested in the details involved.

Proposition 13.1. *Let M be a simply connected positively curved manifold with*

$$\mathrm{symrank}(M, g) > 9(\mathrm{cohom}(M, g) + 1).$$

Then M is tangentially homotopically equivalent to a rank one symmetric space.

Proof. Put $\mathsf{L} = \mathrm{Iso}(M, g)_0$. Let P denote the principal isotropy group of L. By the rank lemma (Proposition 1.4) we have $\mathrm{rank}(\mathsf{P}) > 8(\dim(M/\mathsf{L}) + 1)$. By Corollary 12.1, P_0 can be decomposed into at most $2(\dim(M/\mathsf{L}) + 1)$ factors. Combining both statements we see that P_0 contains a simple normal subgroup H' of rank ≥ 5. Because of Lemma 7.1, H' is contained in a simple normal subgroup $\mathsf{G} \subset \mathsf{L}$. Using Propositions 7.3 and 7.4 we see that the hypothesis of Theorem 5.1 is satisfied for the subaction of G on M. Thus the proposition follows. □

Proof of Theorem 2. In order to establish the theorem by induction it is useful to prove a slightly stronger statement. We introduce notation and say that a nontrivial isometric action of G on a positively curved manifold M is good if G is a simple, connected Lie group, the principal isotropy group H contains a simple subgroup H' of rank at least 2 and either the hypothesis of Theorem 5.1 is satisfied for the action of G or $\mathsf{G}/\mathsf{H}' \cong \mathsf{Spin}(9)/\mathsf{Spin}(7) \cong \mathbb{S}^{15}$.

Consider a connected subgroup $\mathsf{L} \subset \mathrm{Iso}(M, g)$ with

$$\mathrm{rank}(\mathsf{L}) > 3(\dim(M/\mathsf{L}) + 1).$$

We want to prove by induction on $k := \dim(M/\mathsf{L}) > 0$ that for a suitable simple normal subgroup $\mathsf{G} \subset \mathsf{L}$ the subaction of G on M is good. The theorem then follows by application of this statement to $\mathsf{L} = \mathrm{Iso}(M, g)_0$. In case $\mathsf{G}/\mathsf{H} \cong \mathbb{S}^{15}$ it is easy to see that M is fixed-point homogeneous with respect to the subaction of G.

For a proof of the above statement it also useful to drop the hypothesis $\pi_1(M) = 1$. The classification of positively curved homogeneous spaces implies that for a transitive almost faithful action of a Lie group C on a positively curved manifold M there is a good subaction of a simple normal subgroup $\mathsf{G} \subset \mathsf{C}$, provided that $\mathrm{rank}(\mathsf{C}) \geq 4$ and M is not locally isometric to $\mathsf{F}_4/\mathsf{Spin}(8)$, $\mathsf{SU}(5)/\mathsf{S}^1\mathsf{Sp}(2)$ or $\mathrm{Ca}\mathbb{P}^2$. We will use this implicitly in the proof.

Let P denote the principal isotropy group of the action of L. By the rank lemma $\mathrm{rank}(\mathsf{P}) > 2k + 2$. Thus P contains a simple normal subgroup H' of rank ≥ 2 by Corollary 12.1. By Lemma 7.1, H' is contained in a simple normal subgroup $\mathsf{G} \subset \mathsf{L}$. Clearly we may assume that the hypothesis of Theorem 5.1 is not satisfied for the subaction of G and also $(\mathsf{G}, \mathsf{H}') \neq (\mathsf{Spin}(9), \mathsf{Spin}(7))$. If possible, we choose H' such that $\mathsf{G} = \mathsf{F}_4$. For any subgroup $\mathsf{G}' \subset \mathsf{L}$ we let $C(\mathsf{G}')_0$ denote the identity component of the centralizer of G' in L.

By Theorem 10.1 $(\mathsf{G}, \mathsf{H}')$ is one of the pairs listed there. If $(\mathsf{G}, \mathsf{H}')$ is one of the pairs $(\mathsf{G}_2, \mathsf{SU}(3))$ or $(\mathsf{Spin}(7), \mathsf{G}_2)$, then there is an isotropy group $\mathsf{K} \subset \mathsf{L}$ corresponding to a face of the orbit space such that $\mathsf{G} \subset \mathsf{K}$. Notice that $C(\mathsf{G})_0$ acts almost faithfully on a component M' of $\mathrm{Fix}(\mathsf{G})$ with cohomogeneity $k - 1$. Because $\mathrm{rank}(C(\mathsf{G})_0) = \mathrm{rank}(\mathsf{L}) - \mathrm{rank}(\mathsf{G})$ the induction hypothesis implies that there is a good subaction of a simple normal subgroup $\bar{\mathsf{G}} \subset C(\mathsf{G}_0)$ on M'. Consider a principal isotropy group $\bar{\mathsf{H}}$ of the $\bar{\mathsf{G}}$-action in M'. The slice representation of $\bar{\mathsf{H}}$ in M commutes with the action of G. Hence the subaction of $\bar{\mathsf{G}}$ on M is good as well.

Because $k \geq 1$, we have $\mathrm{rank}(\mathsf{L}) \geq 7$. With this in mind it is actually easy to see that $k \geq 2$. Hence $\mathrm{rank}(\mathsf{L}) \geq 10$.

If $(\mathsf{G}, \mathsf{H}')$ is one of the pairs $(\mathsf{F}_4, \mathsf{Spin}(8))$, $(\mathsf{Spin}(10), \mathsf{SU}(4))$ or $(\mathsf{Spin}(8), \mathsf{G}_2)$, then it follows from Lemma 3.1 that there is an isotropy group $\mathsf{K} \subset \mathsf{L}$ with $\mathsf{G} \subset \mathsf{K}$ and K corresponds to a codimension 2 orbit type of the orbit space M/L. We consider the action of $C(\mathsf{G})_0$ on the component M' of $\mathrm{Fix}(\mathsf{G})$. Clearly the cohomogeneity of this action is $k-2$, $\mathrm{rank}(C(\mathsf{G})_0) \geq \mathrm{rank}(\mathsf{L})-5$, and it is easy to see that the kernel of this action can be at most one dimensional. It is also easy to rule out the possibility that M' is one of the exceptional homogeneous spaces of positive curvature. Thus the induction hypothesis implies that there is a good subaction of a simple normal subgroup $\bar{\mathsf{G}} \subset C(\mathsf{G})_0$ on M'. As before it follows that the action of $\bar{\mathsf{G}}$ on M is good as well.

Consider next the case $(\mathsf{G}, \mathsf{H}') = (\mathsf{E}_7, \mathsf{Spin}(8))$ or $(\mathsf{E}_6, \mathsf{Spin}(8))$. It is easy to deduce that there is an isotropy group $\mathsf{K} \subset \mathsf{L}$ corresponding to a codimension 2 stratum of the orbit space with $\mathsf{K} \cap \mathsf{G} = \mathsf{F}_4$. The slice representation of K_0 is effectively only a representation of F_4 (of real type). Furthermore K acts with cohomogeneity two on the normal space of the orbit type of K in M. Consider the fixed-point component M' of F_4. By the previous remark $C(\mathsf{F}_4)_0$ acts with finite kernel on M' and cohomogeneity $k - 2$. Furthermore,

$\mathrm{rank}(C(\mathsf{F}_4)_0) = \mathrm{rank}(\mathsf{L}) - 6$. As before the statement follows from the induction hypothesis.

Thus we may assume that $(\mathsf{G}, \mathsf{H}')$ is one of the pairs $(\mathsf{F}_4, \mathsf{SU}(3))$, $(\mathsf{Spin}(10), \mathsf{SU}(3))$, $(\mathsf{Spin}(9), \mathsf{SU}(3))$, $(\mathsf{Spin}(8), \mathsf{SU}(3))$ or $(\mathsf{Spin}(7), \mathsf{SU}(3))$. Using $\mathrm{rank}(\mathsf{L}) \geq 10$, it is easy to see that $\dim(M/\mathsf{G}) \geq 3$. We can find an isotropy group K_f corresponding to a codimension 1 orbit type in M/L such that $\mathsf{K}_f \cap \mathsf{G}$ contains $\mathsf{SU}(4)$ or G_2. The isotropy representation of $\mathsf{G}/(\mathsf{K}_f \cap \mathsf{G})$ contains an irreducible subrepresentation of dimension 6 or 7 (of real type), and the orbit type of $\mathsf{K}_f \cap \mathsf{G}$ is at least two dimensional in M/G. Therefore we can apply Lemma 3.1 once more to see that there is an isotropy group $\mathsf{K} \subset \mathsf{L}$ corresponding to an orbit type of codimension ≤ 3 in M/L such that $\mathsf{K} \cap \mathsf{G}$ contains $\mathsf{Spin}(7)$. If we restrict the nontrivial part of the slice representation of K to $\mathsf{Spin}(7)$ then it is either given by two 8-dimensional representations or by an 8- and a 7-dimensional representation. Hence $C(\mathsf{Spin}(7))_0$ acts with cohomogeneity $\leq k - 2$ on a component M' of $\mathrm{Fix}(\mathsf{Spin}(7))$, and the kernel of the action is at most one dimensional. Because of $\mathrm{rank}\big(C(\mathsf{Spin}(7))_0\big) \geq \mathrm{rank}(\mathsf{L}) - 4$, the statement can again be deduced from the induction hypothesis. □

Addendum to Theorem 2. *Suppose there is the same hypothesis as in Theorem* 2 *without the assumption that M is simply connected. Then $\pi_1(M)$ is isomorphic to the fundamental group of a space form of dimension $b \leq 4k + 3$.*

Proof. By the previous proof there is a good subaction of a simple normal subgroup $\mathsf{G} \subset \mathrm{Iso}(M, g)_0$. As before let H denote the principal isotropy group of the G-subaction. If $\mathsf{G}/\mathsf{H} \cong \mathbb{S}^{15}$, then M is fixed-point homogeneous with respect to the subaction of G. By Grove and Searle the fundamental group is either trivial or $\mathbb{Z}_2$. Otherwise the subaction of G satisfies the hypothesis of Theorem 5.1, and it follows that $\pi_1(M)$ is the fundamental group of a b-dimensional space form. In order to get the estimate on b one combines Theorem 5.1 with the rank lemma (Proposition 1.4). □

14. Proof of Corollary 3 and Theorem 4

Proposition 14.1. *Let L be a Lie group acting isometrically on a positively curved n-dimensional manifold (M, g) with cohomogeneity k. Suppose also that M is not locally isometric to $\mathsf{F}_4/\mathsf{Spin}(8)$ and $n \geq 18(k+1)^2$. Then there are a normal subgroup $\mathsf{G} \subset \mathsf{L}$ and a number $u \in \{1, 2, 4\}$ such that after possibly replacing G by a cover, $(\mathsf{G}, u) \in \big\{(\mathsf{Sp}(d), 4), (\mathsf{SU}(d), 2), (\mathsf{SO}(d), 1)\big\}$ with $d \geq \frac{n+1}{u(k+1)}$. Furthermore* $\mathrm{rank}(\mathsf{L}) > 3k + 3$.

The estimate on d is sharp. For example, there is a cohomogeneity k action of $\mathsf{Sp}(k+1) \times \mathsf{Sp}(d)$ on $\mathbb{S}^n$ with $n = 4d(k+1) - 1$, $d \geq k + 1$. The estimate also implies $\mathrm{symdeg}(M, g) \geq \dim(\mathsf{G}) \geq 2n$.

Of course Corollary 3 follows immediately. In fact combining further with Theorem 6.1 one sees that the principal isotropy group of the subaction of G contains an $l \times l$ block, where $d - l \leq k+1$. By Theorem 8 c) there is a chain of positively curved manifolds

$$M = M_0 \subset M_1 \subset M_2 \subset$$

with $\dim(M_i) = \dim(M) + hi$ and $h \leq 4k+4$, as was claimed in the introduction.

Proof of Proposition 14.1. We argue by induction on k assuming that L acts almost faithfully and that M is simply connected. The case $k = 0$ follows from the classification of simply connected positively curved homogeneous manifolds. Let P denote the principal isotropy group of the action of L on M.

$$\begin{aligned}\dim(\mathsf{L}) &= \dim(M) - k + \dim(\mathsf{P})\\ &\geq 18(k+1)^2 - k\\ &= \dim(\mathsf{Sp}(3k+2)) + 8k + 8.\end{aligned}$$

We first want to prove $\mathrm{rank}(\mathsf{L}) > 3k+2$. For $k \neq 2, 3$ this is clear as the Lie group $\mathsf{Sp}(3k+2)$ has maximal dimension among all compact groups of rank $3k+2$. If E_8 is a normal subgroup of L, then $k \geq 3$ by Lemma 12.4, and because of $\dim(M) \geq 18(k+1)^2 > 247$, Lemma 12.4, implies $k \geq 4$. This takes care of $k = 2, 3$.

Thus $\mathrm{rank}(\mathsf{L}) \geq 3k+3$. By the rank lemma (Proposition 1.4) $\mathrm{rank}(\mathsf{P}) \geq 2k+2$, and by Corollary 12.1, the center of P_0 has dimension at most $k+1$. Therefore $\dim(\mathsf{P}) \geq 4(k+1)$ and

$$\dim(\mathsf{L}) \geq 18(k+1)^2 + 3k + 4 = \dim(\mathsf{Sp}(3k+3)) + 1.$$

As above we deduce $\mathrm{rank}(\mathsf{L}) > \mathrm{rank}(\mathsf{Sp}(3k+3)) = 3k+3$ for $k \neq 2$. In the case of $k = 2$, one is done as well, because then E_8 cannot be a normal subgroup of L.

From the proof of Theorem 2 we know that there is a simple normal subgroup $\mathsf{G} \subset \mathsf{L}$ such that $\mathsf{H} := \mathsf{P} \cap \mathsf{G}$ contains a simple normal subgroup H' of rank ≥ 2. Furthermore, either $\mathsf{G}/\mathsf{H} \cong \mathbb{S}^{15}$ or the action of G satisfies the hypothesis of Theorem 5.1. If $\mathsf{G}/\mathsf{H} \cong \mathbb{S}^{15}$, then $M' := \mathrm{Fix}(\mathsf{G})$ has codimension 16, L acts on M' with cohomogeneity $k-1$, and the proposition follows from the induction hypothesis.

Otherwise we can choose a number $u \in \{1, 2, 4\}$ such that $(\mathsf{G}, \mathsf{H}', u)$ is one of the triples $(\mathsf{Sp}(h), \mathsf{Sp}(h-r), 4)$, $(\mathsf{SU}(h), \mathsf{SU}(h-r), 2)$ or $(\mathsf{SO}(h), \mathsf{SO}(h-r), 1)$. We first want to consider the special case that the fixed-point set of G is empty. By Theorem 6.1 an $h-1$ block of G is contained in an isotropy group $\mathsf{K} \subset \mathsf{L}$. By inspecting the slice representation of K we see that $r \leq k+1$. Because of Theorem 6.1 the nonexistence of a fixed-point also implies $n < urh$. Thus $h > \frac{n}{ur} \geq \frac{n}{u(k+1)}$ and we are done.

Hence we may assume that G has a fixed-point and $h \le \frac{n}{u(k+1)}$. Let M' denote the fixed-point set of G. Notice that L acts on M'. The cohomogeneity k' of this action is bounded above by $k-r$. The codimension of M' is given by urh. Thus we have for the dimension n' of M'

$$\begin{aligned} n' \,=\, n - urh &\ge \bigl(1 - \tfrac{r}{(k+1)}\bigr)n \\ &\ge 18(k+1)\bigl(k+1-r\bigr) \\ &\ge 18(k'+1)^2 + 18. \end{aligned}$$

Since the induction hypothesis is satisfied, we can find a normal subgroup $\bar{\mathsf{G}} \subset \mathsf{L}$ and $\bar{u}$ such that $(\bar{\mathsf{G}}, \bar{u})$ is one of the pairs $(\mathsf{Sp}(d), 4)$, $(\mathsf{SU}(d), 2)$ or $(\mathsf{SO}(d), 1)$ with

$$\begin{aligned} d &> \tfrac{n'}{\bar{u}(k'+1)} \\ &\ge \bigl(1 - \tfrac{r}{(k+1)}\bigr)\tfrac{n}{\bar{u}(k'+1)} \\ &= \tfrac{n}{\bar{u}(k+1)} \cdot \tfrac{k+1-r}{k'+1} \ge \tfrac{n}{\bar{u}(k+1)}. \end{aligned}$$

□

Proof of Theorem 4. Let $\mathsf{L} = \mathrm{Iso}(M, g)_0$. By Proposition 14.1 there are a simple, connected normal subgroup $\bar{\mathsf{G}} \subset \mathsf{L}$ and a number $u \in \{1, 2, 4\}$ such that after possible replacement of $\bar{\mathsf{G}}$ by a cover the following holds. The pair $(\bar{\mathsf{G}}, u)$ is given by one of the following pairs $(\mathsf{Sp}(\bar{d}), u)$, $(\mathsf{SU}(\bar{d}), u)$ or $(\mathsf{SO}(\bar{d}), u)$ with $\bar{d} \ge \frac{n+1}{u(k+1)} > \frac{18}{u}(k+1)$. As mentioned after Proposition 14.1 it is clear that the principal isotropy group contains an $l \times l$ block, and by Theorem 6.1 we see that the principal isotropy group itself is an $l \times l$ block with $l \ge \bar{d} - k - 1 \ge 5$. In the case of $l = \bar{d} - 1$ the manifold would be fixed-point homogeneous and we would be done by [11]. Next we want to define $\mathsf{G} \subset \bar{\mathsf{G}}$.

If $\bar{\mathsf{G}} = \mathsf{Sp}(\bar{d})$ we put $d = \bar{d} - 1$ and $\mathsf{G} := \mathsf{Sp}(d) \subset \bar{\mathsf{G}}$. If $\bar{\mathsf{G}} = \mathsf{SU}(\bar{d})$ we put $d = \bigl[\frac{\bar{d}-2}{2}\bigr]$ and $\mathsf{G} := \mathsf{Sp}(d) \subset \mathsf{SU}(\bar{d}-2) \subset \bar{\mathsf{G}}$ and if $\bar{\mathsf{G}} = \mathsf{SO}(\bar{d})$ we put $d := \bigl[\frac{\bar{d}-3}{4}\bigr]$ and $\mathsf{G} = \mathsf{Sp}(d) \subset \mathsf{SO}(\bar{d}-3) \subset \bar{\mathsf{G}}$. Using Theorem 6.1 we verify easily that the dimension of the fixed-point set of G is at least 4.

Thus we have proved that there is an action of $\mathsf{G} = \mathsf{Sp}(d)$ whose principal isotropy group is an $l \times l$ block with $l \ge d/2$ and whose fixed-point set has dimension at least 4. Furthermore $d \ge \frac{n+1}{4(k+1)} - 2$.

By [7, Th. 4], the conclusion of Theorem 4 holds if the fixed-point set of any $l \times l$ block $\mathsf{B}_l \subset \mathsf{G}$ is simply connected. Notice that

$$\mathrm{Fix}(\mathsf{B}_l) = \mathrm{Fix}(\mathrm{diag}(-1, \ldots, -1, 1 \ldots, 1)),$$

where $D_l := \mathrm{diag}(-1, \ldots, -1, 1 \ldots, 1)$ is the diagonal matrix with precisely l entries equal to -1. Put $M_l := \mathrm{Fix}(D_{d-l})$. Then $M_0 \subset \cdots \subset M_d = M$ is a chain of totally geodesic submanifolds. Furthermore the codimension of M_i in M_{i+1} is independent of i. And M_{i-1} can be realized as the intersection of two isometric copies of M_i in M_{i+1}, $i \leq d-1$. Using the connectedness lemma we see that M_l is simply connected and thus Theorem 4 follows. □

Concluding remarks

1. In the proof of Theorem 2 the first crucial requirement is to show that the principal isotropy group of the action of $\mathsf{L} := \mathrm{Iso}(M,g)_0$ has a simple subgroup of rank 2. This is no longer true if one only requires $\mathrm{symrank}(M,g) \geq 3(\mathrm{cohom}(M,g)+1)$. In fact there is a positively curved cohomogeneity k manifold M^{8k+7} (a sphere with a deformed metric) whose isometry group has the identity component $(\mathsf{Sp}(1)\mathsf{Sp}(2))^{k+1}$.

2. Theorem 1 as well as Theorem 2 do not remain valid if one relaxes the hypothesis from positive sectional curvature to positive sectional curvature on an open and dense set of points. In fact it was shown in [21] that the projective tangent bundle $P_{\mathbb{C}}T\mathbb{CP}^n$ of $\mathbb{CP}^n$ admits a cohomogeneity 2 metric with positive sectional curvature on an open dense set of points. These manifolds fit in a chain $P_{\mathbb{C}}T\mathbb{CP}^3 \subset P_{\mathbb{C}}T\mathbb{CP}^4 \subset \cdots$ such that each pair satisfies all conclusions of the stability theorem. The limit space is the classifying space $(\mathbb{CP}^\infty)^2$. It would be interesting to know whether this behavior is enforced by the curvature and symmetry properties of the examples.

3. In an earlier version the author gained control of the principal isotropy group of an action of low cohomogeneity on a positively curved manifold M by a rather different and computational approach. The idea is to combine the condition of positive sectional curvature with the Gauss equations at a principal orbit. Although this approach might be less elegant it actually gives another interesting result. In dimensions above $(30(k+l))^2$ any simply connected cohomogeneity k manifold with positive l^{th} Ricci curvature is tangentially homotopically equivalent to a rank one symmetric space. Recall that a Riemannian manifold is said to have positive l^{th} Ricci curvature if for any unit vector v, the sum of the l smallest eigenvalues of $R(\cdot,v)v$, viewed as an endomorphism of $(v)^\perp$, is positive.

4. A result of Straume [18] says that an exotic n-sphere which admits a metric with symmetry degree $\geq \frac{3}{2}(n+1)$ is necessarily given by a Brieskorn variety or equivalently bounds a parallelizable manifold. Notice that for $n \geq 15$ the symmetry degree of the manifold in Theorem 1 is necessarily $\geq \frac{3}{2}(n+1)$.

5. It would be interesting to know whether some kind of analogue of Theorem 4 holds for homotopy $\mathbb{CP}^n$'s or $\mathbb{HP}^n$'s. In this context the following observation might be a starting point. If (M,g) is a positively curved cohomogeneity k manifold (M,g) of dimension $n \geq 18(k+1)^2$ with the homotopy type of $\mathbb{CP}^m$ one can lift the action of $\mathrm{Iso}(M,g)_0$ on M to an action of a group L on the total space Σ^{2m+1} of the principal S^1-bundle whose Euler class is a

generator of $H^2(M, \mathbb{Z})$ such that the actions of S^1 and L commute. With the same proof one can show that Σ^{2m+1} satisfies the conclusion of Theorem 4. Thus the action of S^1 on the homotopy sphere Σ^{2m+1} has to commute with a reasonably large, well understood group action of $\mathsf{Sp}(d) \subset \mathsf{L}$.

6. Up to a small modification Theorem 4 remains valid if one replaces the dimension hypothesis $n \geq 18(k+1)^2$ by the assumption $\mathrm{symrank}(M, g) > 3\,\mathrm{cohom}(M, g) + 3$. More precisely: in that case either the conclusion of Theorem 4 holds or M endowed with the action of $\mathrm{Iso}(M, g)_0$ is equivariantly diffeomorphic to $\mathbb{S}^n$ endowed with a linear action.

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