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POSITIVITY OF ANTICANONICAL DIVISORS FROM THE VIEWPOINT OF FANO CONIC BUNDLES

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Abstract

We give the first examples of flat fiber type contractions of Fano manifolds onto varieties that are not weak Fano, and we prove that these morphisms are Fano conic bundles. We also review some known results about the interaction between the positivity properties of anticanonical divisors of varieties of contractions.

1. Introduction

Given a contraction $\varphi\colon X \to Y$, *i.e.* a surjective morphism with connected fibers between normal complex projective varieties, it is a natural and fundamental problem to try to understand the link between the positivity properties of the anticanonical divisors of our varieties. This matter can be viewed as a Positivity Problem for the anticanonical divisor of Y.

Assume that X, Y are smooth. In [10, Corollary 2.9], Kollár, Miyaoka and Mori proved that when such a morphism is smooth, and X is Fano, then so is Y. Under the same assumption of smoothness of the morphism, in [8, Theorem 1.1] Fujino and Gongyo showed that if X is weak Fano (that is $-K_X$ is nef and big) then so is Y; while in [5, Theorem 1.1] Birkar and Chen proved that if $-K_X$ is semiample, then so is $-K_Y$.

Weaker results are known when φ is not smooth: for instance, in [16, Theorem 2.9] Prokhorov and Shokurov proved that if $\varphi\colon X \to Y$ is a contraction and X is log Fano, then Y is also log Fano, in particular $-K_Y$ is big. We recall that a normal projective variety X is log Fano if there exists an effective $\mathbb{Q}$-divisor $\triangle$ of X such that $-(K_X + \triangle)$ has a positive multiple which is Cartier and ample and the pair $(X, \triangle)$ is klt (see [11] for the definition of klt). Moreover, we refer the reader to [6, §3.6] for an overview and proofs of some results mentioned above.

Let us denote by $\mathcal{N}_1(X)$ the $\mathbb{R}$-vector space of one-cycles with real coefficients, modulo numerical equivalence, whose dimension is the *Picard number* ρ_X. A *Fano conic bundle* $f\colon X \to Y$ is a contraction where X is smooth and Fano, and such that all fibers of f are one-dimensional. We say that a Fano conic bundle $f\colon X \to Y$ is *elementary* if $\rho_X - \rho_Y = 1$, otherwise f is called *non-elementary*. We refer the reader to [17] for a detailed study of non-elementary Fano conic bundles.

In this note we discuss some recent developments about the Positivity Problem introduced above from the viewpoint of Fano conic bundles, and our main goal is to construct some new examples.

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In [19] Wiśniewski posed a question related to our positivity problem: given a Fano conic bundle $f\colon X \to Y$, is Y also Fano? In [19, §5, 5.2] he gave an example of Fano conic bundle onto a variety that is not Fano but it is weak Fano. This example was generalized by Debarre in [6, Example 3.16 (3)]. Moreover, in [6, footnote 14] Debarre observed that Wiśniewski's construction does not seem able to provide an example of Fano conic bundle whose target is not weak Fano. In this paper we overcome this problem, in fact with a modification of Wiśniewski's construction we get the first examples of Fano conic bundles onto varieties that are not weak Fano, by proving the following result:

Theorem 1.1. *For every $m \in \mathbb{Z}$, $m \geq 2$ there exists an elementary Fano conic bundle $f\colon X \to Y$ where $\dim X = 3(m+1)$, and $-K_Y$ is not nef.*

Notice that there exist examples where $f\colon X \to Y$ is a flat and not smooth contraction, with X weak Fano and $-K_Y$ not nef (see [8, Example 4.6]). Theorem 1.1 implies something stronger: $-K_Y$ is not always nef even if X is a Fano manifold.

We conclude with Section 4 where some open problems are presented.

2. Preliminaries

2.1. Notation and terminology. We work over the field of complex numbers. Let X be a smooth projective variety with arbitrary dimension n.

$\mathcal{N}_1(X)$ (respectively, $\mathcal{N}^1(X)$) is the $\mathbb{R}$-vector space of one-cycles (respectively, Cartier divisors) with real coefficients, modulo numerical equivalence.

$\dim \mathcal{N}_1(X) = \dim \mathcal{N}^1(X) =: \rho_X$ is the *Picard number* of X.

Let C be a one-cycle of X, and D a divisor of X. We denote by $[C]$ (respectively, $[D]$) the numerical equivalence class in $\mathcal{N}_1(X)$ (respectively in $\mathcal{N}^1(X)$).

We denote by $\mathrm{Eff}(X)$ the *Effective cone* of X, that is the convex cone inside $\mathcal{N}^1(X)$ spanned by classes of effective divisors.

Let $D \subset X$ be a Cartier divisor. The divisor D is *movable* if there exists $m > 0$, $m \in \mathbb{Z}$ such that $\operatorname{codim} \mathrm{Bs}\,|mD| \geq 2$.

We denote by $\mathrm{Mov}(X)$ the *Movable cone* of X, that is the convex cone of $\mathcal{N}^1(X)$ spanned by classes of movable divisors.

A *contraction* of X is a surjective morphism $\varphi\colon X \to Y$ with connected fibers, where Y is normal and projective.

We denote by $\mathrm{Exc}(\varphi)$ the *exceptional locus* of φ, *i.e.* the locus where φ is not an isomorphism.

2.2. Preliminaries on Fano conic bundles. A contraction $f\colon X \to Y$ is a *conic bundle* if $-K_X$ is f-ample and every fiber of f is one-dimensional. When X is Fano, we call such a contraction *Fano conic bundle*.

The following theorem is due to Ando ([2, Theorem 3.1]) and it is a generalization in higher dimension of Mori's result in dimension 3 (see [13, Theorem 3.5, (3.5.1)]). For the second part we refer the reader to [4, Proposition 1.2 (ii)] and [19, §4].

An important consequence of this theorem is that conic bundles can be easily characterized among fiber type contractions of smooth varieties.

Theorem 2.1 ([2], Theorem 3.1 (ii))**.** *Let X be a smooth projective variety and let $f\colon X \to Y$ be a contraction where $-K_X$ is f-ample and every fiber is one-dimensional. Then Y is*

smooth.

Moreover, there exists a locally free sheaf $\mathcal{E}$ on Y of rank 3 such that the projection $\pi\colon \mathbb{P}(\mathcal{E}) \to Y$ contains X embedded over Y as a divisor whose restriction to every fiber of π is one-dimensional and it is an element of $|\mathcal{O}_{\mathbb{P}^2}(2)|$, and $f = \pi_{|X}$. The pushforward $f_*\mathcal{O}_X(-K_X)$ can be taken as the above $\mathcal{E}$.

Moreover, X is defined by the vanishing of a section $\sigma \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2\xi + \pi^*M))$, where M is a divisor of Y and using the adjunction formula it is easy to check that $M \sim -\det\mathcal{E} - K_Y$.

In [4], Beauville introduced the following definition of the *discriminant divisor* of a conic bundle $f\colon X \to Y$:

$$\triangle_f := \{y \in Y \mid f^{-1}(y) \text{ is singular}\}.$$

Using the same notation introduced above, we recall from [18, §1.7] that

$$\triangle_f \in |(\det\mathcal{E})^{\otimes 2} \otimes \mathcal{O}_Y(3M)|. \tag{1}$$

We can express the regularity conditions of the fibers of f in terms of the properties of $\triangle_f$. Indeed, we obtain the following explicit description of the singular locus of the discriminant divisor (see for instance [3, §2], [18, Proposition 1.8, (5.c)]):

$$\operatorname{Sing}(\triangle_f) = \{y \in Y \mid f^{-1}(y) \text{ is non-reduced}\}. \tag{2}$$

Now, we recall the last results related to $\triangle_f$ that will be needed.

The following results are consequences of [19, Proposition 4.3] and its proof.

Proposition 2.2 ([19])**.** *Let $f\colon X \to Y$ be a Fano conic bundle. Assume that Y is not Fano. Let C be a rational curve of Y such that $-K_Y \cdot C \leq 0$. Then $C \subseteq \operatorname{Sing}(\triangle_f)$.*

Corollary 2.3 ([19])**.** *Let $f\colon X \to Y$ be a Fano conic bundle. If f does not have non-reduced fibers, then Y is Fano. Moreover, if $\dim X \leq 4$ or $\rho_X \leq 2$, then Y is Fano.*

We are going to study our Positivity Problem for the anticanonical divisors from the viewpoint of Fano conic bundles.

We focus on the following problem studied in [19, §4] by Wiśniewski: given a Fano conic bundle $f\colon X \to Y$, is Y Fano or not?

The following result arises from the study of non-elementary Fano conic bundles of [17]. In particular, it is a consequence of [17, Theorem 1.1] and allows us to give an answer to Wiśniewski's question, in many cases.

Theorem 2.4 ([17], Corollary 1.2)**.** *Let $f\colon X \to Y$ be a Fano conic bundle, where Y is not Fano. Then $\rho_X - \rho_Y = 2$ or $\rho_X - \rho_Y = 1$. If $\rho_X - \rho_Y = 2$ then $\rho_Y \geq 3$, and Y has a smooth $\mathbb{P}^1$-fibration.*[1]

Thanks to the above theorem we deduce that the target of a non-elementary Fano conic bundle is often Fano. In the next Section we use Theorem 2.4 to prove that the new examples of Fano conic bundles are elementary.

[1] A smooth $\mathbb{P}^1$-fibration is a smooth morphism with fibers isomorphic to $\mathbb{P}^1$.

3. New examples of elementary Fano conic bundles

In this section we give examples of Fano conic bundles $f\colon X \to Y$, where $-K_Y$ is not nef. Putting our construction in a more general setting, we get the first examples of equidimensional fiber type contractions of Fano varieties onto varieties Y which are not weak Fano. Moreover, since we obtain equidimensional morphisms between smooth projective varieties, these are flat morphisms, and by Corollary 2.3 we deduce that they have non-reduced fibers.

Let $m \in \mathbb{Z}$, $m \geq 2$, and set[2] $Y = \mathbb{P}_{\mathbb{P}^{3m}}(\mathcal{O}_{\mathbb{P}^{3m}} \oplus \mathcal{O}_{\mathbb{P}^{3m}}(2m) \oplus \mathcal{O}_{\mathbb{P}^{3m}}(2m))$.

One has $\rho_Y = 2$, in particular $\mathcal{N}^1(Y)$ is generated by the class of a divisor D associated with the line bundle $\mathcal{O}_Y(1)$ and by the class of the pullback H of a hyperplane in $\mathbb{P}^{3m}$. Let $\pi\colon Y \to \mathbb{P}^{3m}$ be the natural projection. Notice that D is a nef divisor, not ample, and that it is base point free (see for instance [12, §2.3, Lemma 2.3.2]). Since D and H are both nef, not ample divisors of Y and $\rho_Y = 2$, then $\mathrm{Nef}\,(Y) = \langle [D], [H] \rangle$ (see Figure 1).

Denote by $V \subset Y$ the section corresponding to the trivial quotient $\mathcal{O}_{\mathbb{P}^{3m}} \oplus \mathcal{O}_{\mathbb{P}^{3m}}(2m) \oplus \mathcal{O}_{\mathbb{P}^{3m}}(2m) \twoheadrightarrow \mathcal{O}_{\mathbb{P}^{3m}}$, so that $D_{|V} \cong \mathcal{O}_V$. We have:

$$-K_Y = 3D + (1-m)H$$

so that $-K_Y$ is not nef, indeed if C is a line contained in $V \cong \mathbb{P}^{3m}$, then $-K_Y \cdot C = 1 - m < 0$.

The section V is a complete intersection of two divisors $G_1, G_2 \sim D - 2mH$. Indeed by the two possible factorizations of the trivial quotient

$$\mathcal{O}_{\mathbb{P}^{3m}} \oplus \mathcal{O}_{\mathbb{P}^{3m}}(2m) \oplus \mathcal{O}_{\mathbb{P}^{3m}}(2m) \twoheadrightarrow \mathcal{O}_{\mathbb{P}^{3m}} \oplus \mathcal{O}_{\mathbb{P}^{3m}}(2m) \twoheadrightarrow \mathcal{O}_{\mathbb{P}^{3m}},$$

we have the immersions:

$$V \cong \mathbb{P}^{3m} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^{3m}}) \hookrightarrow G_1 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^{3m}} \oplus \mathcal{O}_{\mathbb{P}^{3m}}(2m)) \hookrightarrow Y$$
$$V \hookrightarrow G_2 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^{3m}} \oplus \mathcal{O}_{\mathbb{P}^{3m}}(2m)) \hookrightarrow Y$$

and V is given by the intersection between G_1 and G_2. For $i = 1, 2$ we have $\mathcal{O}_{G_i}(1) = D_{|G_i}$ and:

$$-K_{G_i} = 2D_{|G_i} + (m+1)H_{|G_i}$$

and by the adjunction formula we find that:

$$K_{G_i} = (G_i - 3D + (m-1)H)_{|G_i}$$

so that $(G_i - D + 2mH)_{|G_i} \sim 0$. But $\rho_{G_i} = \rho_Y = 2$ and the restriction $\mathrm{Pic}\,(Y) \to \mathrm{Pic}\,(G_i)$ is an isomorphism, then $G_i \sim D - 2mH$.

Since D is base point free, the map associated to a sufficiently large multiple of D is a contraction, which we denote by $\varphi\colon Y \to Y'$.

Using that $D_{|V} \cong \mathcal{O}_V$, it is easy to check that φ contracts V to a point, hence $V \subseteq \mathrm{Exc}\,(\varphi)$.

Notice that $\mathrm{Exc}\,(\varphi) = V$. Indeed, if C is an irreducible curve of Y such that $\varphi(C)$ is a point, since $\varphi_*(C) = 0$ one has that $D \cdot C = 0$, instead $H \cdot C > 0$, then $G_i \cdot C < 0$ for $i = 1, 2$, and $\mathrm{Exc}\,(\varphi) \subseteq G_1 \cap G_2 = V$. In particular, we observe that φ is a small elementary contraction.

Lemma 3.1. *Notation as above. Then*

$$\mathrm{Bs}\,(|2D - 2mH|) = \mathrm{Bs}\,(|D - mH|) = \mathrm{Bs}\,(|2D - mH|) = V.$$

[2] As usual, we follow Grothendieck's notation: for a vector bundle $\mathcal{E}$, the projectivization $\mathbb{P}(\mathcal{E})$ is the space of *hyperplanes* in the fibers of $\mathcal{E}$.

Proof. For every curve $C \subset V$, we have $D \cdot C = 0$ and $H \cdot C > 0$, thus $(2D - 2mH) \cdot C < 0$, then $V \subseteq \mathrm{Bs}\,(|2D - 2mH|)$.

We know that $\mathrm{Bs}\,(|2D - 2mH|) \subseteq \mathrm{Bs}\,(|D - mH|)$, thus we are left to prove that $\mathrm{Bs}\,(|D - mH|) \subseteq V$.

We have already observed that V is a complete intersection of two divisors of $|D - 2mH|$, hence $\mathrm{Bs}\,(|D - 2mH|) \subseteq V$. Being H base point free, it follows that $\mathrm{Bs}\,(|D - mH|) \subseteq \mathrm{Bs}\,(|D - 2mH|)$. Hence we get the following inclusions:

$$V \subseteq \mathrm{Bs}\,(|2D - 2mH|) \subseteq \mathrm{Bs}\,(|D - mH|) \subseteq \mathrm{Bs}\,(|D - 2mH|) \subseteq V,$$

so that $\mathrm{Bs}\,(|2D - 2mH|) = \mathrm{Bs}\,(|D - mH|) = V$. Using the same method, we observe that $V \subseteq \mathrm{Bs}\,(|2D - mH|)$ and since D is base point free one has $\mathrm{Bs}\,(|2D - mH|) \subseteq \mathrm{Bs}\,(|D - mH|) = V$, thus $\mathrm{Bs}\,(|2D - mH|) = V$. □

Proposition 3.2. *Let Y be as above. Let us consider the locally free sheaf $\mathcal{E} := \mathcal{O}(D)^{\oplus 2} \oplus \mathcal{O}(D + mH)$ on Y of rank 3, and let $Z = \mathbb{P}_Y(\mathcal{E}) \xrightarrow{p} Y$ be the projection. Set $\xi := \mathcal{O}_Z(1)$. Then a general divisor $X \in |2\xi - 2mp^*H|$ is smooth, and $p_{|X}\colon X \to Y$ is an elementary Fano conic bundle.*

Proof. By Lemma 3.1 we deduce that $\mathcal{E} \otimes \mathcal{O}_Y(-mH) = \mathcal{O}_Y(D - mH)^{\oplus 2} \oplus \mathcal{O}_Y(D)$ is globally generated outside V, and by [12, Lemma 2.3.2] the tautological bundle $\xi - mp^*H$ of $\mathbb{P}_Y(\mathcal{E} \otimes \mathcal{O}(-mH))$ is spanned outside $p^{-1}(V)$.

Being $\mathrm{Bs}\,(|2\xi - 2mp^*H|) \subseteq \mathrm{Bs}\,(|\xi - mp^*H|)$, also $|2\xi - 2mp^*H|$ is globally generated outside $p^{-1}(V)$; in particular $|2\xi - 2mp^*H| \neq \emptyset$.

By Bertini's theorem it follows that a general divisor $X \in |2\xi - 2mp^*H|$ is smooth outside $p^{-1}(V)$.

Now we prove that $-K_Z - X$ is ample on Z. This will imply that X is Fano, once that we show the smoothness of X. We have:

$$-K_Z = 3\xi - p^*(K_Y + 3D + mH) = 3\xi + (1 - 2m)p^*H.$$

Being $X \in |2\xi - 2mp^*H|$, one has $-K_Z - X = \xi + p^*H$.

Notice that $\xi + p^*H$ is the tautological bundle of $\mathbb{P}_Y(\mathcal{E} \otimes \mathcal{O}(H)) = \mathbb{P}_Y(\mathcal{O}(D + H)^{\oplus 2} \oplus \mathcal{O}(D + (m + 1)H))$ and being $\mathcal{E} \otimes \mathcal{O}(H)$ sum of ample line bundles of Y, we conclude that $-K_Z - X$ is ample on Z.

Now we show that the fibers of $p\colon X \to Y$ are isomorphic to plane conics.

We observe that the divisor X is given by a section of $H^0(Z, \mathcal{O}_Z(2\xi - 2mp^*H)) \cong H^0(Y, \mathrm{Sym}^2(\mathcal{E} \otimes \mathcal{O}(-mH)))$. Being

$$\mathrm{Sym}^2(\mathcal{E} \otimes \mathcal{O}(-mH))) = \mathcal{O}_Y(2(D - mH))^{\oplus 3} \oplus \mathcal{O}_Y(2D - mH)^{\oplus 2} \oplus \mathcal{O}_Y(2D),$$

X can be represented by a symmetric matrix of sections:

$$S = \begin{pmatrix} s_1 & s_2 & \lambda_1 \\ s_2 & s_3 & \lambda_2 \\ \lambda_1 & \lambda_2 & \sigma \end{pmatrix}$$

with $s_1, s_2, s_3 \in H^0(Y, \mathcal{O}_Y(2(D - mH)))$, $\lambda_1, \lambda_2 \in H^0(Y, \mathcal{O}_Y(2D - mH))$ and $\sigma \in H^0(Y, \mathcal{O}_Y(2D))$. We prove that the matrix S vanishes nowhere.

Since D is base point free and $D_{|V} \cong \mathcal{O}_V$, σ can be chosen such that $\sigma_{|V}$ is constant and non-zero.

Then the possible common zeros of the sections of S are outside V. But by Lemma 3.1 we know that $\mathrm{Bs}\,(|2D - 2mH|) = \mathrm{Bs}\,(|2D - mH|) = V$, hence for a general choice of these sections they have no common zeros and this allows to deduce that a general divisor $X \in |2\xi - 2mp^*H|$ gives a conic fibration over Y.

It remains to show that X is smooth in a neighborhood of $p^{-1}(V) \cap X$. To this end, we take an open subset $U \subset Y$ which trivializes $\mathcal{E}$. If $[z_0 : z_1 : z_2]$ are projective coordinates of the trivialization of $\mathbb{P}(\mathcal{E})$ over U compatible with the splitting of $\mathcal{E}$, then on $p^{-1}(U) \cong U \times \mathbb{P}^2$, the general divisor $X \in |2\xi - 2mp^*H|$ is defined by the equation:

$$F := (z_0, z_1, z_2)\, S\, (z_0, z_1, z_2)^t = 0,$$

namely:

$$F = s_1 z_0^2 + 2s_2 z_0 z_1 + 2\lambda_1 z_0 z_2 + s_3 z_1^2 + 2\lambda_2 z_1 z_2 + \sigma z_2^2 = 0.$$

Using Lemma 3.1, the equation F over V degenerates to $\sigma z_2^2 = 0$. Therefore, to prove the smoothness of a general X we have to compute the differential dF over $W := p^{-1}(V) \cap X = p^{-1}(V) \cap \{z_2 = 0\}$.

Taking the covering of $\mathbb{P}^2$ given by the standard open subsets we can compute dF locally over each $U \times B_i$ where $B_i = \{(z_0 : z_1 : z_2) \in \mathbb{P}^2 \mid z_i \neq 0\} \cong \mathbb{A}^2$ for every $i = 0, 1, 2$.

Take $U \times B_0$, where $B_0 \cong \mathbb{A}^2_{y,z}$, with coordinates $y = \frac{z_1}{z_0}$, $z = \frac{z_2}{z_0}$. We evaluate the differential of this equation over $(U \times B_0) \cap \{z = 0\}$. Considering all the vanishing of the sections of the matrix S (that are regular functions over U) due to Lemma 3.1, this differential becomes:

$$ds_1 + 2yds_2 + y^2 ds_3.$$

In the same way, we consider $U \times B_1$, where $B_1 \cong \mathbb{A}^2_{x,z}$, with coordinates $x = \frac{z_0}{z_1}$, $z = \frac{z_2}{z_1}$. Then we evaluate dF over $(U \times B_1) \cap \{z = 0\}$, getting:

$$x^2 ds_1 + 2xds_2 + ds_3.$$

By these local computations, it follows that:

$$dF_{|W} = z_0^2 ds_1 + 2z_0 z_1 ds_2 + z_1^2 ds_3.$$

We have already observed that $V \subset Y$ is given by a complete intersection of two divisors $G_1, G_2 \in |D - 2mH|$. Let ν_1, ν_2 be the two sections of $H^0(Y, \mathcal{O}_Y(D - 2mH))$ corresponding to these two divisors. Since G_1 and G_2 intersect transversally along V, their differentials $d\nu_1$, $d\nu_2$ are independent everywhere on V.

To prove the smoothess of X, it is enough to show that there exists a divisor in $|2\xi - 2mp^*H|$ which is smooth in a neighborhood of $p^{-1}(V) \cap X$.

To this end, let $\sigma' \in H^0(Y, \mathcal{O}_Y(D))$ be a section such that $\sigma'_{|V}$ is constant and non-zero, and set $s_1 = \sigma'\nu_1$ and $s_2 = s_3 = \sigma'\nu_2$.

Then $ds_{1|V} = \sigma'_{|V} d\nu_{1|V}$, $ds_{2|V} = ds_{3|V} = \sigma'_{|V} d\nu_{2|V}$, and:

$$dF_{|W} = \sigma'_{|V}(z_0^2 d\nu_1 + z_1(2z_0 + z_1)d\nu_2)$$

does not vanish on W. Thus X is smooth, so that $p_{|X}$ is a Fano conic bundle. By Theorem 2.4 we conclude that $\rho_X - \rho_Y = 1$ because $\rho_Y = 2$, hence the statement. □

Using the above construction and results we prove Theorem 1.1. Proof of Theorem 1.1. Setting as in Proposition 3.2. Take $f := p_{|X}$.

We have already proved that $f\colon X \to Y$ is an elementary Fano conic bundle and that $-K_Y$ is not nef. Finally, since $\pi\colon Y \to \mathbb{P}^{3m}$ is a $\mathbb{P}^2$-bundle, $\dim(Y) = 3m + 2$, and $\dim(X) =$

$3(m+1)$. □

Keeping the notation introduced above, we conclude this section with two remarks. In the first one we describe the singular locus of the discriminant divisor of our examples of Fano conic bundles. In the last remark we study the effective and movable cones of the targets of these morphisms.

Remark 3.3. By the proof of Proposition 3.2 we can observe that the fibers of $f\colon X \to Y$ over V are not reduced, so that by (2), it follows that $V \subseteq \mathrm{Sing}\,(\triangle_f)$.

Moreover, by (1) we know that $\triangle_f \sim 2\det\mathcal{E} + 3M$ where in our case $\mathcal{E} := \mathcal{O}(D)^{\oplus 2} \oplus \mathcal{O}(D+mH)$, $M \sim -2mH$, so that $\triangle_f \sim 2(3D-2mH)$, and being $\mathrm{Bs}\,(|2(3D-2mH)|) \subseteq \mathrm{Bs}\,(|3D-2mH|) \subseteq \mathrm{Bs}\,(|D-2mH|) = V$, we get $\mathrm{Sing}\,(\triangle_f) = V$. Finally, if $C \subset V \cong \mathbb{P}^{3m}$ is a line, we can compute that $\triangle_f \cdot C = -4m$.

Remark 3.4. Let us denote by $g\colon Y \dashrightarrow \tilde{Y}$ the flip of $\varphi\colon Y \to Y'$, where we recall that φ is the map associated to a sufficiently large multiple of D. We prove that:

$$\mathrm{Eff}\,(Y) = \mathrm{Mov}\,(Y) = \mathrm{Nef}\,(Y) \cup g^*\,\mathrm{Nef}\,(\tilde{Y}).$$

To this end, we show that the divisor $D-2mH$ gives a fiber type contraction on $\tilde{Y}$, so that $[D-2mH]$ sits in the boundary of the Effective and Movable Cone of Y (see Figure 1 below).

Recall that $G_1 \sim G_2 \sim D-2mH$. Let $i \in \{1,2\}$. First we show that in Y we have $G_i \cdot C = 0$, for every irreducible curve $C \subset G_i$ such that $C \cap V = \emptyset$. To this end, we take one of the two divisor G_i, for simplicity assume that it is G_1. If $C \subset G_1$ is an irreducible curve such that $C \cap V = \emptyset$, by recalling that $G_1 \cap G_2 = V$ and $G_1 \sim G_2$, it follows that $G_2 \cap C = \emptyset$, and $G_1 \cdot C = G_2 \cdot C = 0$.

Recall that G_1 is a $\mathbb{P}^1$-bundle over $\mathbb{P}^{3m}$ and that $V \subset G_1$ is a section. Denote by $\tilde{G}_1 \subset \tilde{Y}$ the transform of G_1 through the flip g.

We observe that $g_{|G_1}\colon G_1 \to \tilde{G}_1$ is the divisorial contraction $\varphi_{|G_1}$ which sends V to a point, and $\rho_{\tilde{G}_1} = 1$. By what we have already shown, $\tilde{G}_1 \cdot \Gamma = 0$ for a general irreducible curve $\Gamma \subset \tilde{G}_1$, and since $\rho_{\tilde{G}_1} = 1$, it follows that $\tilde{G}_1 \cdot \tilde{\Gamma} = 0$ for every irreducible curve $\tilde{\Gamma} \subset \tilde{G}_1$.

Then $\tilde{G}_1$ is a nef divisor of $\tilde{Y}$, and being $\tilde{Y}$ a toric variety, $\tilde{G}_1$ is semiample and gives a

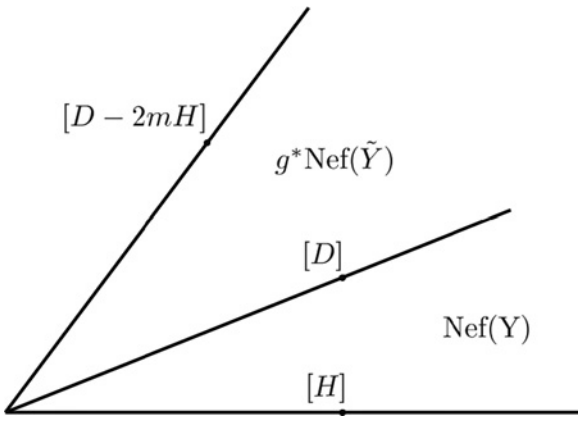

Fig. 1. We denote by $g\colon Y \dashrightarrow \tilde{Y}$ the flip of the small contraction $\varphi\colon Y \to Y'$.
We get $\mathrm{Eff}\,(Y) = \mathrm{Mov}\,(Y) = \mathrm{Nef}\,(Y) \cup g^*\,\mathrm{Nef}\,(\tilde{Y})$.

contraction onto $\mathbb{P}^1$ which sends itself to a point. In Figure 1 we represent the cones of Y.

4. Final Comments and Open Questions

Let us denote by $f\colon X \to Y$ a non-elementary Fano conic bundle, and by $\triangle_f$ its discriminant divisor. Thanks to Theorem 2.4 the positivity properties of $-K_Y$ are well known when $\rho_X - \rho_Y \geq 3$.

When $\rho_X - \rho_Y = 2$, some open questions arise. In particular, we do not know if there exists a Fano conic bundle $f\colon X \to Y$ with non-reduced fibers and $\rho_X - \rho_Y = 2$. Using the classification of Fano 3-folds of [14, 15] and [17, Theorem 1.1] it is easy to check that there are no examples of Fano conic bundles $f\colon X \to S$ where $\dim(X) = 3$, $\rho_X - \rho_Y = 2$ and f has non-reduced fibers. Our questions are the following:

QUESTION 1. Are there examples of Fano conic bundles $f\colon X \to Y$ where $\rho_X - \rho_Y = 2$ and f has non-reduced fibers?

QUESTION 2. Are there examples of Fano conic bundles $f\colon X \to Y$ where $\rho_X - \rho_Y = 2$ and Y is not Fano?

Notice that the two questions are related by Corollary 2.3.

The following corollary is a consequence of some results of Viehweg and Kollár (see [9] and references therein). In particular, it follows applying [9, Theorem 3.30] and [9, Lemma 3.5] when $f\colon X \to Y$ is a Fano conic bundle:

Corollary 4.1. *Let $f\colon X \to Y$ be a Fano conic bundle. Denote by $\triangle_f$ its discriminant divisor. If $C \subset Y$ is an irreducible curve such that $-K_Y \cdot C < 0$, then $C \subset \triangle_f$.*

On the other hand, by Proposition 2.2 we know that *rational* curves $C \subset Y$ with $-K_Y \cdot C < 0$ cover at most the closed subset $\mathrm{Sing}(\triangle_f)$ which is properly contained in $\triangle_f$.

By Corollary 2.3 it seems that there is a close link between the kind of singularities of the fibers of f and the positivity properties of $-K_Y$. This remark induces the following question posed by Ejiri in [7], where we refer the reader to [1, Definition 1.3] for the definition of semi-log canonical singularities.

QUESTION 3. Let X be a smooth Fano variety. Let $f\colon X \to Y$ be an equidimensional fiber type contraction. Assume that f is not a smooth morphism, but its fibers have some mild singularities, for example semi-log canonical singularities. Does $-K_Y$ have some good positivity properties?

As observed in [17, Remark 2.5], when $f\colon X \to Y$ is a Fano conic bundle, Corollary 2.3 gives a positive answer to Ejiri's question. Indeed, the fibers of the conic bundle f have semi-log canonical singularities if and only if they are reduced (see [1, Example 1.4]).

REMARK 4.2. As we observed in the Introduction, by [16, Theorem 2.9] we know that the target of a contraction of a log Fano variety is log Fano, so that in the setting of Question 3 (and thus in the specific case in which f is a Fano conic bundle) one has that $-K_Y$ is always a big divisor of Y.

REMARK 4.3. Notice that if in Question 3 we do not assume that f is equidimensional, it is easy to find counterexamples. For instance in [6, Example 3.16 (2)] we find examples of divisorial contractions $\varphi\colon Z \to Y$, where Z is smooth and Fano, the fibers of φ are smooth, but Y is not Fano. If we take the natural projection $\pi\colon \mathbb{P}^1 \times Z \to Z$, then $\varphi \circ \pi\colon \mathbb{P}^1 \times Z \to Y$ gives a fiber type contraction of a Fano variety, where the fibers are smooth, but Y is not Fano.

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