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Lazhar Dhaouadi, Ahmed Fitouhi

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Abstract

In this paper we study the positivity of the generalized q-translation associated with the q-Bessel Hahn Exton function which is deduced by a new formulation of the Graf's addition formula related to this function.

1 Introduction and Preliminaries

It is well known that the generalized translation operator T associated with the Bessel function of the first kind is positive in the sense if $f > 0$, then $Tf > 0$. This property is easily seen when we write Tf as an integral representation with a kernel involving the area of some triangle ([1],[9]) and has several applications in many mathematical fields such that hypergroup structure, heat equation....

In 2002, the appropriated q-generalized translation for the q-Bessel Hahn Exton function was founded [3] and the problem of its positivity asked. In literature we meet many attempts to show this property in particular case, nerveless they are no definitive response at to day. In [2] the authors prove that for the q-cosinus , the correspondent q-translation operator is positive if $q \in [0, q_0]$ for some q_0. In this work and owing a new formulation of the Graf's addition theorem [7] we give an affirmative answer about this theme by a technic involving some inclusion of sets.

To make this work self containing, we begin by the following preliminaries. Throughout this paper we consider $0 < q < 1$ and we adopt the standard conventional notations of [4]. For complex a We put

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$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{i=0}^{n-1}(1-aq^i), \quad n = 1...\infty.$$

Jackson's q-integral (see [5]) over the interval $[0,\infty[$ is defined by

$$\int_0^\infty f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n).$$

We denote by

$$\mathbb{R}_q^+ = \{q^n, \quad n \in \mathbb{Z}\},$$

and we consider $\mathcal{L}_{q,p,v}$ the space of even functions f defined on $\mathbb{R}_q^+$ such that

$$\|f\|_{q,p,v} = \left[\int_0^\infty |f(x)|^p x^{2v+1} d_qx\right]^{1/p} < \infty.$$

The q-Bessel function of third kind and of order v, called also Hahn-Exton function,is defined by the q-hypergeometric function (see [8])

$$J_v(x,q) = \frac{(q^{v+1};q)_\infty}{(q;q)_\infty} x^v {}_1\phi_1(0,q^{v+1},q,qx^2), \quad \Re(v) > -1,$$

and has a normalized form is given by

$$j_v(x,q) = \frac{(q;q)_\infty}{(q^{v+1};q)_\infty} x^{-v} J_v(x,q) = {}_1\phi_1(0,q^{v+1},q,qx^2) = \sum_{n=0}^{\infty}(-1)^n \frac{q^{\frac{n(n+1)}{2}}}{(q;q)_n(q^{v+1};q)_n} x^{2n}.$$

It's an entire analytic function in z.
In([2],[6]) the q-Bessel Fourier transform $\mathcal{F}_{q,v}$ and its properties were studied in great detail, it is defined as follow

$$\mathcal{F}_{q,v} f(x) = c_{q,v} \int_0^\infty f(t) j_v(xt,q^2) t^{2v+1} d_qt,$$

where

$$c_{q,v} = \frac{1}{1-q}\frac{(q^{2v+2},q^2)_\infty}{(q^2,q^2)_\infty}.$$

There is many way to define the q-Bessel translation operator [2],[3]. One of them can be enounced for suitable function f as follows:

$$T_{q,x}^v f(y) = c_{q,v} \int_0^\infty \mathcal{F}_{q,v}(f)(t) j_v(xt,q^2) j_v(yt,q^2) t^{2v+1} d_qt, \quad \forall x,y \in \mathbb{R}_q^+, \forall f \in \mathcal{L}_{q,1,v}.$$

Now we say that $T^v_{q,x}$ is positive if $T^v_{q,x} f \geq 0$ for $f \geq 0$.
Let us putting the domain of positivity of $T^v_{q,x}$ by

$$Q_v = \{q \in]0,1[, \quad T^v_{q,x} \quad \text{is positive for all} \quad x \in \mathbb{R}_q^+\}.$$

Trough the result of [2], Q_v is not empty. So for $q \in Q_v$ the $q-$convolution product of the two functions f and $g \in \mathcal{L}_{q,1,v}$ is defined by

$$f *_q g(x) = c_{q,v} \int_0^\infty T^v_{q,x} f(y) g(y) y^{2v+1} d_q y.$$

To close this section we present the following results proved in [2] which will be used in the remainder.

Proposition 1

$$|j_v(q^n, q^2)| \leq \frac{(-q^2;q^2)_\infty(-q^{2v+2};q^2)_\infty}{(q^{2v+2};q^2)_\infty} \left\{ \begin{array}{lll} 1 & if & n \geq 0 \\ q^{n^2-(2v+1)n} & if & n < 0 \end{array} \right. .$$

Theorem 1 *The operator $\mathcal{F}_{q,v}$ satisfies*

1. *For all functions $f \in \mathcal{L}_{q,1,v}$,*

$$\mathcal{F}^2_{q,v} f(x) = f(x), \quad \forall x \in \mathbb{R}_q^+.$$

2. *If $f \in \mathcal{L}_{q,1,v}$ and $\mathcal{F}_{q,v} f \in \mathcal{L}_{q,1,v}$ then*

$$\|\mathcal{F}_{q,v} f\|_{q,v,2} = \|f\|_{q,v,2}.$$

2 The Graf's addition formula

The Graf's addition formula for Hahn-Exton q-Bessel function proved by H.T.Koelink and F. Swarttouw [7] plays a central role here . It can be stated as follows.

$$\begin{aligned} & J_v(Rq^{1/2(y+z+v)}, q) J_{x-v}(q^{1/2z}, q) \\ & = \sum_{k \in \mathbb{Z}} J_k(Rq^{1/2(x+y+k)}, q) J_{v+k}(Rq^{1/2(y+k+v)}, q) J_x(q^{1/2(z-k)}, q); \end{aligned}$$

and it is valid when $z \in \mathbb{Z}$ and $R, x, y, v \in \mathbb{C}$ satisfying

$$|R|^2 q^{1+\Re(x)+\Re(y)} < 1, \quad \Re(x) > -1, \quad R \neq 0.$$

This formula has originally been derived for $v, x, y \in \mathbb{Z}$, $R > 0$ by the interpretation of the Hahn-Exton q-Bessel function as matrix elements of irreducible unitary representation of the quantum group of plane motions. If we replace q by q^2 and R by q^r in the previous formula we get:

$$\begin{aligned}&J_v(q^{y+z+v+r}, q^2)J_{x-v}(q^z, q^2)\\&= \sum_{k\in\mathbb{Z}} J_k(q^{x+y+k+r}, q^2)J_{v+k}(q^{y+k+v+r}, q^2)J_x(q^{z-k}, q^2),\end{aligned}$$

and put

$$m = y + z + v + r,$$

so

$$x + y + k + r = m + k + x - z - v,$$

$$y + k + v + r = m + k - z,$$

and we have

$$\begin{aligned}&J_v(q^m, q^2)J_{x-v}(q^z, q^2)\\&= \sum_{k\in\mathbb{Z}} J_k(q^{m+k+x-z-v}, q^2)J_{v+k}(q^{m+k-z}, q^2)J_x(q^{z-k}, q^2).\end{aligned}$$

This last formula is valid for $z \in \mathbb{Z}$ and $r, x, y, v \in \mathbb{C}$ satisfying

$$1 + 2\Re(r) + \Re(x) + \Re(y) = 1+\Re(r)+\Re(m)-\Re(z)-\Re(v) > 0, \quad \Re(x) > -1.$$

In the above sum we replace $z - k$ by k we get

$$\begin{aligned}&J_v(q^m, q^2)J_{x-v}(q^z, q^2)\\&= \sum_{k\in\mathbb{Z}} J_{z-k}(q^{m+x-v-k}, q^2)J_{v+z-k}(q^{m-k}, q^2)J_x(q^k, q^2),\end{aligned}$$

The sum in the second member exists for

$$\forall z \in \mathbb{Z}, \quad \forall m, v, x \in \mathbb{C}, \quad \Re(x) > -1.$$

In fact there exist an infinity complex number $r \in \mathbb{C}$ for which

$$1 + \Re(r) + \Re(m) - \Re(z) - \Re(v) > 0.$$

Now using the definition of the normalized q-Bessel function

$$J_x(q^k, q^2) = \frac{(q^{2x+2}, q^2)_\infty}{(q^2, q^2)_\infty} q^{vk} j_x(q^k, q^2) = (1 - q)c_{q,x}q^{xk} j_x(q^k, q^2),$$

we obtain

$$q^{mv+z(x-v)}(1-q)^2 c_{q,v}c_{q,x-v}j_v(q^m,q^2)j_{x-v}(q^z,q^2)$$
$$=(1-q)c_{q,x}\sum_{k\in\mathbb{Z}}q^{xk}J_{z-k}(q^{x-v+m-k},q^2)J_{v+z-k}(q^{m-k},q^2)j_x(q^k,q^2).$$

Let

$$\lambda=q^n,\quad n\in\mathbb{Z},$$

and replace

$$k\to k+n, m\to m+n, z\to z+n.$$

This implies

$$q^{mv+z(x-v)}(1-q)^2 c_{q,v}c_{q,x-v}j_v(q^m\lambda,q^2)j_{x-v}(q^z\lambda,q^2)$$
$$=(1-q)c_{q,x}\sum_{k\in\mathbb{Z}}q^{xk}J_{z-k}(q^{x-v+m-k},q^2)J_{v+z-k}(q^{m-k},q^2)j_x(q^k\lambda,q^2)$$
$$=(1-q)c_{q,x}\sum_{k\in\mathbb{Z}}q^{2k(x+1)}q^{-k(x+2)}J_{z-k}(q^{x-v+m-k},q^2)J_{v+z-k}(q^{m-k},q^2)j_x(q^k\lambda,q^2).$$

We put

$$E_{v,x}(q^m,q^z,q^k)=\frac{1}{(1-q)^2c_{q,v}c_{q,x-v}}q^{-k(x+2)-mv-z(x-v)}J_{z-k}(q^{x-v+m-k},q^2)J_{v+z-k}(q^{m-k},q^2).$$

The Proposition 1 shows that the function:

$$\lambda\mapsto j_v(q^m\lambda,q^2)j_{x-v}(q^z\lambda,q^2),\quad \forall m,z\in\mathbb{Z},$$

belongs to the space $\mathcal{L}_{q,1,v}$. Theorem 1, part 1 leads to the following statement:

Proposition 2 *For $z,m\in\mathbb{Z}$ and $x,v\in\mathbb{C}$ satisfying*

$$\Re(x-v)>-1,\quad \Re(v)>-1$$

we have

$$j_v(q^m\lambda,q^2)j_{x-v}(q^z\lambda,q^2)=c_{q,x}\int_0^\infty E_{v,x}(q^m,q^z,t)j_x(\lambda t,q^2)t^{2x+1}d_qt,$$

and

$$E_{v,x}(q^m,q^z,q^k)=c_{q,x}\int_0^\infty j_v(q^m\lambda,q^2)j_{x-v}(q^z\lambda,q^2)j_x(q^k\lambda,q^2)\lambda^{2x+1}d_q\lambda.$$

3 Positivity of the q-translation operator

This section is a direct application of the previous one but before any things we recall that the q-translation operator possess the q-integral representation (see [2])

Proposition 3 *Let* $f \in \mathcal{L}_{q,1,v}$ *then*

$$T^v_{q,x} f(y) = \int_0^\infty f(z) D_v(x,y,z) z^{2v+1} d_q z,$$

where

$$D_v(x,y,z) = c^2_{q,v} \int_0^\infty j_v(xt,q^2) j_v(yt,q^2) j_v(zt,q^2) t^{2v+1} d_q t.$$

When q tends to 1^- we obtain at least formally the classical one and the q-kernel $D_v(x,y,z)$ tends to the classical one ([1],[9]) which involves the area of some triangle. Hence the positivity of $T^v_{q,x}$ is subject of those of $D_v(x,y,z)$. A direct consequence of the above result is the fact that if the operator $T^v_{q,x}$ is positive and $f \in \mathcal{L}_{q,1,v}$ then $T^v_{q,x} f \in \mathcal{L}_{q,1,v}$ (in general it is not true without this hypothesis). Indeed

$$\begin{aligned}
&\int_0^\infty |T^v_{q,x} f(y)| y^{2v+1} d_q t \\
&\leq \int_0^\infty T^v_{q,x} |f|(y) y^{2v+1} d_q t = \int_0^\infty |f(z)| \left[\int_0^\infty D_v(x,y,z) y^{2v+1} d_q y \right] z^{2v+1} d_q z.
\end{aligned}$$

Putting

$$\phi : t \mapsto j_v(xt,q^2) j_v(zt,q^2),$$

then we can write

$$D_v(x,y,z) = c_{q,v} \mathcal{F}_{q,v} \phi(y).$$

This gives by the of the inversion formula in Theorem 1 the important result as the classical; one

$$\int_0^\infty D_v(x,y,z) y^{2v+1} d_q y = \mathcal{F}^2_{q,v} \phi(0) = \phi(0) = 1.$$

Then we have

$$\int_0^\infty |T^v_{q,x} f(y)| y^{2v+1} d_q t \leq \|f\|_{q,v,1},$$

and we obtain

$$f \in \mathcal{L}_{q,1,v} \Rightarrow T^v_{q,x} f \in \mathcal{L}_{q,1,v}.$$

In [2] and as a first preamble of this theme the authors proved that

$$\begin{aligned} D_{-\frac{1}{2}}(q^m,q^r,q^k) &= \frac{q^{2(r-m)(k-m)-m}}{(1-q)(q;q)_I}(q^{2(r-m)+1};q)_\infty {}_1\phi_1(0,q^{2(r-m)+1},q;q^{2(k-m)+1}) \\ &= \frac{1}{1-q}q^{-m}J_{2(r-m)}(q^{k-m},q), \end{aligned}$$

which implies that the correspondent domain of $T_{q,x}^{-\frac{1}{2}}$ is given by

$$Q_{-\frac{1}{2}} =]0,q_0],$$

where q_0 is the first zero of the q-hypergeometric function:

$$q \mapsto {}_1\phi_1(0,q,q,q);$$

a second statement is easily given by the Proposition 2 with $x = v = 0$ which gives

$$\begin{aligned} E_{0,0}(q^m,q^z,q^k) &= c_{q,0}\int_0^\infty j_0(q^m\lambda,q^2)j_0(q^z\lambda,q^2)j_0(q^k\lambda,q^2)\lambda d_q\lambda \\ &= \frac{1}{c_{q,0}}D_0(q^m,q^z,q^k), \end{aligned}$$

and then

$$\begin{aligned} D_0(q^m,q^z,q^k) &= c_{q,0}E_{0,0}(q^m,q^z,q^k) \\ &= \frac{1}{(1-q)}q^{-2k}\left[J_{z-k}(q^{m-k},q^2)\right]^2, \end{aligned}$$

Hence we have

$$Q_0 =]0,1[.$$

Now we explicit the kernel in the production formula in terms of D_v.

Proposition 4 *For $n,m,k \in \mathbb{Z}$ and $-1 < v$ we have*

$$E_{v,v}(q^m,q^n,q^k) = (1-q)\sum_{i=0}^{\infty}\frac{(q^{-2v},q^2)_i}{(q^2,q^2)_i}q^{i(2+2v)}D_v(q^m,q^{i+n},q^k).$$

Proof. From the following formula (see[7], §5) "which can be proved in a straightforward way by substitution of the defining series for the q-Bessel functions on both sides, by interchanging summations, and by evaluating the q-binomial series which occurs"

$$J_{x-v}(\lambda, q^2) = \lambda^{-v} \sum_{i=0}^{\infty} \frac{(q^{-2v}, q^2)_i}{(q^2, q^2)_i} q^{i(2+x)} J_x(\lambda q^i, q^2),$$

where $\Re(x-v) > -1$ and $\Re(x) > -1$ we obtain

$$(1-q)c_{q,x-v} j_{x-v}(\lambda, q^2) = (1-q)c_{q,x} \sum_{i=0}^{\infty} \frac{(q^{-2v}, q^2)_i}{(q^2, q^2)_i} q^{i(2+2x)} j_x(\lambda q^i, q^2).$$

Put $x = v$ and change λ by $q^n\lambda$ we obtain

$$j_0(q^n\lambda, q^2) = (1-q)c_{q,v} \sum_{i=0}^{\infty} \frac{(q^{-2v}, q^2)_i}{(q^2, q^2)_i} q^{i(2+2v)} j_v(\lambda q^{i+n}, q^2),$$

which gives from Proposition 2

$$\begin{aligned} E_{v,v}(q^m, q^n, q^k) &= c_{q,v} \int_0^{\infty} j_v(q^m\lambda, q^2) j_0(q^n\lambda, q^2) j_v(q^k\lambda, q^2) \lambda^{2v+1} d_q\lambda \\ &= (1-q) \sum_{i=0}^{\infty} \frac{(q^{-2v}, q^2)_i}{(q^2, q^2)_i} q^{i(2+2v)} \\ &\times \left[c_{q,v}^2 \int_0^{\infty} j_v(q^m\lambda, q^2) j_v(q^{i+n}\lambda, q^2) j_v(q^k\lambda, q^2) \lambda^{2v+1} d_q\lambda \right] \\ &= (1-q) \sum_{i=0}^{\infty} \frac{(q^{-2v}, q^2)_i}{(q^2, q^2)_i} q^{i(2+2v)} D_v(q^m, q^{i+n}, q^k). \end{aligned}$$

We justify the exchange of the signs sum and integral by

$$\begin{aligned} &\sum_{i=0}^{\infty} \frac{|(q^{-2v}, q^2)_i|}{(q^2, q^2)_i} q^{i(2+2v)} \int_0^{\infty} \left| j_v(q^m\lambda, q^2) j_v(q^{i+n}\lambda, q^2) j_v(q^k\lambda, q^2) \lambda^{2v+1} \right| d_q\lambda \\ &\leq \|j_v(., q^2)\|_{q,\infty}^2 \|j_v(., q^2)\|_{q,1,v} q^{-2(v+1)m} \sum_{i=0}^{\infty} \frac{|(q^{-2v}, q^2)_i|}{(q^2, q^2)_i} q^{i(2+2v)} < \infty. \end{aligned}$$

So we obtain the result. ■

Proposition 5 *Let* $-1 < v < 0$ *then*

$$Q_v \subsetneq]0, 1[.$$

Proof. For $m, n, k \in \mathbb{Z}$ and $-1 < v < 0$ we have

$$\begin{aligned} E_{v,v}(q^m, q^n, q^k) &= \frac{1}{(1-q)c_{q,v}} q^{-k(v+2)-mv} J_{n-k}(q^{m-k}, q^2) J_{v+n-k}(q^{m-k}, q^2) \\ &= (1-q)\sum_{i=0}^{\infty} \frac{(q^{-2v}, q^2)_i}{(q^2, q^2)_i} q^{i(2+2v)} D_v(q^m, q^{i+n}, q^k). \end{aligned}$$

In particular for $m = n = k = 0$

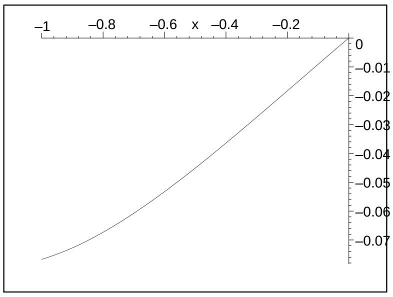
$$E_{v,v}(1,1,1) = \frac{1}{(1-q)c_{q,v}} J_0(1, q^2) J_v(1, q^2).$$

We introduce the following function

$$\phi_v : q \mapsto (q^2, q^2)_\infty^2 J_v(1, q^2).$$

Let $q_1 \simeq 0.658$ the first zero of ϕ_0 and consider the graph of the function

$$v \mapsto \phi_v(q_1), \quad v \in [0, 1].$$



We conclude that for $-1 < v < 0$ we have $\phi_v(q_1) < 0$. Then there exist a very small $\varepsilon > 0$ such that $\phi_v(q_1 - \varepsilon) < 0$ and $\phi_0(q_1 - \varepsilon) > 0$. Hence, this function

$$q \mapsto J_0(1, q^2) J_v(1, q^2) < 0,$$

takes some negative values in the interval $]0, 1[$, then for some entire $i \in \mathbb{N}$

$$\frac{(q^{-2v}, q^2)_i}{(q^2, q^2)_i} q^{i(2+2v)} D_v(1, q^i, 1) < 0.$$

As

$$\frac{(q^{-2v}, q^2)_i}{(q^2, q^2)_i} q^{i(2+2v)} > 0,$$

then $D_v(1, q^i, 1) < 0$ for some entire $i \in \mathbb{N}$. Which prove that

$$Q_v \subsetneq]0, 1[.$$

This finish the proof. ∎

Lemma 1 *For $x \in \mathbb{R}_q^+$ and $\Re(v+t) > -1$ we have*

$$(1-q)c_{q,v} \sum_{n\in\mathbb{Z}} q^{(1+v+t)n} j_v(q^n x, q^2) = \frac{(q^{1+v-t}; q^2)_\infty}{(q^{1+v+t}; q^2)_\infty} x^{-(1+v+t)}.$$

Proof. In [6] the following result was proved

$$\sum_{n\in\mathbb{Z}} q^{(1+t)n} J_v(q^n, q^2) = \frac{(q^{1+v-t}; q^2)_\infty}{(q^{1+v+t}; q^2)_\infty}, \quad \Re(v+t) > -1.$$

Then we get

$$(1-q)c_{q,v} \sum_{n\in\mathbb{Z}} q^{(1+v+t)n} j_v(q^n, q^2) = \frac{(q^{1+v-t}; q^2)_\infty}{(q^{1+v+t}; q^2)_\infty}.$$

Hence for $x = q^k \in \mathbb{R}_q^+$ we have

$$(1-q)c_{q,v} \sum_{n\in\mathbb{Z}} q^{(1+v+t)(n+k)} j_v(q^{n+k}, q^2) = \frac{(q^{1+v-t}; q^2)_\infty}{(q^{1+v+t}; q^2)_\infty}.$$

This finish the proof. ■

Theorem 2 *Let $v \geq x > -1$ then*

$$Q_x \subset Q_v.$$

As consequence

- *If $0 \leq v$ then $Q_v =]0,1[$.*
- *If $-\frac{1}{2} \leq v < 0$ then $]0, q_0] \subset Q_v \subsetneq]0,1[$.*
- *If $-1 < v \leq -\frac{1}{2}$ then $Q_v \subset]0, q_0]$.*

Proof. Let $v > -1$ and $0 < \mu < 1$. We will prove that

$$Q_v \subset Q_{v+\mu},$$

which suffices to show that $Q_x \subset Q_v$ if $x \leq v$. In the following we tack $q \in Q_v$. We have

$$c_{q,v+\mu} j_{v+\mu}(t, q^2) = c_{q,v} \sum_{i=0}^{\infty} \frac{(q^{2\mu}, q^2)_i}{(q^2, q^2)_i} q^{2(v+1)i} j_v(tq^i, q^2),$$

and then

$$\begin{aligned}
& c_{q,v} c_{q,v+\mu}^2 j_v(tx,q^2) j_{v+\mu}(ty,q^2) j_v(tz,q^2) \\
& = c_{q,v}^3 \sum_{i,j,s=0}^{\infty} \frac{(q^{2\mu},q^2)_i}{(q^2,q^2)_i} \frac{(q^{2\mu},q^2)_j}{(q^2,q^2)_j} q^{2(1+v)(i+j)} \\
& \times j_v(tx,q^2) j_v(tq^i y,q^2) j_v(tq^j z,q^2),
\end{aligned}$$

which implies

$$\begin{aligned}
& c_{q,v} T_{v,v+\mu,v}(x,y,z) \\
& = c_{q,v} \sum_{i,j=0}^{\infty} \frac{(q^{2\mu},q^2)_i}{(q^2,q^2)_i} \frac{(q^{2\mu},q^2)_j}{(q^2,q^2)_j} q^{2(1+v)(i+j)} D_v(x,q^i y,q^j z) \geq 0,
\end{aligned}$$

where

$$T_{v,w,\alpha}(x,y,z) = c_{q,w}^2 \int_0^\infty j_v(tx,q^2) j_w(ty,q^2) j_w(tz,q^2) t^{2\alpha+1} d_q t.$$

Note that

$$T_{v,v,v}(x,y,z) = D_v(x,y,z).$$

The exchange of the signs sum and integral is valid since:

$$\begin{aligned}
& \sum_{i,j=0}^{\infty} q^{2(1+v)(i+j)} \int_0^\infty \left|j_v(q^m t,q^2)\right| \left|j_v(q^{n+i} t,q^2)\right| \left|j_v(q^{k+j} t,q^2)\right| t^{2v+1} d_q t \\
& = \sum_{i,j=0}^{\infty} q^{2(1+v)(i+j)} \int_0^1 \left|j_v(q^m t,q^2)\right| \left|j_v(q^{n+i} t,q^2)\right| \left|j_v(q^{k+j} t,q^2)\right| t^{2v+1} d_q t \\
& + \sum_{i,j=0}^{\infty} q^{2(1+v)(i+j)} \int_1^\infty \left|j_v(q^m t,q^2)\right| \left|j_v(q^{n+i} t,q^2)\right| \left|j_v(q^{k+j} t,q^2)\right| t^{2v+1} d_q t.
\end{aligned}$$

Note that

$$\begin{aligned}
& \sum_{i,j=0}^{\infty} q^{2(1+v)(i+j)} \int_0^1 \left|j_v(q^m t,q^2)\right| \left|j_v(q^{n+i} t,q^2)\right| \left|j_v(q^{k+j} t,q^2)\right| t^{2v+1} d_q t \\
& \leq \left\|j_v(.,q^2)\right\|_{q,\infty}^3 \left[\sum_{i,j=0}^{\infty} q^{2(1+v)(i+j)}\right] \int_0^1 t^{2v+1} d_q t < \infty,
\end{aligned}$$

and

$$\sum_{i,j=0}^{\infty} q^{2(1+v)(i+j)} \int_1^{\infty} \left|j_v(q^m t, q^2)\right| \left|j_v(q^{n+i} t, q^2)\right| \left|j_v(q^{k+j} t, q^2)\right| t^{2v+1} d_q t$$

$$= (1-q) \sum_{i,j,r'=0}^{\infty} q^{2(1+v)(i+j-r)} \left|j_v(q^{m-r}, q^2)\right| \left|j_v(q^{n+i-r}, q^2)\right| \left|j_v(q^{k+j-r}, q^2)\right|$$

$$= \sum_{j,r'=0}^{\infty} q^{2(1+v)j} \left|j_v(q^{m-r}, q^2)\right| \left|j_v(q^{k+j-r}, q^2)\right| \times \left[(1-q) \sum_{i=0}^{\infty} q^{2(1+v)(i-r)} \left|j_v(q^{n+i-r}, q^2)\right|\right],$$

where $r' = r - 1$. We write

$$\sum_{i=0}^{\infty} q^{2(1+v)(i-r)} \left|j_v(q^{n+i-r}, q^2)\right| = q^{-2(1+v)n}(1-q) \sum_{i=0}^{\infty} q^{2(n+i-r)} \left|j_v(q^{n+i-r}, q^2)\right|$$

$$= q^{-2(1+v)n}(1-q) \sum_{i=m-r}^{\infty} q^{2(1+v)i} \left|j_v(q^i, q^2)\right|$$

$$< q^{-2(1+v)n} \left\|j_v(., q^2)\right\|_{q,1,v}.$$

Then

$$\sum_{j,r'=0}^{\infty} q^{2(1+v)j} \left|j_v(q^{m-r}, q^2)\right| \left|j_v(q^{k+j-r}, q^2)\right|$$

$$= \sum_{s,r'=0}^{\infty} \left|j_v(q^{m-r}, q^2)\right| \left[\sum_{j=0}^{\infty} q^{2(1+v)j} \left|j_v(q^{k+j-r}, q^2)\right|\right]$$

$$= \sum_{r'=0}^{\infty} \left|j_v(q^{m-r}, q^2)\right| \left[q^{2(1+v)(r-k)} \sum_{j=0}^{\infty} q^{2(k+j-r)} \left|j_0(q^{k+j-r}, q^2)\right|\right]$$

$$= \sum_{s,r'=0}^{\infty} q^{2s} \left|j_v(q^{m-r}, q^2)\right| \left[q^{2(v+1)(r-k)} \sum_{j=k-r}^{\infty} q^{2(1+v)j} \left|j_0(q^j, q^2)\right|\right]$$

$$\leq \frac{q^{-2(v+1)k}}{1-q} \left\|j_v(., q^2)\right\|_{q,1,v} \left\|j_v(., q^2)\right\|_{q,\infty} \sum_{r'=0}^{\infty} q^{2(v+1)r} < \infty.$$

Let $0 < \alpha < \mu$. We introduce the function $A_{\alpha,\mu,v}(x)$ as follows:

$$A_{\alpha,\mu,v}(x) = c_{q,v} \int_0^{\infty} t^{2\alpha} j_{v+\mu}(t, q^2) j_v(xt, q^2) t^{2v+1} d_q t.$$

From the inversion formula in Theorem 1 ($t \mapsto t^{2\alpha} j_{v+\mu}(t, q^2) \in \mathcal{L}_{q,1,v}$) we get

$$c_{q,v} \int_0^\infty A_{\alpha,\mu,v}(x) j_v(xt, q^2) x^{2v+1} d_q x = t^{2\alpha} j_{v+\mu}(t, q^2).$$

Let $x \leq 1$. From Lemma 1 we obtain

$$\begin{aligned}
&A_{\alpha,\mu,v}(x) \\
&= c_{q,v}(1-q) \sum_{n\in\mathbb{Z}} q^{2(\alpha+v+1)n} j_v(xq^n, q^2) j_{v+\mu}(q^n, q^2) \\
&= c_{q,v}(1-q) \sum_{n\in\mathbb{Z}} \left[\sum_{i=0}^{\infty} (-1)^i \frac{q^{i(i+1)}}{(q^{2+2v}, q^2)_i (q^2, q^2)_i}\right] q^{2(\alpha+v+1+i)n} j_{v+\mu}(q^n, q^2) \\
&= c_{q,v}(1-q) \sum_{i=0}^{\infty} (-1)^i \frac{q^{i(i+1)}}{(q^{2+2v}, q^2)_i (q^2, q^2)_i} x^{2i} \left[\sum_{n\in\mathbb{Z}} q^{2(\alpha+v+1+i)n} j_{v+\mu}(q^n, q^2)\right] \\
&= \frac{c_{q,v}}{c_{q,v+\mu}} \sum_{i=0}^{\infty} (-1)^i \frac{q^{i(i+1)}}{(q^{2+2w}, q^2)_i (q^2, q^2)_i} x^{2i} \frac{(q^{2(1+v+\mu)-2(\alpha+v+1+i)}, q^2)_\infty}{(q^{2(\alpha+v+1+i)}, q^2)_\infty} \\
&= \frac{c_{q,v}}{c_{q,v+\mu}} \sum_{i=0}^{\infty} (-1)^i \frac{q^{i(i+1)}}{(q^{2+2v}, q^2)_i (q^2, q^2)_i} x^{2i} \frac{(q^{2(\mu-\alpha)} q^{-2i}, q^2)_\infty}{(q^{2(\alpha+v+1+i)}, q^2)_\infty} \geq 0.
\end{aligned}$$

Note that

$$0 < \mu - \alpha < 1.$$

The exchange of the signs sum is valid since. Indeed, let $q^k = x$ then we have

$$\begin{aligned}
&\sum_{i=0}^{\infty} q^{i(i+1)} q^{2ik} \left[\sum_{n\in\mathbb{Z}} q^{2(\alpha+v+1+i)n} \left|j_{v+\mu}(q^n, q^2)\right|\right] \\
&= \sum_{i=0}^{\infty} q^{i(i+1)} q^{2ik} \left[\sum_{n=0}^{\infty} q^{2(\alpha+v+1+i)n} \left|j_{v+\mu}(q^n, q^2)\right|\right] \\
&+ \sum_{i=0}^{\infty} q^{i(i+1)} q^{2ik} \left[\sum_{n=1}^{\infty} q^{-2(\alpha+v+1+i)n} \left|j_{v+\mu}(q^{-n}, q^2)\right|\right] \\
&\leq \sum_{i=0}^{\infty} q^{i(i+1)} q^{2ik} \left[\sum_{n=0}^{\infty} q^{2(\alpha+v+1)n}\right] \\
&+ \sum_{i=0}^{\infty} q^{i(i+1)} q^{2ik} \left[\sum_{n=1}^{\infty} q^{-2(\alpha+v+1+i)n+n^2+(2v+2\mu+1)n}\right].
\end{aligned}$$

One has to observe that the first sum exist. The second sum also:

$$\sum_{i=0}^{\infty} q^{i(i+1)} q^{2ik} \left[\sum_{n=1}^{\infty} q^{-2(\alpha+v+1+i)n+n^2+(2v+2\mu+1)n} \right]$$
$$= \sum_{n=1}^{\infty} q^{-2(\alpha+v+1+i)n+n^2+(2v+2\mu+1)n} \left[\sum_{i=0}^{\infty} q^{i(i+1)} q^{2ik} \right]$$
$$= \sum_{n=1}^{\infty} q^{-2(\alpha+v+1)n+(2v+2\mu+1)n+(2k+1)n} \left[\sum_{i=0}^{\infty} q^{(i-n)^2} q^{(2k+1)(i-n)} \right]$$
$$= \sum_{n=1}^{\infty} q^{2(\mu-\alpha)n+(2k+1)n} \left[\sum_{i=-n}^{\infty} q^{i^2} q^{(2k+1)i} \right] \leq \sum_{n=1}^{\infty} q^{2(\mu-\alpha)n+(2k+1)n} \left[\sum_{i=-\infty}^{\infty} q^{i^2} q^{(2k+1)i} \right] < \infty.$$

If $x > 1$ then we obtain

$$A_{\alpha,\mu,v}(x)$$
$$= c_{q,v}(1-q) \sum_{n\in\mathbb{Z}} q^{2(\alpha+v+1)n} j_v(xq^n, q^2) j_{v+\mu}(q^n, q^2)$$
$$= c_{q,v}(1-q) \sum_{n\in\mathbb{Z}} \left[\sum_{i=0}^{\infty} (-1)^i \frac{q^{i(i+1)}}{(q^{2+2v}, q^2)_i (q^2, q^2)_i} \right] q^{2(\alpha+v+1+i)n} j_v(q^n x, q^2)$$
$$= c_{q,v}(1-q) \sum_{i=0}^{\infty} (-1)^i \frac{q^{i(i+1)}}{(q^{2+2v+2\mu}, q^2)_i (q^2, q^2)_i} \left[\sum_{n\in\mathbb{Z}} q^{2(\alpha+v+1+i)n} j_v(q^n x, q^2) \right]$$
$$= x^{-2(\alpha+v+1)} \sum_{i=0}^{\infty} (-1)^i \frac{q^{i(i+1)}}{(q^{2+2v+2\mu}, q^2)_i (q^2, q^2)_i} x^{-2i} \frac{(q^{2(1+v)-2(\alpha+v+1+i)}, q^2)_\infty}{(q^{2(\alpha+v+1+i)}, q^2)_\infty}$$
$$= x^{-2(\alpha+v+1)} \sum_{i=0}^{\infty} (-1)^i \frac{q^{i(i+1)}}{(q^{2+2v+2\mu}, q^2)_i (q^2, q^2)_i} x^{-2i} \frac{(q^{-2\alpha} q^{-2i}, q^2)_\infty}{(q^{2(\alpha+v+1+i)}, q^2)_\infty} < 0.$$

Now, we write

$$c_{q,v} \int_0^\infty A_{\alpha,\mu,v}(x) T_{v,v+\mu,v}(x,y,z) x^{2v+1} d_q x$$
$$= c_{q,v} \int_0^\infty A_{\alpha,\mu,v}(x) \left[c_{q,v}^2 \int_0^\infty j_w(xt, q^2) j_v(yt, q^2) j_v(zt, q^2) t^{2v+1} d_q t \right] x^{2v+1} d_q x$$
$$= c_{q,v+\mu}^2 \int_0^\infty \left[c_{q,v} \int_0^\infty A_{\alpha,\mu,v}(x) j_v(xt, q^2) x^{2v+1} d_q x \right] j_{v+\mu}(yt, q^2) j_{v+\mu}(zt, q^2) t^{2v+1} d_q t$$
$$= c_{q,v+\mu}^2 \int_0^\infty j_{v+\mu}(t, q^2) j_{v+\mu}(yt, q^2) j_{v+\mu}(zt, q^2) t^{2(v+\alpha)+1} d_q t = T_{v+\mu,v+\mu,v+\alpha}(1,y,z).$$

To justify the exchange of the signs integrals we write

$$\int_0^\infty \left[\int_0^\infty |A_{\alpha,\mu,v}(x)||j_v(xt,q^2)|x^{2v+1}d_qx\right]|j_v(yt,q^2)||j_v(zt,q^2)|td_qt$$
$$\leq \left[\|j_v(.,q^2)\|^2_{q,\infty}\|j_v(.,q^2)\|_{q,1,v}\|A_{\alpha,\mu,v}\|_{q,1,v}\right]\frac{1}{z}.$$

Not that $x \mapsto A_{\alpha,\mu,v}(x)$ is continued at 0 and

$$|A_{\alpha,\mu,v}(x)| \leq \left[\|j_{v+\mu}(.,q^2)\|_{q,\infty}\|j_v(.,q^2)\|_{q,1,\alpha+v}\right]x^{-2(\alpha+v+1)}, \quad \text{as} \quad x \to \infty.$$

On the other hand

$$\int_0^\infty A_{\alpha,\mu,v}(x)x^{2v+1}d_qx = 0,$$

then we obtain for all $\delta > 0$

$$T_{v+\mu,v+\mu,v+\alpha}(1,y,z) = \int_0^\infty A_{\alpha,\mu,v}(x)\left[T_{v,v+\mu,v}(x,y,z) - \delta\right]x^{2v+1}d_qx.$$

In the following we assume that $0 < y, z \leq 1$. Let

$$\delta_0 = \inf_{0\leq y,z\leq 1} T_{v,v+\mu,v}(1,y,z)$$

Not that δ_0 exist and strictly positive, indeed the following function

$$(y,z) \mapsto T_{v,v+\mu,v}(1,y,z)$$

is continuous on the compact $[0,1]\times[0,1]$. Hence, there exist

$$(y_0,z_0) \in [0,1]\times[0,1]$$

such that

$$\delta_0 = T_{v,v+\mu,v}(1,y_0,z_0) = \inf_{0\leq y,z\leq 1} T_{v,v+\mu,v}(1,y,z) \geq 0.$$

If we assume that $T_{v,v+\mu,v}(1,y_0,z_0) = 0$ then

$$\sum_{i,j=0}^\infty \frac{(q^{2\mu},q^2)_i}{(q^2,q^2)_i}\frac{(q^{2\mu},q^2)_j}{(q^2,q^2)_j}q^{2(1+v)(i+j)}D_v(1,q^iy_0,q^jz_0) = 0,$$

which implies that

$$c_{q,v}\mathcal{F}_{q,v}\left[t \mapsto j_v(t,q^2)j_v(z_0t,q^2)\right](q^iy_0) = D_v(1,q^iy_0,z_0) = 0, \ \forall i \in \mathbb{N}.$$

From Proposition 1 and the fact that there exist $\sigma_0 > 0$ such that

$$\left| j_v(z, q^2) \right| \le \sigma_0 \; e^{|z|}, \qquad \forall z \in \mathbb{C}$$

we see that this function

$$z \mapsto \mathcal{F}_{q,v}\Big[t \mapsto j_v(t, q^2) j_v(z_0 t, q^2) \Big](z)$$

is analytic. Then for all $x \in \mathbb{R}_q$

$$\mathcal{F}_{q,v}\Big[t \mapsto j_v(t, q^2) j_v(z_0 t, q^2) \Big](x) = 0 \Rightarrow j_v(t, q^2) j_v(z_0 t, q^2) = 0, \quad \forall t \in \mathbb{R}_q,$$

but this is absurd. Then $\delta_0 > 0$. Now we have

$$\delta_0 \le T_{v,v+\mu,v}(x, y, z), \qquad \forall 0 < x \le 1.$$

If

$$\delta_0 > T_{v,v+\mu,v}(x, y, z), \qquad \forall x > 1.$$

then

$$\begin{aligned} T_{v+\mu,v+\mu,v+\alpha}(1, y, z) \quad &= \quad \int_0^1 A_{\alpha,\mu,v}\,(x) \Big[T_{v,v+\mu,v}(x, y, z) - \delta_0 \Big] x^{2v+1} d_q x \\ &\quad + \int_1^\infty A_{\alpha,\mu,v}\,(x) \Big[T_{v,v+\mu,v}(x, y, z) - \delta_0 \Big] x^{2v+1} d_q x \ge 0. \end{aligned}$$

Otherwise, there exist $s > 0$ such that

$$\delta_0 > T_{v,v+\mu,v}(x, y, z), \qquad \forall x > s.$$

because

$$|T_{v,v+\mu,v}(x, y, z)| < c^2_{q,v+\mu} \left\| j_{v+\mu}(.q^2) \right\|^2_{q,\infty} \times \left\| j_v(.q^2) \right\|_{q,v,1} x^{-2(v+1)} \to 0, \qquad \text{as } x \to \infty.$$

For $\delta \ge \delta_0$ we obtain

$$\begin{aligned} T_{v+\mu,v+\mu,v+\alpha}(1, y, z) &= \int_0^1 A_{\alpha,\mu,v}\,(x) \Big[T_{v,v+\mu,v}(x, y, z) - \delta \Big] x^{2v+1} d_q x \\ &+ \int_1^s A_{\alpha,\mu,v}\,(x) \Big[T_{v,v+\mu,v}(x, y, z) - \delta \Big] x^{2v+1} d_q x \\ &+ \int_s^\infty A_{\alpha,\mu,v}\,(x) \Big[T_{v,v+\mu,v}(x, y, z) - \delta \Big] x^{2v+1} d_q x \\ &= I_1(\delta) + I_2(\delta) + I_3(\delta). \end{aligned}$$

Note that

$$\begin{cases} \delta \mapsto I_1(\delta) \text{ is a decreesing function tends towards } -\infty \text{ and } I_1(\delta_0) > 0 \\ \delta \mapsto I_2(\delta) \text{ is an increasing function tends towards } +\infty \text{ and } I_2(\delta_0) < 0 \\ \delta \mapsto I_3(\delta) \text{ is an increasing function tends towards } +\infty \text{ and } I_3(\delta_0) > 0 \end{cases}$$

then there exist $\delta > \delta_0$ such that

$$I_1(\delta) + I_2(\delta) = 0$$

which implies

$$T_{v+\mu,v+\mu,v+\alpha}(1, y, z) = I_3(\delta) > 0.$$

In the end

$$\begin{aligned} &\lim_{\alpha\to\mu} \left[T_{v+\mu,v+\mu,v+\alpha}(1, y, z)\right] \\ &= \lim_{\alpha\to\mu} \left[c^2_{q,v+\mu} \int_0^\infty j_{v+\mu}(t, q^2) j_{v+\mu}(yt, q^2) j_{v+\mu}(zt, q^2) t^{2(v+\alpha)+1} d_qt\right] \\ &= c^2_{q,v+\mu} \int_0^\infty j_{v+\mu}(t, q^2) j_{v+\mu}(yt, q^2) j_{v+\mu}(zt, q^2) \Big(\lim_{\alpha\to\mu} t^{2(v+\alpha)+1} \Big) d_qt \\ &= T_{v+\mu,v+\mu,v+\alpha}(1, y, z) = D_{v+\mu}(1, y, z) \geq 0. \end{aligned}$$

It is not hard to justify the exchange of the signs integral and limit, in fact

$$\begin{aligned} &\left| j_{v+\mu}(t, q^2) j_{v+\mu}(yt, q^2) j_{v+\mu}(zt, q^2) t^{2(v+\alpha)+1} \right| \\ &\leq \left\| t \mapsto j_{v+\mu}(yt, q^2) j_{v+\mu}(zt, q^2) t^{2(v+\alpha)+1} \right\|_{q,\infty} \times \left| j_{v+\mu}(t, q^2) \right|. \end{aligned}$$

From the following identity

$$D_{v+\mu}(x, y, z) = x^{-2(v+\mu+1)} D_{v+\mu}\left(1, \frac{y}{x}, \frac{z}{x}\right),$$

we deduce that

$$D_{v+\mu}(x, y, z) \geq 0, \qquad \forall x, y, z \in \mathbb{R}^+_q.$$

The fact that

$$Q_{-\frac{1}{2}} =]0, q_0] \text{ and } Q_0 =]0, 1[$$

leads to the result. ■

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