

POSITIVITY AND STABILIZATION OF FRACTIONAL 2D LINEAR SYSTEMS DESCRIBED BY THE ROESSER MODEL

TADEUSZ KACZOREK, KRZYSZTOF ROGOWSKI

Faculty of Electrical Engineering
Białystok Technical University, Wiejska 45D, 15–351 Białystok, Poland
e-mail: kaczonek@isep.pw.edu.pl, k.rogowski@doktoranci.pb.edu.pl

A new class of fractional 2D linear discrete-time systems is introduced. The fractional difference definition is applied to each dimension of a 2D Roesser model. Solutions of these systems are derived using a 2D \mathcal{Z} -transform. The classical Cayley-Hamilton theorem is extended to 2D fractional systems described by the Roesser model. Necessary and sufficient conditions for the positivity and stabilization by the state-feedback of fractional 2D linear systems are established. A procedure for the computation of a gain matrix is proposed and illustrated by a numerical example.

Keywords: positivity, stabilization, fractional systems, Roesser model, 2D systems.

1. Introduction

The most popular models of two-dimensional (2D) linear systems are the ones introduced by Roesser (1975), Fornasini-Marchesini (1976; 1978) and Kurek (1985). These models were extended to positive systems in (Valcher, 1997; Kaczorek, 1996; 2001; 2005). An overview of 2D linear systems theory is given in (Bose, 1982; 1985; Kaczorek, 1985; Galkowski, 2001), and some recent results in positive systems can be found in the monographs (Farina and Rinaldi, 2000; Kaczorek, 2001). Asymptotic stability of positive 2D linear systems was investigated in (Twardy, 2007; Kaczorek, 2008a; 2008b; 2009a). The problem of the positivity and stabilization of 2D linear systems by state feedback was considered in (Kaczorek, 2009c).

Mathematical fundamentals of fractional calculus are given in the monographs (Oldham and Spanier, 1974; Nashimoto, 1984; Miller and Ross, 1993; Podlubny, 1999). The notion of fractional 2D linear systems was introduced in (Kaczorek, 2008c) and extended in (Kaczorek, 2008d; 2009b). The problem of the positivity and stabilization of 1D fractional systems by state feedback was considered in (Kaczorek, 2009d).

In this paper a new 2D fractional Roesser type model will be introduced and it will be shown that the problem of finding a gain matrix of the state-feedback such that the closed-loop system is positive and asymptotically stable can be reduced to a suitable linear programming problem.

The paper is organized as follows: In Section 2 fractional 2D state equations of the Roesser model are proposed and their solution are derived. The classical Cayley-Hamilton theorem is extended to fractional 2D systems in Section 3. In Section 4 necessary and sufficient conditions for the positivity of 2D fractional systems are established. In Section 5 the problem of finding a gain matrix of the state-feedback such that the closed-loop 2D system is positive and asymptotically stable is solved. The procedure for the computation of the gain matrix is given and illustrated by a numerical example. Concluding remarks are given in Section 6.

2. Fractional 2D state-space equations and their solutions

Let $\mathbb{R}_+^{n \times m}$ be the set of $n \times m$ matrices with all nonnegative elements and $\mathbb{R}_+^n := \mathbb{R}_+^{n \times 1}$. The set of nonnegative integers will be denoted by \mathbb{Z}_+ and the $n \times n$ identity matrix will be denoted by \mathbb{I}_n .

We introduce the following two notions of horizontal and vertical fractional differences of a 2D function.

Definition 1. The α -order horizontal fractional difference of a 2D function x_{ij} , $i, j \in \mathbb{Z}_+$, is defined by

$$\Delta_\alpha^h x_{ij} = \sum_{k=0}^i c_\alpha(k) x_{i-k,j}, \quad (1a)$$

where $\alpha \in \mathbb{R}, n - 1 < \alpha < n \in \mathbb{N} = \{1, 2, \dots\}$ and

$$c_\alpha(k) = \begin{cases} 1 & \text{for } k = 0, \\ (-1)^k \frac{\alpha(\alpha - 1) \cdots (\alpha - k + 1)}{k!} & \text{for } k > 0. \end{cases} \quad (1b)$$

Definition 2. The β -order vertical fractional difference of a 2D function x_{ij} , $i, j \in \mathbb{Z}_+$, is defined by

$$\Delta_\beta^v x_{ij} = \sum_{l=0}^j c_\beta(l) x_{i,j-l}, \quad (2a)$$

where $\beta \in \mathbb{R}, n - 1 < \beta < n \in \mathbb{N}$ and

$$c_\beta(l) = \begin{cases} 1 & \text{for } l = 0, \\ (-1)^l \frac{\beta(\beta - 1) \cdots (\beta - l + 1)}{l!} & \text{for } l > 0. \end{cases} \quad (2b)$$

Lemma 1. (Kaczorek, 2007) If $0 < \alpha < 1$ ($0 < \beta < 1$), then

$$c_\alpha(k) < 0 \quad (c_\beta(k) < 0) \quad \text{for } k = 1, 2, \dots \quad (3)$$

Consider a fractional 2D linear system described by the state equations

$$\begin{bmatrix} \Delta_\alpha^h x_{i+1,j} \\ \Delta_\beta^v x_{i,j+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_{i,j}^h \\ x_{i,j}^v \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{ij}, \quad (4a)$$

$$y_{ij} = [C_1 \quad C_2] \begin{bmatrix} x_{i,j}^h \\ x_{i,j}^v \end{bmatrix} + D u_{ij} \quad i, j \in \mathbb{Z}_+, \quad (4b)$$

where $x_{ij}^h \in \mathbb{R}^{n_1}$, $x_{ij}^v \in \mathbb{R}^{n_2}$ represent a horizontal and a vertical state vector at the point (i, j) , respectively, $u_{ij} \in \mathbb{R}^m$ is an input vector, $y_{ij} \in \mathbb{R}^p$ is an output vector at the point (i, j) , and $A_{11} \in \mathbb{R}^{n_1 \times n_1}$, $A_{12} \in \mathbb{R}^{n_1 \times n_2}$, $A_{21} \in \mathbb{R}^{n_2 \times n_1}$, $A_{22} \in \mathbb{R}^{n_2 \times n_2}$, $B_1 \in \mathbb{R}^{n_1 \times m}$, $B_2 \in \mathbb{R}^{n_2 \times m}$, $C_1 \in \mathbb{R}^{p \times n_1}$, $C_2 \in \mathbb{R}^{p \times n_2}$, $D \in \mathbb{R}^{p \times m}$.

Using Definitions 1 and 2 we may write (4a) as

$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & A_{12} \\ A_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} x_{i,j}^h \\ x_{i,j}^v \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{ij} - \begin{bmatrix} \sum_{k=2}^{i+1} c_\alpha(k) x_{i-k+1,j}^h \\ \sum_{l=2}^{j+1} c_\beta(l) x_{i,j-l+1}^v \end{bmatrix}, \quad (5)$$

where $\bar{A}_{11} = A_{11} + \alpha \mathbb{I}_{n_1}$ and $\bar{A}_{22} = A_{22} + \beta \mathbb{I}_{n_2}$.

From (5) it follows that fractional 2D systems are 2D systems with delays increasing with i and j . From (1b) and (2b) it follows that the coefficients $c_\alpha(k)$ and $c_\beta(l)$ in (5) strongly decrease when k and l increase.

Therefore, in practical problems we may assume that k and l are bounded by some natural numbers L_1 and L_2 . In this case, Eqn. (5) takes the form

$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & A_{12} \\ A_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} x_{i,j}^h \\ x_{i,j}^v \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{ij} - \begin{bmatrix} \sum_{k=2}^{L_1+1} c_\alpha(k) x_{i-k+1,j}^h \\ \sum_{l=2}^{L_2+1} c_\beta(l) x_{i,j-l+1}^v \end{bmatrix}. \quad (6)$$

The boundary conditions for Eqns. (4a), (5) and (6) are given in the form

$$x_{0j}^h \text{ for } j \in \mathbb{Z}_+, \quad x_{i0}^v \text{ for } i \in \mathbb{Z}_+. \quad (7)$$

Theorem 1. The solution to Eqn. (5) with the boundary conditions (7) is given by

$$\begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} = \sum_{p=0}^i T_{i-p,j} \begin{bmatrix} 0 \\ x_{p0}^v \end{bmatrix} + \sum_{q=0}^j T_{i,j-q} \begin{bmatrix} x_{0q}^h \\ 0 \end{bmatrix} + \sum_{p=0}^i \sum_{q=0}^j (T_{i-p-1,j-q} B^{10} + T_{i-p,j-q-1} B^{01}) u_{pq}, \quad (8a)$$

where

$$B^{10} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad B^{01} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \quad (8b)$$

and the transition matrices $T_{pq} \in \mathbb{R}^{n \times n}$ are defined by

$$T_{pq} = \begin{cases} \mathbb{I}_n & \text{for } p = 0, q = 0, \\ T_{pq} & \text{for } p + q > 0 \quad (p, q \in \mathbb{Z}_+), \\ 0 \text{ (zero matrix)} & \text{for } p < 0 \text{ and/or } q < 0, \end{cases} \quad (8c)$$

where

$$\begin{aligned} T_{pq} &= T_{10} T_{p-1,q} - \sum_{k=2}^p \begin{bmatrix} c_\alpha(k) \mathbb{I}_{n_1} & 0 \\ 0 & 0 \end{bmatrix} T_{p-k,q} \\ &+ T_{01} T_{p,q-1} - \sum_{l=2}^q \begin{bmatrix} 0 & 0 \\ 0 & c_\beta(l) \mathbb{I}_{n_2} \end{bmatrix} T_{p,q-l} \end{aligned} \quad (8d)$$

and

$$T_{10} = \begin{bmatrix} \bar{A}_{11} & A_{12} \\ 0 & 0 \end{bmatrix}, \quad T_{01} = \begin{bmatrix} 0 & 0 \\ A_{21} & \bar{A}_{22} \end{bmatrix}. \quad (8e)$$

Proof. Let $X(z_1, z_2)$ be the 2D \mathcal{Z} -transform of x_{ij} defined by

$$X(z_1, z_2) = \mathcal{Z}[x_{ij}] = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} x_{ij} z_1^{-i} z_2^{-j}. \quad (9)$$

Using (9), we obtain (Kaczorek, 1985)

$$\mathcal{Z} [x_{i+1,j}^h] = z_1 [X^h(z_1, z_2) - X^h(0, z_2)], \quad (10a)$$

where

$$X^h(0, z_2) = \sum_{j=0}^{\infty} x_{0j}^h z_2^{-j},$$

$$\mathcal{Z} [x_{i,j+1}^v] = z_2 [X^v(z_1, z_2) - X^v(z_1, 0)], \quad (10b)$$

$$X^v(z_1, 0) = \sum_{i=0}^{\infty} x_{i0}^v z_1^{-i},$$

$$\mathcal{Z} \left[\sum_{k=2}^{i+1} c_{\alpha}(k) x_{i-k+1,j}^h \right] = \sum_{k=2}^{i+1} c_{\alpha}(k) z_1^{-k+1} X^h(z_1, z_2) \quad (10c)$$

since

$$\begin{aligned} \mathcal{Z} [x_{i-k,j}^h] &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} x_{i-k,j}^h z_1^{-i} z_2^{-j} \\ &= \sum_{i=-k}^{\infty} \sum_{j=0}^{\infty} x_{ij}^h z_1^{-i-k} z_2^{-j} \\ &= z_1^{-k} X^h(z_1, z_2). \end{aligned} \quad (10d)$$

Similarly,

$$\mathcal{Z} \left[\sum_{l=2}^{j+1} c_{\beta}(l) x_{i,j-l+1}^v \right] = \sum_{l=2}^{j+1} c_{\beta}(l) z_2^{-l+1} X^v(z_1, z_2) \quad (10e)$$

since

$$\begin{aligned} \mathcal{Z} [x_{i,j-l}^v] &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} x_{i,j-l}^v z_1^{-i} z_2^{-j} \\ &= \sum_{i=0}^{\infty} \sum_{j=-l}^{\infty} x_{ij}^v z_1^{-i} z_2^{-j-l} \\ &= z_2^{-l} X^v(z_1, z_2). \end{aligned} \quad (10f)$$

Taking into account (10), we obtain the 2D \mathcal{Z} -transform of the state-space equation (5),

$$\begin{aligned} & \begin{bmatrix} z_1 X^h(z_1, z_2) - z_1 X^h(0, z_2) \\ z_2 X^v(z_1, z_2) - z_2 X^v(z_1, 0) \end{bmatrix} \\ &= \begin{bmatrix} \bar{A}_{11} & A_{12} \\ A_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} X^h(z_1, z_2) \\ X^v(z_1, z_2) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(z_1, z_2) \\ & \quad - \begin{bmatrix} \sum_{k=2}^{i+1} c_{\alpha}(k) z_1^{-k+1} X^h(z_1, z_2) \\ \sum_{l=2}^{j+1} c_{\beta}(l) z_2^{-l+1} X^v(z_1, z_2) \end{bmatrix}, \end{aligned} \quad (11)$$

where $U(z_1, z_2) = \mathcal{Z}(u_{ij})$.

Premultiplying (11) by the matrix

$$\text{blockdiag}[\mathbb{I}_{n_1} z_1^{-1}, \mathbb{I}_{n_2} z_2^{-1}],$$

we obtain

$$\begin{aligned} & \begin{bmatrix} X^h(z_1, z_2) \\ X^v(z_1, z_2) \end{bmatrix} \\ &= G^{-1}(z_1, z_2) \left\{ \begin{bmatrix} z_1^{-1} B_1 \\ z_2^{-1} B_2 \end{bmatrix} U(z_1, z_2) \right. \\ & \quad \left. + \begin{bmatrix} X^h(0, z_2) \\ X^v(z_1, 0) \end{bmatrix} \right\}, \end{aligned} \quad (12)$$

where

$$G(z_1, z_2) = \begin{bmatrix} \mathbb{G}_{11} & -z_1^{-1} A_{12} \\ -z_2^{-1} A_{21} & \mathbb{G}_{22} \end{bmatrix}, \quad (13a)$$

$$\mathbb{G}_{11} = \mathbb{I}_{n_1} - z_1^{-1} \bar{A}_{11} + \sum_{k=2}^i c_{\alpha}(k) z_1^{-k} \mathbb{I}_{n_1}, \quad (13b)$$

$$\mathbb{G}_{22} = \mathbb{I}_{n_2} - z_2^{-1} \bar{A}_{22} + \sum_{l=2}^j c_{\beta}(l) z_2^{-l} \mathbb{I}_{n_2}. \quad (13c)$$

Let

$$G^{-1}(z_1, z_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} T_{pq} z_1^{-p} z_2^{-q}. \quad (14)$$

Write

$$T_{pq} = \begin{bmatrix} T_{pq}^{11} & T_{pq}^{12} \\ T_{pq}^{21} & T_{pq}^{22} \end{bmatrix}, \quad (15)$$

where T_{pq}^{kl} have the same sizes as the matrices A_{kl} for $k, l = 1, 2$.

From

$$G^{-1}(z_1, z_2) G(z_1, z_2) = G(z_1, z_2) G^{-1}(z_1, z_2) = \mathbb{I}_n,$$

using (14) and (15), it follows that

$$\begin{aligned} & \begin{bmatrix} \mathbb{G}_{11} & -z_1^{-1} A_{12} \\ -z_2^{-1} A_{21} & \mathbb{G}_{22} \end{bmatrix} \\ & \quad \times \left(\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \begin{bmatrix} T_{pq}^{11} & T_{pq}^{12} \\ T_{pq}^{21} & T_{pq}^{22} \end{bmatrix} z_1^{-p} z_2^{-q} \right) \\ & \quad = \begin{bmatrix} \mathbb{I}_{n_1} & 0 \\ 0 & \mathbb{I}_{n_2} \end{bmatrix}. \end{aligned} \quad (16)$$

Comparing the coefficients at the same powers of z_1 and z_2 yields (8c).

Taking into account the expansion (14) and using the inverse 2D \mathcal{Z} -transform of (12) we obtain the formula (8a). \blacksquare

3. Extension of the Cayley-Hamilton theorem

From (13), for the system (6) we have

$$\bar{G}(z_1, z_2) = \begin{bmatrix} \bar{G}_{11} & -z_1^{-1}A_{12} \\ -z_2^{-1}A_{21} & \bar{G}_{22} \end{bmatrix}, \quad (17a)$$

$$\bar{G}_{11} = \mathbb{I}_{n_1} - z_1^{-1}\bar{A}_{11} + \sum_{k=2}^{L_1} c_\alpha(k)z_1^{-k}\mathbb{I}_{n_1}, \quad (17b)$$

$$\bar{G}_{22} = \mathbb{I}_{n_2} - z_2^{-1}\bar{A}_{22} + \sum_{l=2}^{L_2} c_\beta(l)z_2^{-l}\mathbb{I}_{n_2}. \quad (17c)$$

Let

$$\det \bar{G}(z_1, z_2) = \sum_{p=0}^{N_1} \sum_{q=0}^{N_2} a_{N_1-p, N_2-q} z_1^{-p} z_2^{-q}, \quad (18)$$

where $N_1, N_2 \in \mathbb{Z}_+$ are determined by the numbers L_1 and L_2 in (6).

Theorem 2. *Let (18) be the characteristic polynomial of the system (6). Then the matrices T_{pq} satisfy*

$$\sum_{p=0}^{N_1} \sum_{q=0}^{N_2} a_{pq} T_{pq} = 0. \quad (19)$$

Proof. From the definition of the inverse matrix, as well as (14) and (18), we have

$$\begin{aligned} \text{Adj } \bar{G}(z_1, z_2) &= \left(\sum_{p=0}^{N_1} \sum_{q=0}^{N_2} a_{N_1-p, N_2-q} z_1^{-p} z_2^{-q} \right) \\ &\times \left(\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} T_{kl} z_1^{-k} z_2^{-l} \right), \end{aligned} \quad (20)$$

where $\text{Adj } \bar{G}(z_1, z_2)$ is the adjoint matrix of $\bar{G}(z_1, z_2)$. Comparing the coefficients at the same power $z_1^{-N_1} z_2^{-N_2}$ of the equality (20) yields (19) since $\text{Adj } \bar{G}(z_1, z_2)$ has degrees greater than $-N_1$ and $-N_2$, respectively. ■

Theorem 2 is an extension of the well-known classical Cayley-Hamilton theorem to 2D fractional systems described by the Roesser model (5).

4. Positivity of fractional 2D systems described by the Roesser model

Definition 3. The system (4) is called the (internally) positive fractional 2D system if and only if $x_{ij}^h \in \mathbb{R}_+^{n_1}$, $x_{ij}^v \in \mathbb{R}_+^{n_2}$ and $y_{ij} \in \mathbb{R}_+^m$, $i, j \in \mathbb{Z}_+$ for any boundary conditions $x_{0j}^h \in \mathbb{R}_+^{n_1}$, $j \in \mathbb{Z}_+$ and $x_{i0}^v \in \mathbb{R}_+^{n_2}$, $i \in \mathbb{Z}_+$ and all input sequences $u_{ij} \in \mathbb{R}_+^m$, $i, j \in \mathbb{Z}_+$.

Theorem 3. *The fractional 2D system (5) for $\alpha, \beta \in \mathbb{R}$, $0 < \alpha \leq 1$, $0 < \beta \leq 1$ is positive if and only if*

$$\begin{aligned} \begin{bmatrix} \bar{A}_{11} & A_{12} \\ A_{21} & \bar{A}_{22} \end{bmatrix} &\in \mathbb{R}_+^{n \times n}, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \mathbb{R}_+^{n \times m}, \\ \begin{bmatrix} C_1 & C_2 \end{bmatrix} &\in \mathbb{R}_+^{p \times n}, \quad D \in \mathbb{R}_+^{p \times m}. \end{aligned} \quad (21)$$

Proof. (Necessity) Let us assume that the system (5) is positive and $u_{ij} = 0$ for $i, j \in \mathbb{Z}_+$, $x_{i0}^v = 0$, $i \in \mathbb{Z}_+$ and $x_{01}^h = e_{n_1}^{(k)}$, where $e_{n_1}^{(k)}$ is the k -th column of \mathbb{I}_{n_1} . In this case, from (5) we obtain $x_{11}^h = \bar{A}_{11}x_{01}^h = \bar{A}_{11}^{(k)} \in \mathbb{R}_+^{n_1}$, where $\bar{A}_{11}^{(k)}$ denotes the k -th column of the matrix \bar{A}_{11} . For $k = 1, 2, \dots, n_1$ this implies $\bar{A}_{11} \in \mathbb{R}_+^{n_1}$. Assuming $x_{0j}^h = 0$ for $j \in \mathbb{Z}_+$, $u_{ij} = 0$ for $i, j \in \mathbb{Z}_+$ and $x_{i0}^v = e_{n_2}^{(k)}$, where $e_{n_2}^{(k)}$ is the k -th column of \mathbb{I}_{n_2} , we obtain $x_{11}^v = A_{12}x_{i0}^v = A_{12}^{(k)}$, where $A_{12}^{(k)}$ is the k -th column of A_{12} , and this implies $A_{12} \in \mathbb{R}_+^{n_1 \times n_2}$. In a similar way, it can be shown that $A_{21} \in \mathbb{R}_+^{n_2 \times n_1}$ and $\bar{A}_{22} \in \mathbb{R}_+^{n_2 \times n_2}$.

Now, let us assume that boundary conditions are zero $x_{0j}^h = 0$ for $j \in \mathbb{Z}_+$, $x_{i0}^v = 0$ for $i \in \mathbb{Z}_+$ and $u_{01} = e_m^{(k)}$ ($e_m^{(k)}$ is the k -th column of \mathbb{I}_m). Then we have $x_{11}^h = B_1 u_{01} = B_1^{(k)} \in \mathbb{R}_+^{n_1}$, where $B_1^{(k)}$ is the k -th column of the matrix B_1 . This implies $B_1 \in \mathbb{R}_+^{n_1 \times m}$. In a similar way, we may show that $B_2 \in \mathbb{R}_+^{n_2 \times m}$, $C_1 \in \mathbb{R}_+^{p \times n_1}$, $C_2 \in \mathbb{R}_+^{p \times n_2}$ and $D \in \mathbb{R}_+^{p \times m}$.

(Sufficiency) By Lemma 1, $c_\alpha(k) < 0$ for $k = 1, 2, \dots$ and $0 < \alpha \leq 1$ ($c_\beta(l) < 0$ for $l = 1, 2, \dots$ and $0 < \beta \leq 1$). From (15) it follows that, if the conditions of Theorem 3 are met, then $T_{pq} \in \mathbb{R}_+^{n \times n}$ for $p, q \in \mathbb{Z}_+$. Taking this into account for $x_{0j}^h \in \mathbb{R}_+^{n_1}$ ($j \in \mathbb{Z}_+$), $x_{i0}^v \in \mathbb{R}_+^{n_2}$ ($i \in \mathbb{Z}_+$) and $u_{ij} \in \mathbb{R}_+^m$ ($i, j \in \mathbb{Z}_+$), from (8a) we have $x_{ij}^h \in \mathbb{R}_+^{n_1}$ and $x_{ij}^v \in \mathbb{R}_+^{n_2}$ for $i, j \in \mathbb{Z}_+$.

From (4b) we have $y_{ij} \in \mathbb{R}_+^m$ for $i, j \in \mathbb{Z}_+$ since $x_{ij}^h \in \mathbb{R}_+^{n_1}$, $x_{ij}^v \in \mathbb{R}_+^{n_2}$, $u_{ij} \in \mathbb{R}_+^m$ for $i, j \in \mathbb{Z}_+$ and $C_1 \in \mathbb{R}_+^{p \times n_1}$, $C_2 \in \mathbb{R}_+^{p \times n_2}$, $D \in \mathbb{R}_+^{p \times m}$. ■

5. Stabilization of the Roesser model by state feedback

The following theorem will be used in the proof of the main result of this section.

Theorem 4. (Kaczorek, 2008b) *The positive Roesser model*

$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} \quad (22)$$

is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1. The positive 1D system

$$x_{i+1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x_i \quad (23)$$

is asymptotically stable.

2. There exists a strictly positive vector $\lambda \in \mathbb{R}_+^n$ ($n = n_1 + n_2$) such that

$$\begin{bmatrix} A_{11} - \mathbb{I}_{n_1} & A_{12} \\ A_{21} & A_{22} - \mathbb{I}_{n_2} \end{bmatrix} \lambda < \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (24)$$

Lemma 2. If $n-1 < \alpha < n \in \mathbb{N}$ ($n-1 < \beta < n$), then

$$\sum_{k=0}^{\infty} c_{\alpha}(k) = 0 \quad \left(\text{resp. } \sum_{k=0}^{\infty} c_{\beta}(k) = 0 \right). \quad (25)$$

Proof. It is easy to verify that the Taylor series expansion of the function $(1-z)^{\alpha}$ yields

$$(1-z)^{\alpha} = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} z^k. \quad (26)$$

Substituting $z = 1$ into (26) we obtain

$$\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} = \sum_{k=0}^{\infty} c_{\alpha}(k) = 0. \quad \blacksquare$$

Consider the positive fractional Roeser model (5) with the state-feedback

$$u_{ij} = [K_1 \quad K_2] \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix}, \quad (27)$$

where $K = [K_1 \quad K_2] \in \mathbb{R}^{m \times n}$, $K_j \in \mathbb{R}^{m \times n_j}$, $j = 1, 2$ is a gain matrix.

We are looking for a gain matrix K such that the closed-loop system

$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} + B_1 K_1 & A_{12} + B_1 K_2 \\ A_{21} + B_2 K_1 & \bar{A}_{22} + B_2 K_2 \end{bmatrix} \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} - \begin{bmatrix} \sum_{k=2}^{i+1} c_{\alpha}(k) x_{i-k+1,j}^h \\ \sum_{l=2}^{j+1} c_{\beta}(l) x_{i,j-l+1}^v \end{bmatrix} \quad (28)$$

is positive and asymptotically stable.

Theorem 5. The positive fractional closed-loop system (28) is positive and asymptotically stable if and only if there exist a block diagonal matrix

$$\begin{aligned} \Lambda &= \text{blockdiag}[\Lambda_1, \Lambda_2], \\ \Lambda_k &= \text{diag}[\lambda_{k1}, \dots, \lambda_{kn_k}], \\ \lambda_{kj} &> 0, \end{aligned} \quad (29)$$

$k = 1, 2$, $j = 1, \dots, n_k$, and a real matrix

$$D = [D_1 \quad D_2], \quad D_k \in \mathbb{R}^{m \times n_k}, \quad k = 1, 2 \quad (30)$$

satisfying the conditions

$$\begin{bmatrix} \bar{A}_{11}\Lambda_1 + B_1 D_1 & A_{12}\Lambda_2 + B_1 D_2 \\ A_{21}\Lambda_1 + B_2 D_1 & \bar{A}_{22}\Lambda_2 + B_2 D_2 \end{bmatrix} \in \mathbb{R}_+^{n \times n} \quad (31)$$

and

$$\begin{bmatrix} A_{11}\Lambda_1 + B_1 D_1 & A_{12}\Lambda_2 + B_1 D_2 \\ A_{21}\Lambda_1 + B_2 D_1 & A_{22}\Lambda_2 + B_2 D_2 \end{bmatrix} \begin{bmatrix} \mathbf{1}_{n_1} \\ \mathbf{1}_{n_2} \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (32)$$

where $\mathbf{1}_{n_k} = [1 \quad \dots \quad 1]^T \in \mathbb{R}_+^{n_k}$, $k = 1, 2$ (T denotes the transpose). The gain matrix is given by

$$K = [K_1 \quad K_2] = [D_1 \Lambda_1^{-1} \quad D_2 \Lambda_2^{-1}]. \quad (33)$$

Proof. First, we shall show that the closed-loop system is positive if and only if the condition (31) is satisfied. Using (28) and (33), we obtain

$$\begin{aligned} &\begin{bmatrix} \bar{A}_{11} + B_1 D_1 \Lambda_1^{-1} & A_{12} + B_1 D_2 \Lambda_2^{-1} \\ A_{21} + B_2 D_1 \Lambda_1^{-1} & \bar{A}_{22} + B_2 D_2 \Lambda_2^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \bar{A}_{11}\Lambda_1 + B_1 D_1 & A_{12}\Lambda_2 + B_1 D_2 \\ A_{21}\Lambda_1 + B_2 D_1 & \bar{A}_{22}\Lambda_2 + B_2 D_2 \end{bmatrix} \\ &\cdot \begin{bmatrix} \Lambda_1^{-1} & 0 \\ 0 & \Lambda_2^{-1} \end{bmatrix}. \end{aligned} \quad (34)$$

From (34) and (21) it follows that the closed-loop system (28) is positive if and only if the condition (31) is satisfied. Taking into account that $c_{\alpha}(0) = c_{\beta}(0) = 1$ and $c_{\alpha}(1) = -\alpha$, $c_{\beta}(1) = -\beta$, from (25) we have

$$\sum_{k=2}^{\infty} c_{\alpha}(k) = \alpha - 1 \quad \text{and} \quad \sum_{k=2}^{\infty} c_{\beta}(k) = \beta - 1. \quad (35)$$

It is well known (Busłowicz, 2008; Busłowicz and Kaczorek, 2009) that asymptotic stability of the positive discrete-time linear system with delays is independent of the number and values of the delays and it depends only on the sum of the state matrices. Therefore, the positive closed-loop system (28) is asymptotically stable if and only if the positive 1D system with the matrix

$$\begin{aligned} &\begin{bmatrix} \bar{A}_{11} + B_1 K_1 & A_{12} + B_1 K_2 \\ A_{21} + B_2 K_1 & \bar{A}_{22} + B_2 K_2 \end{bmatrix} \\ &- \sum_{k=2}^{\infty} \begin{bmatrix} \mathbb{I}_{n_1} c_{\alpha}(k) & 0 \\ 0 & \mathbb{I}_{n_2} c_{\beta}(k) \end{bmatrix} \end{aligned} \quad (36)$$

is asymptotically stable.

Using (35) as well as $\bar{A}_{11} = A_{11} + \mathbb{I}_{n_1} \alpha$ and $\bar{A}_{22} = A_{22} + \mathbb{I}_{n_2} \beta$, we may write the matrix (36) in the form

$$\begin{bmatrix} A_{11} + \mathbb{I}_{n_1} + B_1 K_1 & A_{12} + B_1 K_2 \\ A_{21} + B_2 K_1 & A_{22} + \mathbb{I}_{n_2} + B_2 K_2 \end{bmatrix}. \quad (37)$$

By Theorem 4, the positive closed-loop system (28) is asymptotically stable if and only if there exists a strictly positive vector $\lambda = [\lambda_1^T, \lambda_2^T]^T \in \mathbb{R}_+^n$ such that

$$\begin{bmatrix} A_{11} + B_1 K_1 & A_{12} + B_1 K_2 \\ A_{21} + B_2 K_1 & A_{22} + B_2 K_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (38)$$

Taking into account that $\lambda_k = \Lambda_k \mathbf{1}_k$, $k = 1, 2$, and using (33) and (38) we obtain

$$\begin{aligned} & \begin{bmatrix} A_{11} + B_1 K_1 & A_{12} + B_1 K_2 \\ A_{21} + B_2 K_1 & A_{22} + B_2 K_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \\ &= \begin{bmatrix} A_{11} + B_1 D_1 \Lambda_1^{-1} & A_{12} + B_1 D_2 \Lambda_2^{-1} \\ A_{21} + B_2 D_1 \Lambda_1^{-1} & A_{22} + B_2 D_2 \Lambda_2^{-1} \end{bmatrix} \\ & \quad \times \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} \mathbf{1}_{n_1} \\ \mathbf{1}_{n_2} \end{bmatrix} \quad (39) \\ &= \begin{bmatrix} A_{11} \Lambda_1 + B_1 D_1 & A_{12} \Lambda_2 + B_1 D_2 \\ A_{21} \Lambda_1 + B_2 D_1 & A_{22} \Lambda_2 + B_2 D_2 \end{bmatrix} \\ & \quad \times \begin{bmatrix} \mathbf{1}_{n_1} \\ \mathbf{1}_{n_2} \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Therefore, the positive closed-loop system is asymptotically stable if and only if the condition (32) is met. ■

If the conditions of Theorem 5 are satisfied, then the gain matrix can be computed by the use of the following procedure.

Procedure

Step 1. Choose a block diagonal matrix (29) and a real matrix (30) satisfying the conditions (31) and (32).

Step 2. Using the formula (33), compute the gain matrix K .

Theorem 6. *The positive fractional Roesser model is unstable if at least one diagonal entry of the matrix*

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad (40)$$

is positive.

Proof. From (37) for $K_1 = 0$ and $K_2 = 0$, for the positive fractional Roesser model we have

$$\begin{bmatrix} A_{11} + \mathbb{I}_{n_1} & A_{12} \\ A_{21} & A_{22} + \mathbb{I}_{n_2} \end{bmatrix}. \quad (41)$$

If at least one diagonal entry of the matrix (40) is positive, then at least one diagonal entry of the matrix (41) is greater than 1 and this implies that the positive fractional Roesser model is unstable. ■

Example 1. Given the fractional Roesser model with $\alpha = 0.4, \beta = 0.5$ and

$$\begin{aligned} A_{11} &= \begin{bmatrix} -0.5 & -0.1 \\ 0.1 & 0.01 \end{bmatrix}, & A_{12} &= \begin{bmatrix} -0.1 & -0.1 \\ 0.2 & 0.1 \end{bmatrix}, \\ A_{21} &= \begin{bmatrix} -0.3 & -0.1 \\ 0.2 & 0.1 \end{bmatrix}, & A_{22} &= \begin{bmatrix} -1 & -0.1 \\ 0.4 & 0.1 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} -0.2 \\ 0.1 \end{bmatrix}, & B_2 &= \begin{bmatrix} -0.3 \\ 0.2 \end{bmatrix}. \quad (42) \end{aligned}$$

We wish to find a gain matrix $K = [K_1, K_2], K_p \in \mathbb{R}^{1 \times 2}, p = 1, 2$ such that the closed-loop system is positive and asymptotically stable.

The fractional Roesser model (5) with (42) is not positive since the matrix

$$\begin{bmatrix} \bar{A}_{11} & A_{12} \\ A_{21} & \bar{A}_{22} \end{bmatrix} = \begin{bmatrix} -0.1 & -0.1 & -0.1 & -0.1 \\ 0.1 & 0.41 & 0.2 & 0.1 \\ -0.3 & -0.1 & -0.5 & -0.1 \\ 0.2 & 0.1 & 0.4 & 0.6 \end{bmatrix} \quad (43)$$

and the matrices B_1, B_2 have negative entries, and it is unstable since the matrix

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} -0.5 & -0.1 & -0.1 & -0.1 \\ 0.1 & 0.01 & 0.2 & 0.1 \\ -0.3 & -0.1 & -1 & -0.1 \\ 0.2 & 0.1 & 0.4 & 0.1 \end{bmatrix} \quad (44)$$

has two positive diagonal entries.

Using our Procedure, we obtain what follows.

Step 1. We choose

$$\begin{aligned} \Lambda &= \text{blockdiag}[\Lambda_1, \Lambda_2], \\ \Lambda_1 &= \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix}, & \Lambda_2 &= \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix} \quad (45) \end{aligned}$$

and

$$D = [D_1, D_2], \quad D_1 = D_2 = \begin{bmatrix} -0.4 & -0.2 \end{bmatrix}, \quad (46)$$

which satisfy the conditions (31) and (32) since

$$\begin{aligned} \bar{A}_{11} \Lambda_1 + B_1 D_1 &= \begin{bmatrix} 0.04 & 0 \\ 0 & 0.144 \end{bmatrix}, \\ A_{12} \Lambda_2 + B_1 D_2 &= \begin{bmatrix} 0.06 & 0.01 \\ 0 & 0.01 \end{bmatrix}, \\ A_{21} \Lambda_1 + B_2 D_1 &= \begin{bmatrix} 0 & 0.02 \\ 0 & 0 \end{bmatrix}, \\ \bar{A}_{22} \Lambda_2 + B_2 D_2 &= \begin{bmatrix} 0.02 & 0.03 \\ 0 & 0.14 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} & \begin{bmatrix} A_{11} \Lambda_1 + B_1 D_1 & A_{12} \Lambda_2 + B_1 D_2 \\ A_{21} \Lambda_1 + B_2 D_1 & A_{22} \Lambda_2 + B_2 D_2 \end{bmatrix} \\ & \quad \times \begin{bmatrix} \mathbf{1}_{n_1} \\ \mathbf{1}_{n_2} \end{bmatrix} = \begin{bmatrix} -0.05 \\ -0.006 \\ -0.03 \\ -0.01 \end{bmatrix}. \end{aligned}$$

Step 2. From (33) we obtain the gain matrix $K = [K_1, K_2]$,

$$K_1 = \begin{bmatrix} -0.4 & -0.2 \end{bmatrix} \begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix} = \begin{bmatrix} -1 & -0.5 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -0.4 & -0.2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3.33 \end{bmatrix} = \begin{bmatrix} -2 & -0.67 \end{bmatrix}.$$

The closed-loop system is positive since the matrices

$$\begin{aligned} \bar{A}_{11} + B_1 K_1 &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.36 \end{bmatrix}, \\ A_{12} + B_1 K_2 &= \begin{bmatrix} 0.3 & 0.033 \\ 0 & 0.033 \end{bmatrix}, \\ A_{21} + B_2 K_1 &= \begin{bmatrix} 0 & 0.05 \\ 0 & 0 \end{bmatrix}, \\ \bar{A}_{22} + B_2 K_2 &= \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.467 \end{bmatrix} \end{aligned}$$

have all nonnegative entries.

The closed-loop system is asymptotically stable since its characteristic polynomial

$$\begin{aligned} \det \begin{bmatrix} \mathbb{I}_{n_1} z - (A_{11} + B_1 K_1) & -(A_{12} + B_1 K_2) \\ -(A_{21} + B_2 K_1) & \mathbb{I}_{n_2} z - (A_{22} + B_2 K_2) \end{bmatrix} \\ = z^4 + 0.773z^3 + 0.173z^2 + 0.01z + 0.0002 \end{aligned}$$

has positive coefficients.

6. Concluding remarks

A new class of 2D fractional linear systems was introduced. Fractional 2D state equations of linear systems were given and their solutions were derived using the 2D \mathcal{Z} -transform. The classical Cayley-Hamilton theorem was extended to 2D fractional systems described by the Roesser model. Necessary and sufficient conditions for the positivity and stabilization by state feedback of fractional 2D linear systems were established. A procedure for the computation of the gain matrix was proposed and illustrated by a numerical example.

These deliberations can be easily extended to fractional 2D linear systems with delays described by the Roesser model. An extension of this study to fractional 2D continuous-time systems is an open problem.

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References

- Bose, N. K. (1982). *Applied Multidimensional Systems Theory*, Van Nostrand Reinhold Co., New York, NY.
- Bose, N. K. (1985). *Multidimensional Systems Theory Progress, Directions and Open Problems*, D. Reidel Publishing Co., Dordrecht.
- Busłowicz, M. (2008). Simple stability conditions for linear positive discrete-time systems with delays, *Bulletin of the Polish Academy of Sciences: Technical Sciences* **56**(4): 325–328.
- Busłowicz, M. and Kaczorek, T. (2009). Simple conditions for practical stability of positive fractional discrete-time linear systems, *International Journal of Applied Mathematics and Computer Science* **19**(2): 263–269, DOI: 10.2478/v10006-009-0022-6.
- Farina, E. and Rinaldi, S. (2000). *Positive Linear Systems; Theory and Applications*, J. Wiley, New York, NY.
- Fornasini, E. and Marchesini, G. (1976). State-space realization theory of two-dimensional filters, *IEEE Transactions on Automatic Control* **AC-21**(4): 484–491.
- Fornasini, E. and Marchesini, G. (1978). Double indexed dynamical systems, *Mathematical Systems Theory* **12**(1): 59–72.
- Galkowski, K. (2001). *State Space Realizations of Linear 2D Systems with Extensions to the General nD (n > 2) Case*, Springer-Verlag, London.
- Kaczorek, T. (1985). *Two-Dimensional Linear Systems*, Springer-Verlag, London.
- Kaczorek, T. (1996). Reachability and controllability of non-negative 2D Roesser type models, *Bulletin of the Polish Academy of Sciences: Technical Sciences* **44**(4): 405–410.
- Kaczorek, T. (2001). *Positive 1D and 2D Systems*, Springer-Verlag, London.
- Kaczorek, T. (2005). Reachability and minimum energy control of positive 2D systems with delays, *Control and Cybernetics* **34**(2): 411–423.
- Kaczorek, T. (2007). Reachability and controllability to zero of positive fractional discrete-time systems, *Machine Intelligence and Robotic Control* **6**(4): 139–143.
- Kaczorek, T. (2008a). Asymptotic stability of positive 1D and 2D linear systems, in K. Malinowski and L. Rutkowski (Eds), *Recent Advances in Control and Automation*, Akademicka Oficyna Wydawnicza EXIT, Warsaw, pp. 41–52.
- Kaczorek, T. (2008b). Asymptotic stability of positive 2D linear systems, *Proceedings of the 13th Scientific Conference on Computer Applications in Electrical Engineering, Poznań, Poland*, pp. 1–5.
- Kaczorek, T. (2008c). Fractional 2D linear systems, *Journal of Automation, Mobile Robotics & Intelligent Systems* **2**(2): 5–9.
- Kaczorek, T. (2008d). Positive different orders fractional 2D linear systems, *Acta Mechanica et Automatica* **2**(2): 51–58.

- Kaczorek, T. (2009a). LMI approach to stability of 2D positive systems, *Multidimensional Systems and Signal Processing* **20**(1): 39–54.
- Kaczorek, T. (2009b). Positive 2D fractional linear systems, *International Journal for Computation and Mathematics in Electrical and Electronic Engineering, COMPEL* **28**(2): 341–352.
- Kaczorek, T. (2009c). Positivity and stabilization of 2D linear systems, *Discussiones Mathematicae, Differential Inclusions, Control and Optimization* **29**(1): 43–52.
- Kaczorek, T. (2009d). Stabilization of fractional discrete-time linear systems using state feedbacks, *Proceedings of the LogiTrans Conference, Szczyrk, Poland*, pp. 2–9.
- Kurek, J. (1985). The general state-space model for a two-dimensional linear digital systems, *IEEE Transactions on Automatic Control* **AC-30**(2): 600–602.
- Miller, K. S. and Ross, B. (1993). *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, NY.
- Nashimoto, K. (1984). *Fractional Calculus*, Descartes Press, Koriyama.
- Oldham, K. B. and Spanier, J. (1974). *The Fractional Calculus*, Academic Press, New York, NY.
- Podlubny, I. (1999). *Fractional Differential Equations*, Academic Press, San Diego, CA.
- Roesser, R. (1975). A discrete state-space model for linear image processing, *IEEE Transactions on Automatic Control* **AC-20**(1): 1–10.
- Twardy, M. (2007). An LMI approach to checking stability of 2D positive systems, *Bulletin of the Polish Academy of Sciences: Technical Sciences* **55**(4): 385–395.
- Valcher, M. E. (1997). On the internal stability and asymptotic behavior of 2D positive systems, *IEEE Transactions on Circuits and Systems—I* **44**(7): 602–613.



Tadeusz Kaczorek received the M.Sc., Ph.D. and D.Sc. degrees in electrical engineering from the Warsaw University of Technology in 1956, 1962 and 1964, respectively. In the years 1968–69 he was the dean of the Electrical Engineering Faculty, and in the period of 1970–73 he was a deputy rector of the Warsaw University of Technology. In 1971 he became a professor and in 1974 a full professor at the Warsaw University of Technology. Since 2003 he

has been a professor at Białystok Technical University. In 1986 he was elected a corresponding member and in 1996 a full member of the Polish Academy of Sciences. In the years 1988–1991 he was the director of the Research Centre of the Polish Academy of Sciences in Rome. In 2004 he was elected an honorary member of the Hungarian Academy of Sciences. He has been granted honorary doctorates by several universities. His research interests cover the theory of systems and automatic control systems theory, especially singular multidimensional systems, positive multidimensional systems, and singular positive 1D and 2D systems. He has initiated research in the field of singular 2D and positive 2D systems. He has published 21 books (six in English) and over 850 scientific papers. He has also supervised 67 Ph.D. theses. He is the editor-in-chief of the *Bulletin of the Polish Academy of Sciences: Technical Sciences* and a member of editorial boards of about ten international journals.



Krzysztof Rogowski received his M.Sc. degree in electrical engineering from Białystok Technical University, Poland, in 2007. Currently he is a Ph.D. student at the Faculty of Electrical Engineering of the same university. His research interests focus on positive and fractional 1D and 2D systems, computer simulation and analysis.

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