Tôhoku Math. Journ.
32 (1980), 557-566.

POSITIVELY INVARIANT SETS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

KENSUKE SAWANO

(Received July 31, 1979)

1. **Introduction.** For ordinary differential equations, many authors have discussed necessary and sufficient conditions for a closed set in the n-dimensional Euclidean space R^n to be positively invariant. Yorke [11] has discussed this problem by using a non-Lipschitzian Liapunov function which is lower-semicontinuous. For an autonomous system, Brezis [1] obtained a result under the assumption that the right hand side of the system is locally Lipschitzian, and his proof depends essentially on this assumption. Crandall [2] obtained a similar result by applying the method of polygonal approximations. For a nonautonomous system, Hartman [5] also considered an approximation which is different from the one considered in [2].

The purpose of this article is to discuss the same question for functional differential equations with infinite delay. Seifert [10] also discussed this question under the assumption that a closed set is convex. In Section 2, we introduce an abstract phase space B which satisfies some general hypotheses slightly different from those considered in [4]. We consider a subset Ω in $R \times R^n$ such that the cross section $\Omega_t = \{y \in R^n; (t, y) \in \Omega\}$ is convex for all $t \in R$ and that the cross section Ω_t satisfies a continuity condition in the sense of Hausdorff metric. We discuss the properties of Ω which play an important role in Section 3. In Section 3, we state the main theorem. We give the necessary and sufficient condition that, for any initial value (σ, ϕ) in $R \times B$ such that $\phi(t-\sigma) \in \Omega_t$ for all $t \leqq \sigma$, there exists at least one solution $x(t)$ through (σ, ϕ) which is defined on its right maximal interval of existence and satisfies $(t, x(t)) \in \Omega$ there. Special approximate solutions are needed to prove the theorem. The construction of the solutions, although analogous to the one in [5], is much more complicated for functional differential equations. The proof of the theorem is given in Section 4. The case where the delay is finite has been considered in [7] and [8] by a different approach.

2. Preliminaries. Let R^n be an n-dimensional real linear vector space, and let $R = R^1$. We denote by B a real linear vector space of functions mapping $(-\infty, 0]$ into R^n with a semi-norm $|\cdot|$. No confusion will occur if we use the same symbol $|\cdot|$ to denote the norm in R^n. For elements ϕ and ψ in B, $\phi = \psi$ means that $\phi(\theta) = \psi(\theta)$ for all θ in $(-\infty, 0]$. Then the quotient space $B^* = B/|\cdot|$ is a normed linear space with the norm naturally induced by the semi-norm. The topology of B is defined by the semi-norm, that is, a family $\{V(\phi, \varepsilon); \phi \in B, \ \varepsilon > 0\}$ is an open base, where $V(\phi, \varepsilon) = \{\psi \in B; |\phi - \psi| < \varepsilon\}$. B with this topology is a pseudo-metric space.

For any ϕ in B and any $\beta \geqq 0$, let ϕ^β be the restriction of ϕ to the interval $(-\infty, -\beta]$. This is a function mapping $(-\infty, -\beta]$ into R^n. Denote the space of such functions ϕ^β by B^β and define a semi-norm $|\cdot|_\beta$ in B by

$$|\eta|_\beta = \inf\{|\psi|; \psi \in B, \psi^\beta = \eta\}, \quad \eta \in B^\beta .$$

If we let $|\phi|_\beta = |\phi^\beta|_\beta$ for $\phi \in B$, then $|\cdot|_\beta$ is also a semi-norm in B.

For an R^n-valued function x defined on $(-\infty, \sigma)$, we define the function x_t for each $t \in (-\infty, \sigma)$ by the relation $x_t(\theta) = x(t + \theta)$, $-\infty < \theta \leqq 0$.

Let D be an open set in $R \times B$ and let $f: D \to R^n$ be a given continuous function. A functional differential equation on D is the relation

$$x'(t) = f(t, x_t) , \tag{1}$$

where $x'(t)$ stands for the right hand derivative of $x(t)$. For (σ, ϕ) in D, an R^n-valued function x defined on $(-\infty, \sigma + A)$ with $0 < A \leqq \infty$ is said to be a solution of (1) through (σ, ϕ) if $x_\sigma = \phi$ and if x is continuously differentiable and satisfies (1) for all $t \in [\sigma, \sigma + A)$.

We make the following hypotheses on the space B.

(B1) For an $A > 0$, let $x: (-\infty, A) \to R^n$ be a function such that x_0 is in B and x is continuous on $[0, A)$. Then x_t is in B for all t in $[0, A)$ and x_t is continuous in t.

(B2) There is a continuous function $K(\beta) > 0$ such that

$$|\phi| \leqq K(\beta) \sup_{-\beta \leqq \theta \leqq 0} |\phi(\theta)| + |\phi|_\beta$$

for all ϕ in B and for all β in $[0, \infty)$.

Under the hypotheses (B1) and (B2), there exists a solution of (1) through (σ, ϕ) in D. This was proved by Kaminogo [4].

For (σ, ϕ) in D, let $Q(\sigma, \phi)$ be the collection of (T, x), where $T > 0$ and x is a solution of (1) through (σ, ϕ) defined on $(-\infty, \sigma + T)$. We introduce a partial order $\leqq$ in $Q(\sigma, \phi)$ in the following way. For elements (T^1, x^1) and (T^2, x^2) in $Q(\sigma, \phi)$, we write $(T^1, x^1) \leqq (T^2, x^2)$ when $T^1 \leqq T^2$ and the restriction of x^2 to the interval $(-\infty, \sigma + T^1)$ is equal to x^1. Then Zorn's lemma implies the existence of a maximal element (T, x) in $Q(\sigma, \phi)$, and x is called a right maximal solution of (1) through (σ, ϕ) and the interval $(-\infty, \sigma + T)$ is called the right maximal interval of existence of x.

Under the hypotheses (B1) and (B2), we have the following.

LEMMA 1. *For any ϕ in B and constants $A > 0$, $L > 0$, let $F_A^L(\phi)$ be a set of functions $u: (-\infty, A] \to R^n$ such that $u_0 = \phi$ and $|u(t) - u(s)| \leqq L|t - s|$ on $[0, A]$. Then the set $\Gamma = \{u_t; u \in F_A^L(\phi), t \in [0, A]\}$ is compact in B.*

For the proof, see Lemma 2.1 of Hale and Kato [4], though the phase space considered in [4] is slightly different from ours.

Let Ω be a set in $R \times R^n$ such that the cross section $\Omega_t = \{y \in R^n; (t, y) \in \Omega\}$ is nonempty for all $t \in R$. Assume that Ω satisfies the following continuity condition (C).

(C) For any $\varepsilon > 0$ and any $t \in R$, there is a $\delta = \delta(\varepsilon, t) > 0$ such that if $|t - s| < \delta$, then

$$\inf \{r > 0; U(\Omega_t, r) \supset \Omega_s \text{ and } U(\Omega_s, r) \supset \Omega_t\} < \varepsilon ,$$

where $U(\Omega_t, r)$ is an r-neighborhood of Ω_t.

LEMMA 2. *If Ω_t is a closed set in R^n for any $t \in R$ and the condition* (C) *is satisfied, then Ω is a closed set in $R \times R^n$.*

PROOF. If the conclusion is false, then there is a sequence $\{(t_k, y_k)\}$ in Ω such that $(t_k, y_k) \to (t_o, y_o) \notin \Omega$ as $k \to \infty$. Since $y_o \notin \Omega_{t_o}$ and Ω_{t_o} is closed, we see that $U(y_o, \varepsilon_o) \cap \Omega_{t_o}$ is empty for some $\varepsilon_o > 0$. On the other hand, if k is large, $U(y_k, \varepsilon_o/3)$ contains a point $z_k \in \Omega_{t_o}$ since the condition (C) implies that $\Omega_{t_k} \subset U(\Omega_{t_o}, \varepsilon_o/3)$ for sufficiently large k. Moreover, $|y_k - y_o| < \varepsilon_o/3$ if k is large. Thus for sufficiently large k, we have

$$|y_o - z_k| \leqq |y_o - y_k| + |y_k - z_k| < \varepsilon_o/3 + \varepsilon_o/3 < \varepsilon_o ,$$

a contradiction to the emptiness of $U(y_o, \varepsilon_o) \cap \Omega_{t_o}$, and we are done.

From now on, let $|y| = (\sum_{i=1}^n y_i^2)^{1/2}$ for $y = (y_1, \cdots, y_n)$ in R^n.

LEMMA 3. *Suppose that Ω_t is closed convex for all $t \in R$ and that the condition* (C) *is satisfied. For a continuous function $p(t): [\sigma, \infty) \to R^n$, let $d(p(t), \Omega_t) = \inf \{|p(t) - y|; y \in \Omega_t\}$. Then there is a continuous function $g(t): [\sigma, \infty) \to R^n$ such that $g(t) \in \Omega_t$ and $d(p(t), \Omega_t) = |p(t) - g(t)|$.*

PROOF. Since Ω_t is closed, there exists a $g(t) \in \Omega_t$ with $d(p(t), \Omega_t) = |p(t) - g(t)|$ for each $t \in [\sigma, \infty)$. We show that $g(t)$ is uniquely determined for each t. Otherwise, there would exist a $z \in \Omega_s$ for some $s \in [\sigma, \infty)$ such that $z \neq g(s)$ and $|p(s)-z| = d(p(s), \Omega_s) = |p(s)-g(s)|$. Set $d(p(s), \Omega_s) = r$ and let $S(p(s), r)$ denote the sphere in R^n with radius r and center $p(s)$. Then $g(s)$ and z belong not only to Ω_s but also to $S(p(s), r)$. Since Ω_s is convex, the segment $\lambda g(s) + (1-\lambda)z$ with $0 \leqq \lambda \leqq 1$ belongs to Ω_s. We see immediately that $|p(s) - \{\lambda g(s) + (1-\lambda)z\}| < r$ for $0 < \lambda < 1$, which contradicts $d(p(s), \Omega_s) = r$.

Next the continuity of $d(p(t), \Omega_t)$ in t will be proved. For any $t, s \in [\sigma, \infty)$, we have

$$(2) \quad |d(p(t), \Omega_t) - d(p(s), \Omega_s)| \leqq |d(p(t), \Omega_t) - d(p(t), \Omega_s)| + |d(p(t), \Omega_s) - d(p(s), \Omega_s)| .$$

For any $\varepsilon > 0$ and any fixed t in $[\sigma, \infty)$, there exists a $\delta_1 = \delta_1(t, \varepsilon) > 0$ such that if $|t-s| < \delta_1$, then

$$(3) \quad |d(p(t), \Omega_s) - d(p(s), \Omega_s)| < \varepsilon/2 ,$$

because we have $|d(p(t), \Omega_s) - d(p(s), \Omega_s)| \leqq |p(t) - p(s)|$. Let $d(p(t), \Omega_s) = |p(t) - u^s|$ for $u^s \in \Omega_s$. Then, by the condition (C), there exists a $\delta_2 = \delta_2(t, \varepsilon) > 0$ such that if $|t-s| < \delta_2$, then $U(u^s, \varepsilon/2)$ contains a point v^s in Ω_t and $U(g(t), \varepsilon/2)$ contains a point w^s in Ω_s. Therefore we have

$$d(p(t), \Omega_t) \leqq |p(t) - v^s| \leqq |p(t) - u^s| + |u^s - v^s| \leqq d(p(t), \Omega_s) + \varepsilon/2$$

and

$$d(p(t), \Omega_s) \leqq |p(t) - w^s| \leqq |p(t) - g(t)| + |g(t) - w^s| \leqq d(p(t), \Omega_t) + \varepsilon/2 ,$$

which then imply that if $|t-s| < \delta_2$, we have

$$(4) \quad |d(p(t), \Omega_t) - d(p(t), \Omega_s)| \leqq \varepsilon/2 .$$

Combining (3) and (4), the right hand side of (2) is less than ε if $|t-s| < \delta$, where $\delta = \min\{\delta_1, \delta_2\}$. Thus $d(p(t), \Omega_t)$ is continuous in t.

Finally we show that $g(t)$ is continuous. Suppose that $g(t)$ is not continuous at $t = t_o \geqq \delta$. Then there exists an $\varepsilon_o > 0$ and a sequence $\{t_k\}$ such that $t_k \to t_o$ as $k \to \infty$ and that $|g(t_k) - g(t_o)| \geqq \varepsilon_o$ for all $k = 1, 2, \cdots$. Since $p(t)$ and $d(p(t), \Omega_t)$ are continuous in t, the sequence $\{g(t_k)\}$ is bounded, and hence we may assume that the sequence is convergent. Set $\lim_{k\to\infty} g(t_k) = z_o$. Then $z_o \in \Omega_{t_o}$ by Lemma 2. Moreover, since $d(p(t), \Omega_t) = |p(t) - g(t)|$ and $p(t)$ are continuous in t, we have

$$|p(t_o) - z_o| = \lim_{k\to\infty} |p(t_k) - g(t_k)| = \lim_{k\to\infty} d(p(t_k), \Omega_{t_k}) = d(p(t_o), \Omega_{t_o}) .$$

Thus $z_0 = g(t_0)$ because of the uniqueness of $g(t)$. On the other hand, $|g(t_k) - g(t_0)| \geqq \varepsilon_0$ implies $|g(t_0) - z_0| \geqq \varepsilon_0$, which contradicts $z_0 = g(t_0)$. This proves that $g(t)$ is continuous and completes the proof.

3. The main result. Consider a system

$$x'(t) = f(t, x_t)\ , \tag{5}$$

where $f: R \times B \to R^n$ is a continuous function.

THEOREM. *Assume that Ω_t is closed convex for all $t \in R$ and the condition* (C) *is satisfied. Then the following two statements are equivalent:*

(i) *For any $(\sigma, \phi) \in R \times B$ with $\phi(t - \sigma) \in \Omega_t$ for all $t \leqq \sigma$, there exists at least one solution x of* (5) *through (σ, ϕ) defined on its right maximal interval of existence and satisfying $(t, x(t)) \in \Omega$ on the interval.*

(ii) *For any $(\sigma, \phi) \in R \times B$ with $\phi(t - \sigma) \in \Omega_t$ for all $t \leqq \sigma$, it holds that*

$$\lim_{h \to 0+} d(\phi(0) + hf(\sigma, \phi), \Omega_{\sigma+h})/h = 0\ .$$

We prove this theorem in the next section. In the rest of this section, we consider special approximate solutions under the condition (ii).

Let $(\sigma, \phi) \in R \times B$ be such that $\phi(t - \sigma) \in \Omega_t$ for all $t \leqq \sigma$. Since f is continuous at (σ, ϕ), there are positive constants r, A and δ such that $|f| \leqq r$ on $[\sigma, \sigma + A] \times V(\phi, \delta)$. Let $L = \max\{K(\beta); 0 \leqq \beta \leqq A\} > 0$. Define $\tilde{\phi}$ by

$$\tilde{\phi}(t) = \begin{cases} \phi(t - \sigma)\ , & t \leqq \sigma\ , \\ \phi(0)\ , & t \geqq \sigma\ . \end{cases}$$

Then $\tilde{\phi}_t$ belongs to B for all $t \geqq \sigma$ by the hypothesis (B1) and $\tilde{\phi}_\sigma = \phi$. Furthermore, by the hypothesis (B1), there is an $\alpha = \alpha(\sigma, \phi)$ with $0 < \alpha \leqq A$ such that

$$3Lr\alpha + |\tilde{\phi}_t - \phi| < \delta \quad \text{for all} \quad t \in [\sigma, \sigma + \alpha]\ . \tag{6}$$

The set W defined by

$$W = \{(t, u_t); \sigma \leqq t \leqq \sigma + \alpha, u_\sigma = \phi \text{ and } |u(t) - u(s)| \leqq 2r|t - s| \text{ on } [\sigma, \sigma + \alpha]\}$$

is compact in $R \times B$ by Lemma 1.

Let ε, $0 < \varepsilon < r$, be given. Since W is compact, there is an $\eta(\varepsilon, W) > 0$ such that

$$|f(t, \phi^1) - f(t, \phi^2)| < \varepsilon \tag{7}$$

if $(t, \phi^1) \in W$ and $|\phi^1 - \phi^2| < \eta(\varepsilon, W)$, where we can assume that

$$\eta(\varepsilon, W) < Lr\alpha \,. \tag{8}$$

Now consider the set $Q_\varepsilon(\sigma, \phi)$ which consists of all (T, x), where $0 < T \leqq \alpha$ and x is a function mapping $(-\infty, \sigma + T]$ into R^n with the following properties:

(I) $x_\sigma = \phi$, $x(\sigma + T) \in \Omega_{\sigma+T}$ and $d(x(t), \Omega_t) < \eta(\varepsilon, W)L^{-1}$ for all $t \in [\sigma, \sigma + T]$.

(II) $|x(t) - x(t')| \leqq 2r|t - t'|$ on $[\sigma, \sigma + T]$.

(III) $|\dot{x}(t) - f(t, x_t)| \leqq 3\varepsilon$ for almost all $t \in [\sigma, \sigma + T]$, where $\dot{x}(t)$ is the derivative of $x(t)$.

(IV) Every subinterval of $[\sigma, \sigma + T]$ of length ε contains a point s such that $(s, x(s)) \in \Omega$.

LEMMA 4. *The set $Q_\varepsilon(\sigma, \phi)$ is nonempty for any small $\varepsilon > 0$.*

PROOF. By Lemma 3, there is a continuous mapping $g: [\sigma, \infty) \to R^n$ such that $d(\phi(0) + hf(\sigma, \phi), \Omega_{\sigma+h}) = |\phi(0) + hf(\sigma, \phi) - g(\sigma + h)|$ and $g(\sigma + h) \in \Omega_{\sigma+h}$ for all $h \geqq 0$. For S with $0 < S \leqq \varepsilon$, define a function y by

$$y(t) = \begin{cases} \phi(t - \sigma)\,, & t \leqq \sigma\,, \\ \phi(0) + \{(g(\sigma + S) - \phi(0))/S\}(t - \sigma)\,, & \sigma < t \leqq \sigma + S\,. \end{cases}$$

We show that (S, y) belongs to $Q_\varepsilon(\sigma, \phi)$ if S is sufficiently small.

The condition (ii) implies that there is a δ_1 with $0 < \delta_1 \leqq \varepsilon$ such that

$$|(g(\sigma + h) - \phi(0))/h - f(\sigma, \phi)| < \varepsilon \tag{9}$$

for all $h \in (0, \delta_1]$. Hence if $S \leqq \delta_1$, we have

$$\begin{aligned} |y(t) - y(t')| &= |(g(\sigma + S) - \phi(0))/S||t - t'| \\ &\leqq (|f(\sigma, \phi)| + \varepsilon)|t - t'| \leqq 2r|t - t'| \end{aligned} \tag{10}$$

on $[\sigma, \sigma + S]$. Then by the hypothesis (B2), we have $|y_t - \phi| \leqq |y_t - \tilde{\phi}_t| + |\tilde{\phi}_t - \phi| \leqq 2rL(t - \sigma) + |\tilde{\phi}_t - \phi|$ for all $t \in [\sigma, \sigma + S]$. Hence the continuity of f implies that there is a δ_2 with $0 < \delta_2 \leqq \delta_1$ such that $|f(\sigma, \phi) - f(t, y_t)| < \varepsilon$ for all $t \in [\sigma, \sigma + S]$ if $S \leqq \delta_2$. From this and (9), it follows that

$$\begin{aligned} |\dot{y}(t) - f(t, y_t)| &\leqq |\{g(\sigma + S) - \phi(0)\}/S - f(\sigma, \phi)| + |f(\sigma, \phi) - f(t, y_t)| \\ &\leqq 2\varepsilon \end{aligned} \tag{11}$$

for all $t \in [\sigma, \sigma + S]$ if $S \leqq \delta_2$.

Since $g(\sigma) = \phi(0) = y(\sigma)$ and $y(t)$ satisfies (10) for $S \leqq \delta_2$, there is a δ_3 with $0 < \delta_3 \leqq \delta_2$ such that $|g(t) - y(t)| < \eta(\varepsilon, W)L^{-1}$ on $[\sigma, \sigma + S]$ if $S \leqq \delta_3$. Therefore we have

$$d(y(t),\ \Omega_t) \leqq |y(t) - g(t)| < \eta(\varepsilon,\ W)L^{-1} \tag{12}$$

for all $t \in [\sigma, \sigma + S]$ if $S \leqq \delta_3$. From (10), (11) and (12), it follows that $y(t)$ satisfies (I), (II) and (III) if $S = \delta_3$. The condition (IV) is also satisfied because $0 < \delta_3 \leqq \varepsilon$. This completes the proof.

LEMMA 5. *There is an element* (α, x) *in* $Q_\varepsilon(\sigma, \phi)$ *for any small* $\varepsilon > 0$.

PROOF. Introduce a partial order $\leqq$ in $Q_\varepsilon(\sigma, \phi)$ as follows. For elements (T^1, x^1) and (T^2, x^2) in $Q_\varepsilon(\sigma, \phi)$, we write $(T^1, x^1) \leqq (T^2, x^2)$ when $T^1 \leqq T^2$ and the restriction of x^2 to the interval $(-\infty, \sigma + T^1]$ is equal to x^1. First, we show that there is a maximal element. $Q_\varepsilon(\sigma, \phi)$ is non-empty by Lemma 4. Let $E = \{(T^\lambda, x^\lambda); \lambda \in \Lambda\}$ be any totally ordered set in $Q_\varepsilon(\sigma, \phi)$. Set $J = \sup\{T^\lambda; \lambda \in \Lambda\}$. If $(T^\lambda, x^\lambda) \leqq (T^\mu, x^\mu)$ for $\lambda, \mu \in \Lambda$, we see that

$$|x^\lambda(\sigma + T^\lambda) - x^\mu(\sigma + T^\mu)| = |x^\mu(\sigma + T^\lambda) - x^\mu(\sigma + T^\mu)| \leqq 2r|T^\lambda - T^\mu|$$

by the condition (II). Hence $\lim_{T^\lambda \to J} x^\lambda(\sigma + T^\lambda) = p$ exists, and $p \in \Omega_{\sigma+J}$ by Lemma 2. Define $x^*(t)$ by

$$x^*(t) = \begin{cases} x^\lambda(t)\,, & t \leqq \sigma + T^\lambda,\ \lambda \in \Lambda\,, \\ p\,, & t = \sigma + J\,. \end{cases}$$

Then (J, x^*) is in $Q_\varepsilon(\sigma, \phi)$ and is the supremum of E. Therefore there is a maximal element (T, x) in $Q_\varepsilon(\sigma, \phi)$ by Zorn's lemma.

Next, we prove that $T = \alpha$ for the maximal element (T, x) obtained above. Suppose that $T < \alpha$. By Lemma 3, there is a continuous mapping $g_1: [\sigma, \sigma + T] \to R^n$ such that $d(x(t), \Omega_t) = |x(t) - g_1(t)|$ and $g_1(t) \in \Omega_t$ for all $t \in [\sigma, \sigma + T]$. Let $\xi: (-\infty, \sigma + T] \to R^n$ be a function such that $\xi_\sigma = \phi$ and $\xi(t) = g_1(t)$ on $[\sigma, \sigma + T]$. Then $\xi_t \in B$ for all $t \in [\sigma, \sigma + T]$ by the hypothesis (B1). Recall that $|x(t) - \xi(t)| = |x(t) - g_1(t)| < \eta(\varepsilon, W)L^{-1}$ on $[\sigma, \sigma + T]$ by (I). Since $x(t)$ satisfies (I) and (II), it follows from the hypothesis (B2) and (6), (8) that

$$\begin{aligned}
|\xi_{\sigma+T} - \phi| &\leqq |\xi_{\sigma+T} - \tilde{\phi}_{\sigma+T}| + |\tilde{\phi}_{\sigma+T} - \phi| \\
&\leqq L \sup_{-T \leqq \theta \leqq 0} |\xi(\sigma + T + \theta) - \phi(0)| + |\tilde{\phi}_{\sigma+T} - \phi| \\
&\leqq L \sup_{-T \leqq \theta \leqq 0} \{|\xi(\sigma + T + \theta) - x(\sigma + T + \theta)| + |x(\sigma + T + \theta) - \phi(0)|\} \\
&\qquad + |\tilde{\phi}_{\sigma+T} - \phi| \\
&\leqq L\{\eta(\varepsilon,\ W)L^{-1} + 2rT\} + |\tilde{\phi}_{\sigma+T} - \phi| \leqq 3Lr\alpha + |\tilde{\phi}_{\sigma+T} - \phi| < \delta\,.
\end{aligned}$$

Therefore we have

$$|f(\sigma + T,\ \xi_{\sigma+T})| \leqq r\,. \tag{13}$$

Since $\xi_{\sigma+T}(t-\sigma-T)\in\Omega_t$ for all $t\leqq\sigma+T$ and $\xi(\sigma+T)=x(\sigma+T)$, we have

$$\lim_{h\to 0+} d(x(\sigma+T)+hf(\sigma+T,\xi_{\sigma+T}),\Omega_{\sigma+T+h})/h=0 \tag{14}$$

by the condition (ii). Again by Lemma 3, there is a continuous function $g_2(t)\colon[\sigma+T,\infty)\to R^n$ such that

$$\begin{aligned} &d(x(\sigma+T)+hf(\sigma+T,\xi_{\sigma+T}),\Omega_{\sigma+T+h})\\ &\qquad=|x(\sigma+T)+hf(\sigma+T,\xi_{\sigma+T})-g_2(\sigma+T+h)|\end{aligned}$$

and $g_2(\sigma+T+h)\in\Omega_{\sigma+T+h}$ for all $h\geqq 0$. Then by (14), there is a δ_1 with $0<\delta_1\leqq\varepsilon$ such that

$$|f(\sigma+T,\xi_{\sigma+T})-\{g_2(\sigma+T+h)-x(\sigma+T)\}/h|\leqq\varepsilon \tag{15}$$

for all $h\in(0,\delta_1]$.

Let S be a constant such that $0<S<\alpha-T$ and $S\leqq\delta_1$, and define a function y by

$$y(t)=\begin{cases}x(t)\,, & t\leqq\sigma+T\,,\\ x(\sigma+T)+\{(g_2(\sigma+T+S)-x(\sigma+T))/S\}(t-\sigma-T)\,, & \sigma+T\leqq t\leqq\sigma+T+S\,.\end{cases}$$

We show that $(T+S,y)$ belongs to $Q_\varepsilon(\sigma,\phi)$ if S is sufficiently small. Since $y(t)=x(t)$ for $t\leqq\sigma+T$, it is sufficient to consider the case $t\geqq\sigma+T$. By (13) and (15) and as in the proof of Lemma 4, we can find a δ_2 with $0<\delta_2\leqq\delta_1$ such that $y(t)$ satisfies (I), (II) and (IV) for $S\leqq\delta_2$.

To show (III), define another function z by $z(t)=\xi(t)$ on $(-\infty,\sigma+T]$ and $z(t)=y(t)$ on $[\sigma+T,\sigma+T+S]$, where $S=\delta_2$. Then $y_\sigma=z_\sigma=\phi$ and $\sup\{|z(t)-y(t)|;t\in[\sigma,\sigma+T+S]\}=\sup\{|x(t)-\xi(t)|;t\in[\sigma,\sigma+T]\}<\eta(\varepsilon,W)L^{-1}$, and hence

$$|y_t-z_t|\leqq L\sup_{-(t-\sigma)\leqq\theta\leqq 0}|y(t+\theta)-z(t+\theta)|<L\eta(\varepsilon,W)L^{-1}=\eta(\varepsilon,W)$$

for all $t\in[\sigma,\sigma+T+S]$ by the hypothesis (B2). (t,y_t) belongs to the compact set W for all $t\in[\sigma,\sigma+T+S]$ since $y(t)$ satisfies (II) on $[\sigma,\sigma+T+S]$. Thus we have

$$|f(t,y_t)-f(t,z_t)|<\varepsilon\quad\text{on}\quad[\sigma,\sigma+T+S] \tag{16}$$

by (7). Since $z(t)$ is $2r$-Lipschitzian in $t\in[\sigma+T,\sigma+T+S]$, we see that $|z_t-\xi_{\sigma+T}|=|z_t-z_{\sigma+T}|$ is small if $t-(\sigma+T)>0$ is sufficiently small by the hypotheses (B1) and (B2). Therefore the continuity of f implies that there is a δ_3 with $0<\delta_3\leqq\delta_2$ such that

$$|f(t,z_t)-f(\sigma+T,\xi_{\sigma+T})|<\varepsilon \tag{17}$$

for all $t \in [\sigma + T, \sigma + T + S]$ if $S \leqq \delta_3$. Let $S = \delta_3$. Then it follows from (15), (16) and (17) that

$$\begin{aligned} |\dot{y}(t) - f(t, y_t)| &\leqq |\{g_2(\sigma + T + S) - x(\sigma + T)\}/S - f(\sigma + T, \xi_{\sigma+T})| \\ &\quad + |f(\sigma + T, \xi_{\sigma+T}) - f(t, z_t)| + |f(t, z_t) - f(t, y_t)| \\ &\leqq 3\varepsilon \end{aligned}$$

for all $t \in [\sigma + T, \sigma + T + S]$.

Consequently, we obtain an element $(T + S, y)$ in $Q_\varepsilon(\sigma, \phi)$ such that $(T, x) \leqq (T + S, y)$ and $(T, x) \neq (T + S, y)$, which contradicts the maximality of (T, x). Thus T should be equal to α, and we are done.

4. The proof of the theorem. It is easily proved that (i) implies (ii), and so we prove the converse.

Let $\{\varepsilon_k\}$ be a sequence such that $\varepsilon_k > 0$ and $\varepsilon_k \to 0$ as $k \to \infty$. Let $(\sigma, \phi) \in R \times B$ be such that $\phi(t - \sigma) \in \Omega_t$ for all $t \leqq \sigma$. By Lemma 5, there exists an $\alpha > 0$ such that the set $Q_{\varepsilon_k}(\sigma, \phi)$ contains an element (α, x^k) for each k. Since the sequence of the functions $\{x^k(t)\}$ is uniformly bounded and equicontinuous on $[\sigma, \sigma + \alpha]$, we may assume that the sequence converges uniformly to a continuous function $x(t)$ on $[\sigma, \sigma + \alpha]$ as $k \to \infty$. Let $x(t) = \phi(t - \sigma)$ for $t \leqq \sigma$. Then x_t belongs to B for all $t \in [\sigma, \sigma + \alpha]$ by the hypothesis (B1). Also, $|x_t^k - x_t| \to 0$ as $k \to \infty$ for all $t \in [\sigma, \sigma + \alpha]$ by the hypothesis (B2). Since (t, x_t^k) belongs to the compact set W for all $t \in [\sigma, \sigma + \alpha]$ by (II), we have $|f(t, x_t^k)| \leqq C$ for all $t \in [\sigma, \sigma + \alpha]$ and all k, where C is a constant. Hence, applying Lebesgue's dominant convergence theorem, we see by (II) and (III) that

$$\begin{aligned} x(t) &= \lim_{k\to\infty} x^k(t) = \lim_{k\to\infty} \left\{\phi(0) + \int_\sigma^t \dot{x}^k(s)ds\right\} \\ &= \phi(0) + \lim_{k\to\infty}\left\{\int_\sigma^t f(s, x_s^k)ds + \int_\sigma^t [\dot{x}^k(s) - f(s, x_s^k)]ds\right\} \\ &= \phi(0) + \int_\sigma^t f(s, x_s)ds \end{aligned}$$

for all $t \in [\sigma, \sigma + \alpha]$. Thus $x(t)$ is a solution of (5) through (σ, ϕ).

By (IV), for each $t \in [\sigma, \sigma + \alpha]$ and k, there is a point $s^k \in [\sigma, \sigma + \alpha]$ such that $|t - s^k| \leqq \varepsilon_k$ and $(s^k, x^k(s^k)) \in \Omega$. Then, by (II), we have $|x(t) - x^k(s^k)| \leqq |x(t) - x^k(t)| + |x^k(t) - x^k(s^k)| \leqq |x(t) - x^k(t)| + 2r\varepsilon_k$, which implies $\lim_{k\to\infty} (s^k, x^k(s^k)) = (t, x(t))$, and hence $(t, x(t)) \in \Omega$ by Lemma 2. Consequently, $x(t)$ is the solution of (5) through (σ, ϕ) such that $(t, x(t)) \in \Omega$ on $(-\infty, \sigma + \alpha]$.

Now let $Q(\sigma, \phi, \Omega)$ be the set defined by

$$Q(\sigma, \phi, \Omega) = \{(T, y) \in Q(\sigma, \phi); (t, y(t)) \in \Omega \text{ on } (-\infty, \sigma + T)\}\,.$$

Then $Q(\sigma, \phi, \Omega)$ is nonempty because $(\alpha, x) \in Q(\sigma, \phi, \Omega)$. Introducing the same partial order in $Q(\sigma, \phi, \Omega)$ as in $Q(\sigma, \phi)$, we obtain a maximal element (T, y) in $Q(\sigma, \phi, \Omega)$ by Zorn's lemma. We show that the (T, y) is also a maximal element in $Q(\sigma, \phi)$. Otherwise, y can be extended up to $t = \sigma + T$, and then $(t, y(t)) \in \Omega$ for all $t \leqq \sigma + T$ by Lemma 2. Clearly, $y_{\sigma+T}$ belongs to B by the hypothesis (B1). Therefore, by applying the condition (ii) to $(\sigma + T, y_{\sigma+T})$ and by the same argument as above we obtain an element (α', z) in $Q(\sigma + T, y_{\sigma+T}, \Omega)$. Then $(T + \alpha', z)$ is in $Q(\sigma, \phi, \Omega)$, $(T + \alpha', z) \geqq (T, y)$ and $(T + \alpha', z) \neq (T, y)$. This contradicts the maximality of (T, y) in $Q(\sigma, \phi, \Omega)$. Thus (T, y) is in $Q(\sigma, \phi, \Omega)$ and is the maximal element in $Q(\sigma, \phi)$, that is, y is the solution of (5) through (σ, ϕ) defined on its right maximal interval of existence $(-\infty, \sigma + T)$ and satisfying $(t, y(t)) \in \Omega$ there.

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MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, 980 JAPAN