POSITIVITY AND REACHABILITY OF FRACTIONAL ELECTRICAL CIRCUITS

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Abstract: Conditions for the positivity of fractional linear electrical circuits composed of resistors, coils, condensators and voltage (current) sources are established. It is shown that: 1) the fractional electrical circuit composed of resistors, coils and voltage source is positive for any values of their resistances, inductances and source voltages if and only if the number of coils is less or equal to the number of its linearly independent meshes, 2) the fractional electrical circuit is not positive for any values of its resistances, capacitances and source voltages if each its branch contains resistor, capacitor and voltage source, It is also shown that the fractional positive electrical circuits of R, C, e type are reachable if and only if the conductances between their nodes are zero and the fractional positive electrical circuits of R, L, e type are reachable if and only if the resistances belonging to two meshes are zero.

1. INTRODUCTION

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive systems theory is given in the monographs (Farina and Rinaldi 2000; Kaczorek 2002). Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc..

The notion of controllability and observability and the decomposition of linear systems have been introduced by Kalman (1960, 1963). These notions are the basic concepts of the modern control theory (Antsaklis, Michel 2006; Kaczorek 1999, 2002; Kailath 1980; Rosenbrock 1970; Wolovich 1970). They have been also extended to positive linear systems (Farina and Rinaldi 2000; Kaczorek 2002). The decomposition of the pair (*A,B*) and (*A,C*) of the positive discrete-time linear system has been addressed in Kaczorek (2010b).

The reachability of linear systems is closely related to the controllability of the systems. Specially for positive linear systems the conditions for the controllability are much stronger than for the reachability (Kaczorek 2002). Tests for the reachability and controllability of standard and positive linear systems are given in Kaczorek (2008b, 2002; Klamka 1991). The stability, robust stability and stabilization of positive linear systems have been investigated in (Busłowicz 2008a, 2008b, 2008c, 2009, 2010; Kaczorek 2002, 2011c). Analysis of fractional electrical circuits has been addressed in Kaczorek (2010a, 2011a, 2011b).

In this paper the necessary and sufficient conditions for the positivity and reachability of fractional linear electrical circuits composed of resistors, coils, condensators (supercondensators) and voltage (current) sources will be established.

The paper is organized as follows. In section 2 the state equations of the fractional linear electrical circuits and their solutions are presented. The positive fractional linear electrical circuits composed of resistors, condensators, coils and voltage sources are analyzed in section 3. The reachability of the fractional positive electrical circuits is investigated in section 4. Concluding remarks are given in section 5.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n \times m}$ – the set of $n \times m$ real matrices, $\Re_+^{n \times m}$ – the set of $n \times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n \times 1}$, M_n – the set of $n \times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n \times n$ identity matrix.

2. FRACTIONAL STATE EQUATIONS AND THEIR SOLUTIONS

In this paper the following Caputo definition of the derivative-integral of fractional order will be used (Kaczorek 2008a, 2011c)

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \tag{2.1}$$

$n-1 < \alpha < n$, $n \in N = \{1,2,...\}$ where

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \ \text{Re}(x) > 0 \tag{2.2}$$

is the gamma function and

$$f^{(n)}(\tau) = \frac{d^n f(\tau)}{d\tau^n} \tag{2.3}$$

is the classical *n* order derivative.

Let the current $i_C(t)$ in a supercondensator (shortly condensator) with the capacity C be the α order derivative of its charge $q(t)$ (Kaczorek 2010a, 2011c)

$$i_C(t) = \frac{d^\alpha q(t)}{dt^\alpha}, \; 0 < \alpha < 1 \tag{2.4}$$

Using $q(t) = C u_C(t)$ we obtain

$$i_C(t) = C \frac{d^\alpha u_C(t)}{dt^\alpha} \tag{2.5}$$

where $u_C(t)$ is the voltage on the condensator.

Similarly, let the voltage $u_L(t)$ on coil (inductor) with the inductance L be the β order derivative of its magnetic flux $\Psi(t)$ (Kaczorek 2010a, 2011c)

$$u_L(t) = \frac{d^\beta \Psi(t)}{dt^\beta}, \; 0 < \beta < 1 \tag{2.6}$$

Taking into account that $\Psi(t) = L i_L(t)$ we obtain

$$u_L(t) = L \frac{d^\beta i_L(t)}{dt^\beta} \tag{2.7}$$

where $i_L(t)$ is the current in the coil.

Consider an electrical circuit composed of resistors, n capacitors and m voltage sources. Using the equation (2.5) and the Kirchhoff's laws we may describe the transient states in the electrical circuit by the fractional differential equation

$$\frac{d^\alpha x(t)}{dt^\alpha} = A x(t) + B u(t), \; 0 < \alpha < 1 \tag{2.8}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$. The components of the state vector $x(t)$ and input vector $u(t)$ are the voltages on the condensators and source voltages respectively.

Consider an electrical circuit composed of resistors, n coils and m sources. Similarly, using the equation (2.6) and the Kirchhoff's laws we may describe the transient states in the electrical circuit by the fractional differential equation

$$\frac{d^\beta x(t)}{dt^\beta} = A x(t) + B u(t), \; 0 < \beta < 1 \tag{2.9}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$. In this case the components of the state vector $x(t)$ are the currents in the coils.

Theorem 2.1. Solution of the equation (2.8) satisfying the initial condition $x(0) = x_0$ has the form

$$x(t) = \Phi_0(t) x_0 + \int_0^t \Phi(t-\tau) B u(\tau) d\tau \tag{2.10}$$

where

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \;\; \Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}, \;\; 0 < \alpha < 1 \tag{2.11}$$

Proof is given in Kaczorek (2008a, 2011c) .

Now let us consider electrical circuit composed of resistors, capacitors, coils and voltage (current) source. As the state variables (the components of the state vector $x(t)$ we choose the voltages on the capacitors and the currents in the coils. Using the equations (2.5), (2.7) and Kirchhoff's laws we may write for the fractional linear circuits in the transient states the state equation

$$\begin{bmatrix} \dfrac{d^\alpha x_C}{dt^\alpha} \\ \dfrac{d^\beta x_L}{dt^\beta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_C \\ x_L \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u , \; 0 < \alpha, \beta < 1 \tag{2.12a}$$

where the components $x_C \in \Re^{n_1}$ are voltages on the condensators, the components $x_L \in \Re^{n_2}$ are currents in the coils and the components of $u \in \Re^m$ are the source voltages

$$A_{ij} \in \Re^{n_i \times n_j}, \; B_i \in \Re^{n_i \times m}, \; i,j = 1,2 . \tag{2.12b}$$

Some examples of electrical circuits described by the equation (2.12) are given in (Kaczorek 2010c, 2011c).

Theorem 2.2. The solution of the equation (2.12) for $0 < \alpha < 1$, $0 < \beta < 1$ with initial conditions

$$x_C(0) = x_{10} \text{ and } x_L(0) = x_{20} \tag{2.13}$$

has the form

$$x(t) = \Phi_0(t) x_0 + \int_0^t [\Phi_1(t-\tau) B_{10} + \Phi_2(t-\tau) B_{01}] u(\tau) d\tau \tag{2.14a}$$

where

$$x(t) = \begin{bmatrix} x_C(t) \\ x_L(t) \end{bmatrix}, \quad x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \quad B_{10} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad B_{01} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$

$$T_{kl} = \begin{cases} I_n & \text{for } k = l = 0 \\ \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} & \text{for } k = 1, l = 0 \\ \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} & \text{for } k = 0, l = 1 \\ T_{10} T_{k-1,l} + T_{01} T_{k,l-1} & \text{for } k+l > 0 \end{cases} \tag{2.14b}$$

$$\Phi_0(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} T_{kl} \frac{t^{k\alpha+l\beta}}{\Gamma(k\alpha+l\beta+1)}$$

$$\Phi_1(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} T_{kl} \frac{t^{(k+1)\alpha+l\beta-1}}{\Gamma[(k+1)\alpha+l\beta]}$$

$$\Phi_2(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} T_{kl} \frac{t^{k\alpha+(l+1)\beta-1}}{\Gamma[k\alpha+(l+1)\beta]}$$

Proof is given in Kaczorek (2010c, 2011c).

The extension of Theorem 2.2 to systems consisting of n subsystems with different fractional order is given in Kaczorek (2011b).

3. POSITIVE FRACTIONAL ELECTRICAL CIRCUITS

Definition 3.1. The fractional electrical circuit (2.8) (or (2.9), (2.12)) is called the (internally) positive fractional system if the state vector $x(t) \in \Re_+^n$, $t \geq 0$ for any initial conditions $x_0 \in \Re_+^n$ and all $u(t) \in \Re_+^m$, $t \geq 0$.

Definition 3.2. A square real matrix $A = [a_{ij}]$ is called the Metzler matrix if its off-diagonal entries are nonnegative, i.e. $a_{ij} \geq 0$ for $i \neq j$ (Kaczorek, 2002, 2011c).

Theorem 3.1. The fractional electrical circuit (2.8) is (internally) positive if and only if

$$A \in M_n, \quad B \in \Re_+^{n \times m} \tag{3.1}$$

where M_n is the set of $n \times n$ Metzler matrices.

Proof is given in Kaczorek (2002, 2011c).

From Theorem 3.1 applied to the fractional circuit (2.12) it follows that the electrical circuit is positive if and only if

$$A_{kk} \in M_{n_k}, \quad k = 1,2; \quad A_{12} \in \Re_+^{n_1 \times n_2}, \quad A_{21} \in \Re_+^{n_2 \times n_1},$$

$$B_1 \in \Re_+^{n_1 \times m}, \quad B_2 \in \Re_+^{n_2 \times m} \tag{3.2}$$

3.1. Fractional R, C, e type electrical circuits

Theorem 3.2. The fractional electrical circuit is not positive if each its branch contains resistors, condensator and voltage source.

The proof is similar to the proof of Theorem 3.1 in Kaczorek (2011a).

Consider the fractional electrical circuit shown on Figure 3.1 with given conductances $G_k, k = 0, 1, \ldots, n$; capacitances C_j, $j = 1, \ldots, n$ and source voltages e.
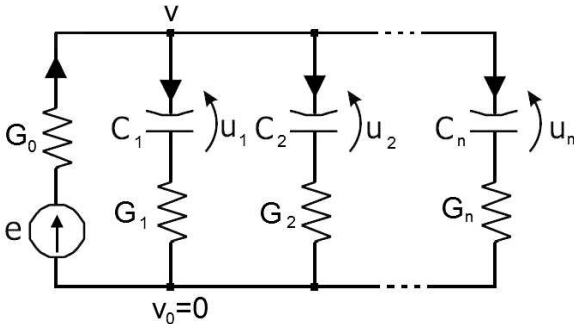

Fig. 3.1. Fractional electrical circuit

Using (2.5) and the Kirchhoff's laws we may write the equations

$$C_k \frac{d^\alpha u_k}{dt^\alpha} = G_k(v - u_k), \quad k = 1,\ldots,n \tag{3.3}$$

and

$$G_0(e - v) = \sum_{j=1}^{n} G_j(v - u_j). \tag{3.4}$$

From (3.4) we have

$$v = \frac{1}{G}\left(G_0 e + \sum_{j=1}^{n} G_j u_j \right), \quad G = \sum_{i=0}^{n} G_i. \tag{3.5}$$

Substitution of (3.5) into (3.3) yields

$$\frac{d^\alpha}{dt^\alpha}\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = A\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + Be \tag{3.6}$$

where

$$A = \begin{bmatrix} -\dfrac{G_1 G - G_1^2}{C_1 G} & \dfrac{G_1 G_2}{C_1 G} & \cdots & \dfrac{G_1 G_n}{C_1 G} \\ \dfrac{G_2 G_1}{C_2 G} & -\dfrac{G_2 G - G_2^2}{C_2 G} & \cdots & \dfrac{G_2 G_n}{C_2 G} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{G_n G_1}{C_n G} & \dfrac{G_n G_2}{C_n G} & \cdots & -\dfrac{G_n G - G_n^2}{C_n G} \end{bmatrix},$$

$$B = \begin{bmatrix} \dfrac{G_0 G_1}{C_1 G} \\ \vdots \\ \dfrac{G_0 G_n}{C_n G} \end{bmatrix}. \tag{3.7}$$

From (3.7) it follows that $A \in M_n$ and $B \in \Re_+^n$. Therefore, the following theorem has been proved.

Theorem 3.3. The fractional electrical circuit shown on Fig. 3.1 is positive for any values of the conductances $G_k, k = 0, 1, \ldots, n$; capacitances C_j, $j = 1, \ldots, n$ and source voltage e.

In general case let us consider the fractional electrical circuit composed of q conductances $G_k, k = 1, \ldots, q$; r capacitances C_i, $i = 1, \ldots, r$ and m source voltages e_j, $j = 1, \ldots, m$. Let n be the number of linearly independent nodes of the electrical circuit and $n > r$.

Using the Kirchhoff's laws we may write the equation

$$\frac{d^\alpha}{dt^\alpha}\begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} = A_r\begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} + A_n\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + B_m\begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix} \tag{3.8}$$

where u_i is the voltage on the i-th ($i = 1, \ldots, r$) capacitor, v_j is the voltage of the j-th node ($j = 1,\ldots,n$), $A_r \in \Re^{r \times r}$ is the diagonal Metzler matrix, $A_n \in \Re^{r \times n}$ and $B_m \in \Re^{r \times m}$.

Using the node method we obtain

$$G\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = -F\begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} - H\begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix} \tag{3.9}$$

where $G \in \Re^{n \times n}$ is a Metzler matrix, $F \in \Re^{n \times r}$ and $H \in \Re^{n \times m}$.

Taking into account that $-G^{-1} \in \Re_+^{n \times n}$ from (3.9) we obtain

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = -G^{-1}F\begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} - G^{-1}H\begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix}. \tag{3.10}$$

Substitution of (3.10) into (3.8) yields

$$\frac{d^\alpha}{dt^\alpha}\begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} = A\begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} + B\begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix} \tag{3.11}$$

where

$$A = A_r - A_n G^{-1} F , \quad B = B_m - A_n G^{-1} H . \tag{3.12}$$

The electrical circuit described by the equation (3.11) is positive if and only if the matrix A is a Metzler matrix and the matrix B has nonnegative entries. Therefore, the following theorem has been proved.

Theorem 3.4. The linear electrical circuit composed of q resistors, r capacitors and m source voltages is positive if and only if $r < n$ and

$$A_r - A_n G^{-1} F \in M_r , \quad B_m - A_n G^{-1} H \in \Re_+^{r \times m} . \tag{3.13}$$

3.2. Fractional R, L, e type electrical circuits

Consider the electrical circuit shown on Figure 3.2 with given resistances R_1, R_2, R_3 inductances L_1, L_2, L_3 and source voltages e_1, e_2.
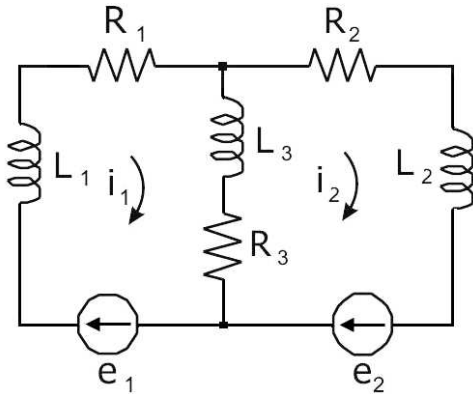

Fig. 3.2. Fractional electrical circuit

Using (2.7) and the mesh method for the electrical circuit we obtain the following equations

$$\begin{bmatrix} L_{11} & -L_{12} \\ -L_{21} & L_{22} \end{bmatrix} \frac{d^\beta}{dt^\beta}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -R_{11} & R_{12} \\ R_{21} & -R_{22} \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{3.14a}$$

where

$$\begin{aligned} R_{11} &= R_1 + R_3, \quad R_{12} = R_{21} = R_3, \quad R_{22} = R_2 + R_3, \\ L_{11} &= L_1 + L_3, \quad L_{12} = L_{21} = L_3, \quad L_{22} = L_2 + L_3. \end{aligned} \tag{3.14b}$$

Note that the inverse matrix

$$L^{-1} = \begin{bmatrix} L_{11} & -L_{12} \\ -L_{21} & L_{22} \end{bmatrix}^{-1} = \frac{1}{L_1(L_2 + L_3) + L_2 L_3}\begin{bmatrix} L_{22} & L_{12} \\ L_{21} & L_{11} \end{bmatrix} \tag{3.15}$$

has positive entries. From (3.14) we have

$$\frac{d^\beta}{dt^\beta}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{3.16}$$

where

$$A = L^{-1}\begin{bmatrix} -R_{11} & R_{12} \\ R_{21} & -R_{22} \end{bmatrix} = \frac{1}{L_1(L_2 + L_3) + L_2 L_3}$$

$$\cdot\begin{bmatrix} -L_2(R_1 + R_3) - L_3 R_1 & L_2 R_3 - L_3 R_2 \\ L_1 R_3 - L_3 R_1 & -L_1(R_2 + R_3) - L_3 R_2 \end{bmatrix}, \tag{3.17}$$

$$B = L^{-1} \in \Re_+^{2 \times 2} .$$

From (3.17) it follows that $A \in M_2$ if and only if

$$L_2 R_3 \geq L_3 R_2 \text{ and } L_1 R_3 \geq L_3 R_1 . \tag{3.18}$$

Therefore, the fractional electrical circuit is positive if and only if $A \in M_2$ i.e. the condition (3.18) is met.

In general case let us consider the fractional n-mesh electrical circuit with given resistances $R_k, k = 1, \dots, q$, inductances $L_1, \dots, L_r$ for $r \geq n$ and $m \leq n$ mesh source voltages $e_{jj}, j = 1, \dots, m$. Denote by $i_1, \dots, i_n$ the mesh currents. In a similar way as for the electrical circuit shown on Fig 3.2 using the mesh method we obtain the equation

$$L\frac{d^\beta}{dt^\beta}\begin{bmatrix} i_1 \\ \vdots \\ i_n \end{bmatrix} = A'\begin{bmatrix} i_1 \\ \vdots \\ i_n \end{bmatrix} + \begin{bmatrix} e_{11} \\ \vdots \\ e_{mm} \end{bmatrix} \tag{3.19a}$$

where

$$L = \begin{bmatrix} L_{11} & -L_{12} & \dots & -L_{1,n} \\ -L_{21} & L_{22} & \dots & -L_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ -L_{n,1} & -L_{n,2} & \dots & L_{nn} \end{bmatrix},$$

$$\tag{3.19b}$$

$$A' = \begin{bmatrix} -R_{11} & R_{12} & \dots & R_{1,n} \\ R_{21} & -R_{22} & \dots & R_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n,1} & R_{n,2} & \dots & -R_{nn} \end{bmatrix} .$$

Note that $-L \in M_n$, $A' \in M_n$ and $L^{-1} \in \Re_+^{n \times n}$. Premultiplying (3.19a) by L^{-1} we obtain

$$\frac{d^\beta}{dt^\beta}\begin{bmatrix} i_1 \\ \vdots \\ i_n \end{bmatrix} = A\begin{bmatrix} i_1 \\ \vdots \\ i_n \end{bmatrix} + B\begin{bmatrix} e_{11} \\ \vdots \\ e_{mm} \end{bmatrix} \tag{3.20a}$$

where

$$A = L^{-1} A', \quad B = L^{-1} \in \Re_+^{n \times n} . \tag{3.20b}$$

The fractional electrical circuit is positive if and only if the matrix $L^{-1}A'$ is a Metzler matrix, i.e.

$$L^{-1} A' \in M_n . \tag{3.21}$$

Therefore, the following theorem has been proved.

Theorem 3.4. The fractional linear electrical circuit composed of resistors, coils and voltage sources is positive for $r > n$ if its resistances and inductances satisfy the condition (3.21).

Remark 3.1. In the case $r = n$ if it is possible to choose the n linearly independent meshes so that to each mesh belongs only one coil. Then the matrix $L = diag[L_1, \dots, L_n]$ and the condition (3.21) is met for any values of the resistances and inductances of the electrical circuit.

Remark 3.2. Note that it is impossible to choose the n linearly independent meshes so that to each mesh belongs only one coil if all branches belonging to the same node contain the coils. In this case we can eliminate one of the branch currents using the fact that the sum of the currents in the coils is equal to zero.

From Theorem 3.4 and Remark 3.1 we have the following important theorem.

Theorem 3.5. The fractional linear electrical circuit composed of resistors, coils and voltage sources is positive for almost all values of the resistances, inductances and source voltages if and only if the number of coils is less or equal to the number of its linearly independent meshes and the directions of the mesh currents are consistent with the directions of the mesh source voltages.

3.3. Fractional R, L, C type electrical circuits

Consider the fractional electrical circuit shown on Figure 3.3 with given resistance R, inductance L, capacitance C and source voltage e.
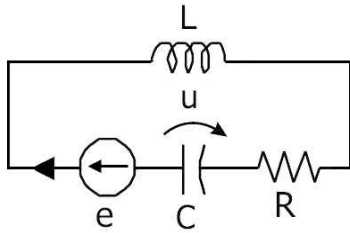

Fig. 3.3. Fractional electrical circuit

Using the Kirchhoff's laws we can write the equations

$$i = C \frac{d^\alpha u}{dt^\alpha}$$
$$e = Ri + L \frac{d^\beta i}{dt^\beta} + u \qquad (3.22)$$

which can be written in the form

$$\begin{bmatrix} \dfrac{d^\alpha u}{dt^\alpha} \\ \dfrac{d^\beta i}{dt^\beta} \end{bmatrix} = A \begin{bmatrix} u \\ i \end{bmatrix} + Be \qquad (3.23a)$$

where

$$A = \begin{bmatrix} 0 & \dfrac{1}{C} \\ -\dfrac{1}{L} & -\dfrac{R}{L} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \dfrac{1}{L} \end{bmatrix}. \qquad (3.23b)$$

The matrix A has negative off-diagonal entry $(-1/L)$ and it is not a Metzler matrix for any values of R, L, C. Therefore, the fractional electrical circuit is not positive one for any values of the resistances R, inductance L, capacitance C.

In general case we have the following theorem.

Theorem 3.6. The fractional electrical circuits of R, L, C type is not positive for almost all values of its resistances, inductances, capacitances and source voltages if at least one its branch contains inductance and capacitance.

Proof. It is well-known that the linear independent meshes of the electrical circuits can be chosen so that the branch containing the inductance L and capacitance C belongs to the first one. The equation for the first mesh contains the following term

$$e_{11} = L \frac{d^\beta i_1}{dt^\beta} + u_1 + \dots \qquad (3.24)$$

where e_{11} and i_1 are the source voltage and current of the first mesh and u_1 is the voltage on the capacitance C. From (3.24) and $i_1 = C((d^\alpha u_1)/dt^\alpha)$ it follows that the matrix A of the electrical circuit has at least one negative off-diagonal entry. Therefore the matrix A is not a Metzler matrix and the electrical circuit is not positive one.

Consider the electrical circuit shown on Fig. 3.4 with given resistances $R_k, k = 1, \dots, n$, inductances $L_2, L_4 \dots, L_{n_2}$, capacitances $C_1, C_3 \dots, C_{n_1}$ and source voltages $e_1, e_2 \dots, e_n$.
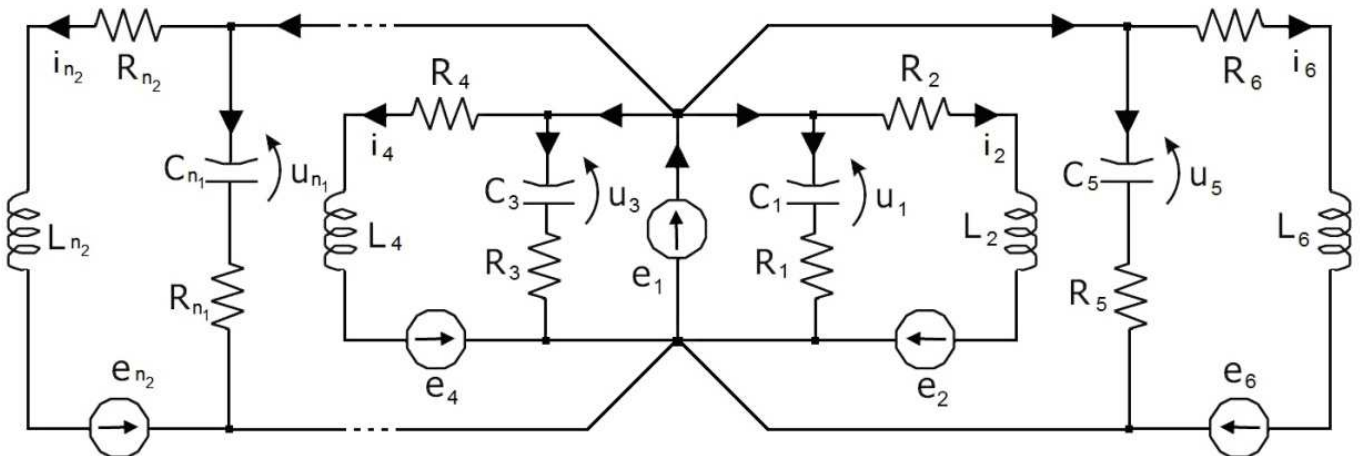

Fig. 3.4. Fractional electrical circuit

Using the Kirchhoff's laws we can write the equations

$$e_1 = R_k C_k \frac{d^\alpha u_k}{dt^\alpha} + u_k \quad \text{for} \quad k = 1,3,...,n_1 \qquad (3.25a)$$

$$e_1 + e_j = L_j \frac{d^\beta i_j}{dt^\beta} + R_j i_j \quad \text{for} \quad j = 2,4,...,n_2 \qquad (3.25b)$$

which can be written in the form

$$\begin{bmatrix} \dfrac{d^\alpha u}{dt^\alpha} \\ \dfrac{d^\beta i}{dt^\beta} \end{bmatrix} = A \begin{bmatrix} u \\ i \end{bmatrix} + Be \qquad (3.26a)$$

where

$$u = \begin{bmatrix} u_1 \\ u_3 \\ \vdots \\ u_{n_1} \end{bmatrix}, \quad i = \begin{bmatrix} i_2 \\ i_4 \\ \vdots \\ i_{n_2} \end{bmatrix}, \quad u = \begin{bmatrix} e_1 \\ e_2 \\ e_4 \\ \vdots \\ e_n \end{bmatrix} \qquad (3.26b)$$

and

$$A = \text{diag}[-\frac{1}{R_1 C_1}, -\frac{1}{R_3 C_3}, ..., -\frac{1}{R_{n_1} C_{n_1}}, -\frac{R_2}{L_2}, -\frac{R_4}{L_4}, ..., -\frac{R_{n_2}}{L_{n_2}},] \in M_n,$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \Re_+^{n \times \left(\frac{n_2}{2}+1\right)}, \quad B_1 = \begin{bmatrix} \dfrac{1}{R_1 C_1} & 0 & 0 & ... & 0 \\ \dfrac{1}{R_3 C_3} & 0 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & ... & \vdots \\ \dfrac{1}{R_{n_1} C_{n_1}} & 0 & 0 & ... & 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} \dfrac{1}{L_2} & \dfrac{1}{L_2} & 0 & ... & 0 \\ \dfrac{1}{L_4} & 0 & \dfrac{1}{L_4} & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{1}{L_{n_2}} & 0 & 0 & ... & \dfrac{1}{L_{n_2}} \end{bmatrix}. \qquad (3.26c)$$

The electrical circuit described by the equation (3.26) is positive for all value of the resistances $R_k, k = 1, ..., n$, inductances L_k, $k = 2, 4, ..., n_2$, capacitances C_k, $k = 1, 3, ..., n_1$. Therefore, the following theorem has been proved.

Theorem 3.7. The fractional linear electrical circuit of the structure shown on Fig. 3.4 is positive for any values of its resistances, inductances and capacitances.

4. REACHABILITY OF FRACTIONAL POSITIVE LINEAR ELECTRICAL CIRCUITS

Consider the fractional positive linear electrical circuit described by the equations (2.8), (2.9) and (2.12).

Definition 4.1. The fractional positive electrical circuit (2.8) is called reachable in time t_f if for any given final state $x_f \in \Re_+^n$ there exists an input $u(t) \in \Re_+^m$, for $t \in [0, t_f]$ that steers the state of the circuit from zero initial state $x(0) = 0$ to the final state x_f, i.e. $x(t_f) = x_f$. If every state $x_f \in \Re_+^n$ is reachable in time t_f, then the circuit is called reachable in time t_f. The fractional positive electrical circuit is called reachable if for every $x_f \in \Re_+^n$ there exist time t_f and input $u(t) \in \Re_+^m$, for $t \in [0, t_f]$ which steers the state of the circuit from $x(0) = 0$ to x_f.

A real square matrix is called monomial if each its row and each its column contains only one positive entry and the remaining entries are zero.

Theorem 4.1. The fractional positive electrical circuit (2.8) is reachable in time t_f if the matrix

$$R(t_f) = \int_0^{t_f} \Phi(\tau) BB^T \Phi^T(\tau) d\tau, \; t_f > 0 \qquad (4.1)$$

is monomial. The input that steers the state of the electrical circuit in time t_f from $x(0) = 0$ to the state x_f is given by the formula

$$u(t) = B^T \Phi^T(t_f - t) R^{-1}(t_f) x_f \quad \text{for} \; t \in [0, t_f]. \qquad (4.2)$$

The proof is given in Kaczorek (2010a).

Theorem 4.2. If the matrix $A = diag[a_1, a_2, ... a_n] \in \Re^{n \times n}$ and $B \in \Re_+^{n \times m}$ for $m = n$ are monomial matrices then the fractional positive electrical circuit (2.8) is reachable.

Proof. From (2.11) it follows that if A is diagonal then the matrix $\Phi(t)$ and $\Phi(t)B$ are also monomial for monomial matrix B. From (4.1) written in the form

$$R(t_f) = \int_0^{t_f} \Phi(\tau) B [\Phi(\tau) B]^T d\tau \qquad (4.3)$$

it follows that the matrix (4.3) is monomial. Therefore, by Theorem 4.1 the fractional system is reachable.

Example 4.1. Consider the fractional electrical circuit shown on Figure 4.1 with given conductances $G_1, G_2, G'_1, G'_2, G_{12}$, capacitance C_1, C_2 and source voltages e_1, e_2.
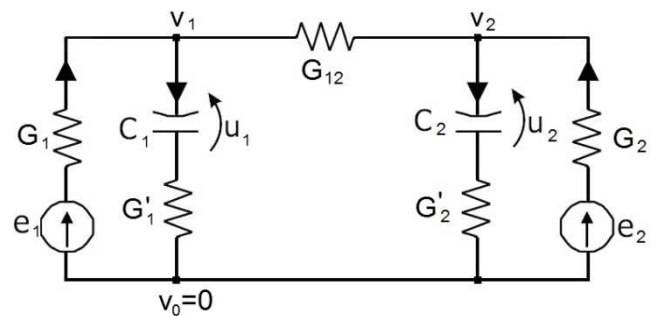

Fig. 4.1. Fractional electrical circuit

Using the Kirchhoff's laws we can write the equations

$$C_k \frac{d^\alpha u_k}{dt^\alpha} = G'_k (v_k - u_k), \quad k = 1,2 \qquad (4.4)$$

and

$$G \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -\begin{bmatrix} G'_1 & 0 \\ 0 & G'_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \qquad (4.5)$$

where

$$G = \begin{bmatrix} -(G_1 + G'_1 + G_{12}) & G_{12} \\ G_{12} & -(G_2 + G'_2 + G_{12}) \end{bmatrix} \quad (4.6)$$

is an Metzler matrix and $-G^{-1} \in \Re_+^{2\times2}$. From (4.5) we obtain

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -G^{-1}\begin{bmatrix} G'_1 & 0 \\ 0 & G'_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - G^{-1}\begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \quad (4.7)$$

Substitution of (4.7) into

$$\frac{d^\alpha}{dt^\alpha}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{G'_1}{C_1} & 0 \\ 0 & -\dfrac{G'_2}{C_2} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \dfrac{G'_1}{C_1} & 0 \\ 0 & \dfrac{G'_2}{C_2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (4.8)$$

we obtain

$$\frac{d^\alpha}{dt^\alpha}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \quad (4.9)$$

where

$$A = \begin{bmatrix} -\dfrac{G'_1}{C_1} & 0 \\ 0 & -\dfrac{G'_2}{C_2} \end{bmatrix} - \begin{bmatrix} \dfrac{G'_1}{C_1} & 0 \\ 0 & \dfrac{G'_2}{C_2} \end{bmatrix}G^{-1}\begin{bmatrix} G'_1 & 0 \\ 0 & G'_2 \end{bmatrix},$$

$$\quad (4.10)$$

$$B = -\begin{bmatrix} \dfrac{G'_1}{C_1} & 0 \\ 0 & \dfrac{G'_2}{C_2} \end{bmatrix}G^{-1}\begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}.$$

From (4.10) it follows that A is a Metzler matrix and the matrix B has nonnegative entries. Therefore, the fractional electrical circuit is positive for all values of the conductances and capacitances.

We shall show that the fractional positive electrical circuit shown on Fig 4.1 is reachable if and only if $G_{12} = 0$.

Note that the matrix (4.6) is diagonal if and only if $G_{12} = 0$. In this case from (4.10) it follows that A is a diagonal Metzler matrix and B is a diagonal matrix with positive diagonal entries. Therefore, by Theorem 4.2 the fractional positive electrical circuit is reachable.

In general case let us consider the fractional electrical circuit shown on Fig 4.2 with conductances $G_k, G'_k, G_{kj}, \; k, j = 1, \ldots, n$; capacitances $C_k, \; k = 1, \ldots, n$ and source voltages $e_k, \; k = 1, \ldots, n$.
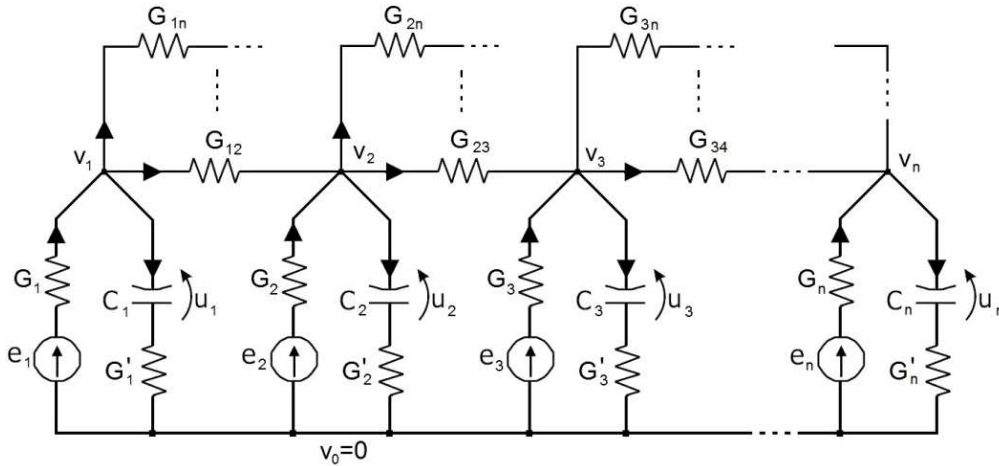

Fig. 4.2. Fractional electrical circuit

Theorem 4.3. The fractional electrical circuit shown on Fig. 4.2 is positive for all values of the conductances, capacitances and source voltages.
Proof. Using the Kirchhoff's laws and the node method for the electrical circuit we may write the equations

$$\frac{d^\alpha}{dt^\alpha}\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = -C^{-1}G'\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + C^{-1}G'\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad (4.11a)$$

and

$$\bar{G}\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = -G'\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} - G\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} \quad (4.11b)$$

where

$$C^{-1} = \text{diag}[C_1^{-1}, \ldots, C_n^{-1}], \quad G' = \text{diag}[G_1', \ldots, G_n'],$$

$$G = \text{diag}[G_1, \ldots, G_n], \quad (4.11c)$$

$$\bar{G} = \begin{bmatrix} -G_{11} & G_{12} & \ldots & G_{1,n} \\ G_{12} & -G_{22} & \ldots & G_{2,n} \\ \vdots & \vdots & \ldots & \vdots \\ G_{1,n} & G_{2,n} & \ldots & -G_{n,n} \end{bmatrix}$$

G_{ii} is the sum of conductances of all branches belonging to the i-th node, $i = 1, \ldots, n$.

The matrix $\bar{G} \in M_n$ and $-\bar{G}^{-1}$ has nonnegative entries. Substituting (4.11b) into (4.11a) we obtain

$$\frac{d^\alpha}{dt^\alpha}\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = A\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + B\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} \qquad (4.12a)$$

where

$$A = -C^{-1}G'[I_n + \overline{G}^{-1}G'] \in M_n \qquad (4.12b)$$

and

$$B = -C^{-1}G'\overline{G}^{-1}G \in \Re_+^{n\times n} \qquad (4.12c)$$

since the matrices C^{-1}, G', G and $-\overline{G}^{-1}$ have nonnegative entries. Therefore, the electrical circuit is positive.

Theorem 4.4. The fractional positive electrical circuit shown on Fig. 4.2 is reachable if and only if

$$G_{k,j} = 0 \text{ for } k \neq j \text{ and } k, j = 1,...n.. \qquad (4.13)$$

Proof. The matrix $\overline{G}$ defined by (4.11c) is diagonal if and only if the condition (4.13) is met. In this case the matrices $\overline{G}^{-1}G'$, A and B are also diagonal and from (4.12) we obtain

$$\frac{d^\alpha u_k}{dt^\alpha} = \frac{1}{C_k}(G_k'G_{k,k}^{-1}G_k' - G_k)u_k + \frac{1}{C_k}G_kG_{k,k}^{-1}G_k e_k, \ k=1,...,n \qquad (4.14a)$$

where

$$G_{k,k} = G_k + G'_k, \ k = 1,...,n. \qquad (4.14b)$$

Note that the subsystem (4.14a) is reachable. Therefore, the positive electrical circuit is reachable if and only if the condition (4.13) is satisfied.

Example 4.2. Consider the fractional electrical circuit shown on Figure 4.3 with given resistances R_1, R_2, R_3, inductances L_1, L_2 and source voltages e_1, e_2.
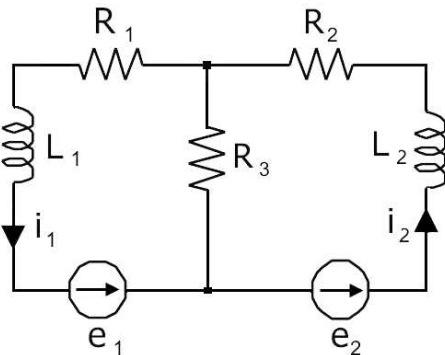

Fig. 4.3. Fractional electrical circuit

Using the Kirchhoff's laws we can write the equations

$$e_1 = R_3(i_1 - i_2) + R_1i_1 + L_1\frac{d^\beta i_1}{dt^\beta}$$
$$\qquad (4.15)$$
$$e_2 = R_3(i_2 - i_1) + R_2i_2 + L_2\frac{d^\beta i_2}{dt^\beta}$$

which can be written in the form

$$\frac{d^\beta}{dt^\beta}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \qquad (4.16a)$$

where

$$A = \begin{bmatrix} -\dfrac{R_1 + R_3}{L_1} & \dfrac{R_3}{L_1} \\ \dfrac{R_3}{L_2} & -\dfrac{R_2 + R_3}{L_2} \end{bmatrix}, \quad B = \begin{bmatrix} \dfrac{1}{L_1} & 0 \\ 0 & \dfrac{1}{L_2} \end{bmatrix}. \qquad (4.16b)$$

The fractional electrical circuit is positive since the matrix A is Metzler matrix and the matrix B has nonnegative entries.

We shall show that the fractional positive circuit is reachable if $R_3 = 0$. In this case

$$A = \begin{bmatrix} -\dfrac{R_1}{L_1} & 0 \\ 0 & -\dfrac{R_2}{L_2} \end{bmatrix}. \qquad (4.17)$$

and

$$e^{A\tau} = \begin{bmatrix} e^{-\frac{R_1}{L_1}\tau} & 0 \\ 0 & e^{-\frac{R_2}{L_2}\tau} \end{bmatrix} \qquad (4.18)$$

and from (4.1) we obtain

$$R_f = \int_0^{t_f} e^{A\tau}BB^Te^{A^T\tau}d\tau = \int_0^{t_f}\begin{bmatrix} \dfrac{1}{L_1^2}e^{-\frac{2R_1}{L_1}\tau} & 0 \\ 0 & \dfrac{1}{L_2^2}e^{-\frac{2R_2}{L_2}\tau} \end{bmatrix}d\tau \qquad (4.19)$$

The matrix (4.19) is monomial and by Theorem 4.1 the fractional positive electrical circuit is reachable if $R_3 = 0$.

Now let us consider the fractional n-mesh electrical circuit with given resistances R_k, $k = 1,...,q$, inductances L_i, $i = 1,...,n$ and m-mesh source voltages e_{jj}, $j = 1,...,m$. It is assumed that to each linearly independent mesh belongs only one inductance. In this case the matrix L defined by (3.19b) is diagonal one and the condition (3.21) is met.

Theorem 4.5. The fractional positive n-meshes electrical circuit with only one inductance in each linearly independent mesh is reachable if

$$R_{ij} = 0 \text{ for } i \neq j, \ i, j = 1,...,n \qquad (4.20)$$

where R_{ij} are entries of the matrix A' defined by (3.19b).

Proof. If the condition (4.20) is met then the Metzler matrix A' is diagonal. The matrix L defined by (3.19b) is also diagonal since by assumption only one inductance belongs to each linearly independent mesh. In this case the matrix $A = L^{-1}A'$ is diagonal Metzler matrix and $B = L^{-1} \in \Re_+^{n\times n}$ is also diagonal. For diagonal Metzler matrix A and diagonal B the matrix $e^{A\tau}B$ is also diagonal and the matrix R_f defined by (4.1) is monomial. By Theorem 4.1 the positive electrical circuit is reachable.

Remark 4.1. The condition (4.20) is met if the resistance of the branch belonging to two linearly independent meshes is zero. This result is consistent with the one obtained in Example 4.2.

Consider the fractional electrical circuit shown on Fig. 4.4 with given resistances R_k, $k = 1, \dots, 5$, inductances L_1, L_2, capacitance C and source voltage e.

Using the Kirchhoff's laws we can write the equations

$$e_1 = R_1 i_1 + L_1 \frac{d^\beta i_1}{dt^\beta} - R_5 i_2 + (R_3 + R_5) i_3$$

$$L_2 \frac{d^\beta i_2}{dt^\beta} + u + (R_2 + R_3) i_3 - R_2 i_1 = 0 \qquad (4.21a)$$

$$C \frac{d^\alpha u}{dt^\alpha} = i_2$$

and

$$(R_2 + R_4) i_1 + (R_4 + R_5) i_2 - (R_2 + R_3 + R_4 + R_5) i_3 = 0 . \quad (4.21b)$$

From (4.21b) we have

$$i_3 = \frac{(R_2 + R_4) i_1 + (R_4 + R_5) i_2}{R_2 + R_3 + R_4 + R_5} \qquad (4.22)$$

Substituting (4.22) into (4.21a) we obtain

$$\begin{bmatrix} \dfrac{d^\beta i_1}{dt^\beta} \\ \dfrac{d^\beta i_2}{dt^\beta} \\ \dfrac{d^\alpha u}{dt^\alpha} \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \\ u \end{bmatrix} + Be \qquad (4.23a)$$

where

$$A = \begin{bmatrix} -\dfrac{R_1}{L_1} - \dfrac{(R_2 + R_4)(R_3 + R_5)}{L_1(R_2 + R_3 + R_4 + R_5)} & \dfrac{R_2 R_5 - R_3 R_4}{L_1(R_2 + R_3 + R_4 + R_5)} & 0 \\ \dfrac{R_2 R_5 - R_3 R_4}{L_2(R_2 + R_3 + R_4 + R_5)} & -\dfrac{(R_2 + R_3)(R_4 + R_5)}{L_2(R_2 + R_3 + R_4 + R_5)} & -\dfrac{1}{L_2} \\ 0 & \dfrac{1}{C} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \dfrac{1}{L_1} \\ 0 \\ 0 \end{bmatrix}. \qquad (4.23b)$$
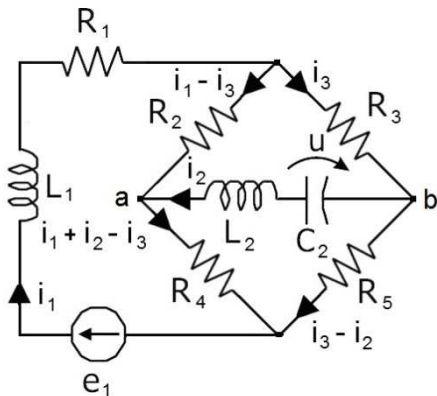

Fig. 4.4. Fractional electrical circuit

From (4.23b) it follows that the matrix A is not a Metzler matrix if

$$R_2 R_5 = R_3 R_4 \qquad (4.24)$$

If the condition (4.24) is met then the voltage between the points a, b is equal to zero and $u_2 = 0, L_2 \frac{di_2}{dt} = 0$, $i_2 = 0$. In this case the equation (4.23a) takes the form

$$\frac{di_1}{dt} = \left(-\frac{R_1}{L_1} - \frac{(R_2 + R_4)(R_3 + R_5)}{L_1(R_2 + R_3 + R_4 + R_5)} \right) i_1 + \frac{1}{L_1} e_1 \qquad (4.25)$$

The fractional electrical circuit described by the equation (4.25) is positive. Therefore, we have the following corollary.

Corollary 4.1. If the resistances of the electrical circuit satisfy the condition (4.24) then the fractional electrical circuit is positive.

In general case we have.

Corollary 4.2. Fractional nonpositive electrical circuit for some special choice of the parameters (resistances) can be positive one.

Using (4.23b) it is easy to check that

$$\text{rank}[B \quad AB \quad A^2 B] = 3 \qquad (4.26)$$

if and only if the condition (4.24) is not satisfied. Therefore, we have the following corollary.

Corollary 4.3. The fractional standard (nonpositive) electrical circuit shown on Fig. 4.4 is reachable if and only if the condition (4.24) is not satisfied.

From (4.25) it follows that the reduced fractional positive electrical circuit is reachable.

These considerations can be extended for general case of R, L, C, e type electrical circuits.

5. CONCLUDING REMARKS

The conditions for the positivity of fractional linear electrical circuits composed of resistors, coils, condensators and voltage (current) sources have been established. It has been shown that:

1. The fractional electrical circuits composed of resistors coils and voltage sources (shortly called R, L, e type) are positive for any values of their resistances, induc-

tances and source voltages if and only if the number of coils is less or equal to the number of its linearly independent meshes (Theorem 3.5).

2. The fractional electrical circuits composed of resistors, condensators and voltage sources (shortly called *R*, *C*, *e* type) are not positive for any values of its resistances, capacitances and voltage sources if each their branch contains resistor capacitor and voltage source (Theorem 3.2).

3. The fractional nonpositive electrical circuits of the *R*, *L*, *C*, *e* type can be positive for some special choice of their parameters (Corollary 4.2).

The conditions for the reachability of the fractional positive electrical circuits have been established. It has been shown that the fractional positive electrical circuit of *R*, *C*, *e* type are reachable if and only if the conductances between their nodes are zero (Theorem 4.4) and the fractional positive electrical circuits of *R*, *L*, *e* type are reachable if and only if the resistances belonging to two meshes are zero (Theorem 4.5). The fractional standard (nonpositive) electrical circuits of *R*, *C*, *L*, *e* type are usually reachable and are unreachable only for some special choice of the parameters.

The considerations have been illustrated by examples of linear electrical circuits.

Some of these results can be also extended for the controllability and observability of the fractional linear electrical circuit. Open problem are extension of these considerations for the following classes of the fractional systems:

1. disturbed parameters linear systems;
2. nonlinear electrical circuits.

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