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Positivity of Wightman Functionals and the Existence of Local Nets

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Abstract. The paper is concerned with the existence of a local net of von Neumann algebras associated with a given Wightman field. For fields satisfying a generalized H-bound the existence of such a net is shown to be equivalent to a certain positivity property of the Wightman distributions.

1. Introduction

The connection of Wightman quantum field theory [25, 20] with the theory of local nets of C^*- or von Neumann algebras [19, 2, 3] has been the subject of a number of investigations during the past 25 years, cf. e.g. [11, 17, 15, 4, 5, 13, 16, 23, 28, 14, 26, 1, 29, 12, 30]. The present note is concerned with one aspect of this problem, viz. to formulate conditions on the Wightman distributions that ensure the existence of a corresponding local net of von Neumann algebras on the Hilbert space of the field.

Before we proceed it is necessary to make precise what it means to associate a Wightman field to a local net of von Neumann algebras. For notational simplicity we shall here only deal with the case of a single, hermitian, scalar field Φ. By a local net of von Neumann algebras we mean an assignment $R \mapsto \mathscr{A}(R)$ of regions R in Minkowski space $\mathbb{R}^d$ to von Neumann algebras $\mathscr{A}(R)$ on the Hilbert space of the field such that the usual conditions of isotony, locality and covariance are fulfilled [2, 3, 19]. It is convenient and for most purposes sufficient to restrict the choice of regions R to the following types: Closed double cones K, wedge domains W (bounded by two light-like hyperplanes), and causal complements, K^c and W^c of such domains.

A field can be associated to a net in different ways, cf. [14]. We shall use the following simple notion:

1.1. Definition. A Wightman field Φ is associated to a local net $\mathscr{A}$ of von Neumann algebras if each field operator $\Phi(f)$ has an extension to a closed operator, $\Phi(f)_e \subset \Phi(f^*)^*$, that is affiliated with the von Neumann algebra $\mathscr{A}(R)$ if the support of the test function f is contained in the interior of R.

In the simplest case the field operators $\Phi(f)$ are essentially self adjoint for f real valued, and bounded functions of $\Phi(f)^{**}$ and $\Phi(g)^{**}$ commute if the test functions f and g have space-like separated supports. In particular this holds if certain growth conditions on the Wightman functions [11, 17] or the Schwinger functions [13, 18] are fulfilled. The von Neumann algebra $\mathscr{A}(R)$ can then be defined as the algebra generated by bounded functions of $\Phi(f)^{**}$ with $\operatorname{supp} f \subset R$. This is a special case of the definition above, for if $\Phi(f)^{**}$ is self-adjoint for all real valued test functions f, then the closed operator $\Phi(g)^{**} \subset \Phi(g^*)^*$ is for all g affiliated with the von Neumann algebra generated by $\Phi(\operatorname{Re} g)^{**}$. In general self adjointness of the field operators cannot be expected, however. (Counterexamples are provided by Wick polynomials of free fields.)

As a preparation for our discussion we now recall some results of Bisogniano and Wichman [4, 5] and Driessler, Summers, and Wichmann [14]. The first concerns the relation between extensions of operator families and their weak commutants.

Let $\mathscr{P}$ be a family of operators with a common dense domain of definition $\mathscr{D}$ in a Hilbert space $\mathscr{H}$, such that if $X \in \mathscr{P}$, then also $X^*|\mathscr{D} =: X^\dagger \in \mathscr{P}$. The *weak commutant*, $\mathscr{P}^w$, of $\mathscr{P}$ is defined as the set of all bounded operators C on $\mathscr{H}$ such that $\langle \phi, CX\psi \rangle = \langle X^\dagger \phi, C\psi \rangle$ for all $\phi, \psi \in \mathscr{D}$.

1.2. Lemma. *Let $\mathscr{P}$ be as above and $\mathscr{A}$ a von Neumann algebra. The following are equivalent:*

1. *Every $X \in \mathscr{P}$ has a closed extension, X_e, affiliated with $\mathscr{A}$ and such that $X_e \subset X^{\dagger *}$.*
2. *$\mathscr{A}' \subset \mathscr{P}^w$.*

For a proof see [5, Lemma 10]. We remark also that the extension of operator families with the aid of weak commutants is a basic ingredient of a decomposition theory for states on algebras of unbounded operators [8, 9].

As a simple consequence of Lemma 1.2 we note that if the domain $\mathscr{D}$ is invariant under the operator family $\mathscr{P}$, then we may just as well assume that $\mathscr{P}$ is an algebra, because the weak commutant of $\mathscr{P}$ and of the algebra generated by $\mathscr{P}$ are identical.

Now suppose Φ is a Wightman field with a cyclic vacuum Ω. Denote by $\mathscr{P}(R)$ the algebra generated by the field operators $\Phi(f)$ with $\operatorname{supp} f \subset R$ and domain $\mathscr{D}_0 := \mathscr{P}(\mathbb{R}^d)\Omega$. The weak commutant $\mathscr{P}(R)^w$ is a weakly closed, $*$-invariant subspace of bounded operators on the Hilbert space of Φ, but in general it is not a von Neumann algebra. The problem of associating Φ to a local net simplifies considerably if it is known that $\mathscr{P}(K)^w$ is an algebra for all double cones K. In this case it follows from Lemma 1.2 that every operator in $\mathscr{P}(K)$ has a closed extension affiliated with the von Neumann algebra $\mathscr{A}_{\min}(K) := (\mathscr{P}(K)^w)'$. The question is then whether this "minimal net" $\mathscr{A}_{\min}$ is a local net. A criterium for this is due to Bisogniano and Wichman:

1.3. Theorem. *Suppose $\mathscr{P}(K)^w$ is an algebra for all double cones K. The following are equivalent:*

1. *There exists a local net to which the field is associated in the sense of the definition above.*

2. *The linear span of* $\cup\{\mathscr{P}(K^c)^w\Omega \mid K$ *double cone*$\}$ *is dense in the Hilbert space.*
3. $\mathscr{P}(K^c)^w\Omega$ *is dense for all double cones* K.

This result can be extracted from [4, Theorem 6], cf. [12] and [28]. An essential part of the proof is the identification of the PCT-operator, combined with a rotation and possibly a translation, as the modular operator of a wedge algebra $\mathscr{P}(W)^w$.

In view of this result it is gratifying that a mild regularity condition on the field ensures that $\mathscr{P}(K)^w$ is an algebra [14]:

1.4. Definition. Let Φ be a Wightman field and let H denote its Hamiltonian. The field satisfies a *generalized H-bound* if there exists a nonnegative number $\alpha < 1$, such that $\Phi(f)^{**}e^{-H^\alpha}$ is a bounded operator for all f.

1.5. Lemma [14, Lemma 3.4]. *If* Φ *satisfies a generalized H-bound, then* $\mathscr{P}(K)^w$ *is a von Neumann algebra for all double cones* K.

We remark that $\mathscr{P}(K)^w$ is also known to be an algebra if the operator family $\mathscr{P}(K)$ is essentially self adjoint in the sense of [21], i.e.

$$\cap\{D(A^*) \mid A \in \mathscr{P}(K)\} = \cap\{D(A^{**}) \mid A \in \mathscr{P}(K)\},$$

cf. [21, Lemma 4.6].

In [12] Buchholz uses the Wightman distributions to define a certain seminorm on the Hilbert space of the field and show that condition (2) of Theorem 1.3 is fulfilled if and only if this seminorm is a norm. In the sequel we propose an alternative criterion, stated in terms of a positivity property of the Wightman distributions. The idea can be described briefly as follows: By the Reeh-Schlieder theorem we know that $\mathscr{P}(K)\Omega$ is dense in $\mathscr{H}$ for all double cones K. Now the algebra $\mathscr{P}(K)$ belongs to the *unbounded* commutant of $\mathscr{P}(K^c)$ so it seems natural to try to approximate the elements of $\mathscr{P}(K)$ by bounded operators in $\mathscr{P}(K^c)^w$. A sufficient condition for this to work is that the field operators $\Phi(f)$ with $\operatorname{supp} f \subset K$ have self adjoint extensions $\hat{\Phi}(f)$ such that the bounded functions of $\hat{\Phi}(f)$ commute with $\mathscr{P}(K^c)$. It should be noted that extensions leading out of the Hilbert space of the field are also admitted. In fact, one is looking for operators that commute weakly with $\mathscr{P}(K^c)$, and such operators can be obtained by combining bounded functions of $\hat{\Phi}(f)$ with the projector on the Hilbert space of the field. Extending $\Phi(f)$ to a self adjoint operator with the required commutation properties is a special case of a *noncommutative moment problem.* General formalisms for dealing with such problems have been developed in [15, 23, 1, 29, 10]. For the special case at hand a simple criterion due to Powers [22] applies after a suitable modification.

2. Centrally Positive States

In this section we adapt Powers' result [22, Sect. V] to our needs.

2.1. Definition. Let $\mathfrak{A}$ be a *-algebra with unit and A_0 a hermitian element in $\mathfrak{A}$. If ω is a state on $\mathfrak{A}$, we say that ω is *centrally positive* with respect to A_0 if ω is positive on all elements of the form $\sum_n A_0^n A_n$, $A_n \in \mathfrak{A}$ such that $\sum_n \lambda^n A_n \in \mathfrak{A}^+$ for all $\lambda \in \mathbb{R}$.

2.2. Remark. Powers defines in [22] the concept of a *centrally positive operator*. In the cyclic case considered here, however, it seems more natural to think of central positivity as a property of the state; anyway ω is centrally positive with respect to A_0 in the sense of Definition 2.1 if and only if $\pi_\omega(A_0)$ is a centrally positive operator in the sense of [22]. Here π_ω denotes the GNS-representation of $\mathfrak{A}$ defined by ω on the cyclic domain $\mathscr{D}_\omega$.

The connection of central positivity with self adjoint extensions is established in the next proposition. We recall the meaning of an extension leading out of the original Hilbert space: Suppose X is a hermitian linear operator on a dense domain $\mathscr{D}$ in a Hilbert space $\mathscr{H}$. An operator $\hat{X}$ with domain $\hat{\mathscr{D}}$ in a Hilbert space $\hat{\mathscr{H}}$ is an extension of X if $\mathscr{H}$ is isometrically embedded in $\hat{\mathscr{H}}$ as a subspace with $\mathscr{D} \subset \hat{\mathscr{D}} \cap \mathscr{H}$ and $\hat{X}|\mathscr{D} = X$. A bounded operator C on $\hat{\mathscr{H}}$ is said to commute weakly on $\mathscr{D}$ with an (unbounded) operator $Y: \mathscr{D} \to \mathscr{H}$, if $\mathscr{D} \subset D(Y^*)$ and $\langle \phi, CY\psi \rangle = \langle Y^*\phi, C\psi \rangle$ for all $\phi, \psi \in \mathscr{D}$. Note that this implies that $\varrho(C) := EC|\mathscr{H}$, with E the projector $\hat{\mathscr{H}} \to \mathscr{H}$ commutes weakly with Y on $\mathscr{D}$.

2.3. Proposition. *Suppose ω is a state on $\mathfrak{A}$ and $A_0 \in \mathfrak{A}_h$. The following are equivalent:*

1. *$\pi_\omega(A_0)$ has a self adjoint extension $\hat{\pi}_\omega(A_0)$, evbntually in an extended Hilbert space $\hat{\mathscr{H}}$, such that all bounded functions of $\hat{\pi}_\omega(A_0)$ commute weakly with $\pi_\omega(\mathfrak{A})$ on $\mathscr{D}_\omega$.*
2. *ω is centrally positive with respect to A_0.*

Proof. We first that (1) implies (2). In fact, if (1) holds, we can by the spectral theorem approximate

$$\omega\left(\sum_n A_0^n A_n\right) = \left\langle \Omega_\omega, \sum_n \hat{\pi}_\omega(A_0)^n \pi_\omega(A_n)\Omega_\omega \right\rangle$$

by

$$\sum_i \left\langle \Omega_\omega, E_i \sum_n \lambda_i^n \pi_\omega(A_n)\Omega_\omega \right\rangle,$$

where the E_i are spectral projectors of $\hat{\pi}_\omega(A_0)$ and $\lambda_i \in \mathbb{R}$. Moreover, the E_i commute weakly with $\pi_\omega(\mathfrak{A})$ on $\mathscr{D}$, so if $\sum_n \lambda_i^n A_n \in \mathfrak{A}^+$ for all i, this is obviously nonnegative.

To prove that (2) implies (1) we use the same method as Powers [22, Theorem 5.3]. First, we claim that if ω is centrally positive with respect to A_0, then $\pi_\omega(A_0)$ is in the centre of $\pi_\omega(\mathfrak{A})$, i.e. $\omega(B^*A_0AB\rangle = \omega(B^*AA_0B)$ for all $A, B \in \mathfrak{A}$. It suffices to prove this for hermitian A. We have in any case

$$\omega(B^*A_0AB) = \omega(B^*AA_0B)^*,$$

so we have only to show that $\omega(B^*A_0AB)$ is a real number. But

$$B^*A_0AB = \tfrac{1}{4}\{B^*(A_0^2 + 2A_0A + A^2)B - B^*(A_0^2 - 2A_0A + A^2)B\},$$

and

$$\omega(B^*(A_0^2 \pm 2A_0A + A^2)B) \in \mathbb{R}_+$$

by central positivity.

Next we define $\mathfrak{F}$ as the algebra of all mappings $\mathbb{R}\to\mathfrak{A}$ of the form $\lambda\mapsto\sum_n F_n(\lambda)A_n$ with $A_n\in\mathfrak{A}$ and F_n polynomially bounded continuous functions on $\mathbb{R}$. By $\mathfrak{P}$ we denote the subalgebra of all such mappings where the functions F_n are polynomials. Both $\mathfrak{F}$ and $\mathfrak{P}$ are $*$-algebras in an obvious way. The hermitian part of $\mathfrak{F}$ respectively $\mathfrak{P}$ is denoted by $\mathfrak{F}_h$ respectively $\mathfrak{P}_h$. We define $\mathfrak{F}^+\subset\mathfrak{F}_h$ as the cone of the mappings such that

$$\sum_n F_n(\lambda)A_n\in\mathfrak{A}^+ \quad \text{for all} \quad \lambda\in\mathbb{R}.$$

The centrally positive linear functionals on $\mathfrak{A}$ are in a one-to-one correspondence with the linear functionals on $\mathfrak{P}$ that are positive on $\mathfrak{P}\cap\mathfrak{F}^+$. In fact, if ω is centrally positive on $\mathfrak{A}$, then

$$\omega'\left(\sum_n P_n(\cdot)A_n\right):=\omega\left(\sum_n P_n(A_0)A_n\right)$$

defines a positive functional ω' on $\mathfrak{P}$. Using that $\pi_\omega(A_0)$ commutes with $\pi_\omega(\mathfrak{A})$ it is straightforward to check that the map $\omega\mapsto\omega'$ is a bijection between centrally positive functionals on $\mathfrak{A}$ and positive functionals on $\mathfrak{P}$ with respect to the cone $\mathfrak{P}\cap\mathfrak{F}^+$. In the following we identify ω with ω'.

We now appeal to a variant of the Hahn-Banach theorem [24, Corollary 2.8, p. 82] and extend ω to a positive linear functional $\hat{\omega}$ on $\mathfrak{F}$. This is possible because $\mathfrak{P}_h$ is cofinal in $\mathfrak{F}_h$ with respect to the cone $\mathfrak{F}^+$: If F_n is for $n=1,2,\ldots$ a real valued continuous function with $|F_n|\leqq P_n$, P_n polynomial, and $A_n\subset\mathfrak{A}_h$, then

$$\pm\sum_n F_n(\cdot)A_n\leqq\tfrac{1}{2}\sum_n P_n(\cdot)(1+A_n^2)$$

in the order defined by $\mathfrak{F}^+$.

The functional $\hat{\omega}$ defines via GNS-construction a representation $\hat{\pi}_\omega$ of $\mathfrak{F}$, that extends the representation π_ω of $\mathfrak{P}$. In this representation $\hat{\pi}_\omega(A_0)\pm i$ has an inverse on $\mathscr{D}_{\hat{\omega}}$, namely $\hat{\pi}_\omega((\cdot\pm i)^{-1})$. Here $(\cdot\pm i)^{-1}\in\mathfrak{F}$ denotes the function $\lambda\mapsto(\lambda\pm i)^{-1}$. Hence $\hat{\pi}_\omega(A_0)$ is essentially self adjoint on $\mathscr{D}_{\hat{\omega}}$. Moreover, since $(\cdot\pm i)^{-1}$ commutes with $\mathfrak{P}$, we have also that all polynomially bounded functions of $\hat{\pi}_\omega(A_0)$ commute strongly with $\pi_\omega(\mathfrak{P})$ on $\mathscr{D}_{\hat{\omega}}$, and hence weakly on $\mathscr{D}_\omega$. □

2.4. Remark. Using the formalism of [10] it can be shown that for centrally positive states not only the operator $\pi_\omega(A_0)$, but all the operators $\pi_\omega(A)$, $A\in\mathfrak{A}_h$, have self adjoint extensions in an enlarged Hilbert space, such that bounded functions of $\hat{\pi}_\omega(A_0)$ commute with bounded functions of $\hat{\pi}_\omega(A)$ for all $A\in\mathfrak{A}_h$. We shall not need this strengthening of Proposition 2.1 here, however.

3. Central Positivity and the Existence of Local Nets

The sequence of Wightman distributions W_n of the field Φ defines a positive linear functional $\mathscr{W}$ (Wightman functional) on the tensor algebra built over the space of test functions for the field operators [7]. The choice of test function space is largely irrelevant, but for definiteness we shall take Schwartz' space $\mathscr{S}=\mathscr{S}(\mathbb{R}^d)$. The tensor algebra is denoted by $\underline{\mathscr{S}}$. Its elements are (finite) sequences $\underline{f}=(f_0,f_1,\ldots)$

with $f_0 \in \mathbb{C}$, $f_n \in \mathscr{S}(\mathbb{R}^{nd})$ for $n \geqq 1$. With the natural algebraic operations $\underline{\mathscr{S}}$ is a $*$-algebra [7]. The Wightman functional $\mathscr{W}$ is defined by $\mathscr{W}(\underline{f}) = \sum_n W_n(f_n)$.

We can now state our main result:

3.1. Theorem. *Let Φ be a Wightman field such that $\mathscr{P}(K)^w$ is an algebra for all double cones K. (This holds in particular if the field satisfies a generalized H-bound.) Let $\mathscr{W}$ denote the corresponding Wightman state on $\underline{\mathscr{S}}$. The following are equivalent:*

1. *The field is associated with a local net in the sense of Definition 1.1.*
2. *For any real valued test function f with support in the interior of a double cone K, the state $\mathscr{W}$ is centrally positive with respect to f when restricted to the subalgebra of $\underline{\mathscr{S}}$ generated by f and the test functions with support in K^c.*
3. *There exists a test function f with a strictly nonvanishing Fourier transform and support in the interior of a wedge W such that $\mathscr{W}$ is centrally positive with respect to f when restricted to the subalgebra of $\underline{\mathscr{S}}$ generated by f and test functions with support in W^c.*

Proof. If (1) holds, then $\Phi(f)$ has a closed, symmetric extension $\Phi(f)_e$ affiliated with a local von Neumann algebra $\mathscr{A}(K) \subset \mathscr{P}(K^c)^w$. Let E_1 and E_2 denote the projectors on a basis in $\mathbb{C}^2$ and consider the operator $\tilde{\Phi}(f) = \Phi(f)_e \otimes (E_1 - E_2)$ on $D(\Phi(f)_e) \otimes \mathbb{C}^2 \subset \mathscr{H} \otimes \mathbb{C}^2$. This operator has symmetric defect indices, and the defect projectors are equivalent modulo $\mathscr{A}(K) \otimes M_2(\mathbb{C})$. Hence $\tilde{\Phi}(f)$ has a self adjoint extension affiliated with

$$\mathscr{A}(K) \otimes M_2(\mathbb{C}) \subset \mathscr{P}(K^c)^w \otimes M_2(\mathbb{C}) \subset (\mathscr{P}(K^c) \otimes 1)^w .$$

Statement (2) then follows immediately from Proposition 2.3. The implication (2) $\Rightarrow$ (3) is obvious.

We now come to (3) $\Rightarrow$ (1). Translational invariance implies that, for all a in a neighbourhood $\mathscr{O}$ of $0 \in \mathbb{R}^d$, $\mathscr{W}$ is centrally positive with respect to the translated test function f_a in the algebra generated by f_a and test functions with support in $(W + \mathscr{O})^c$. The domain $W + \mathscr{O}$ can be included in a wedge domain which we shall again denote by W for convenience. Hence we can assume that $\mathscr{W}$ is centrally positive with respect to f_a in the algebra generated by f_a and test functions with support in W^c.

By [14, Lemma 4.7] we have that vectors of the form

$$\Phi(f_{a_1}) \dots \Phi(f_{a_n})\Omega$$

with $a_i \in \mathscr{O}$ span a dense set in $\mathscr{H}$. By Theorem 1.3 and Lemma 1.4 it suffices to show that all such vectors belong to the closure of $\mathscr{P}(W^c)\Omega$. Now since $\mathscr{W}$ is centrally positive with respect to f_{a_i} for $i = 1, \dots, n$ it follows from Proposition 2.3 that for each i the operator $\Phi(f_{a_i})$ has a self adjoint extension $\hat{\Phi}(f_{a_i})$ in a Hilbert space $\hat{\mathscr{H}}_i$ such that bounded functions of $\hat{\Phi}(f_{a_i})$ commute weakly with $\mathscr{P}(W^c)$ on $\mathscr{D}_0$. Denoting the projector $\hat{\mathscr{H}}_i \to \mathscr{H}$ by E_i it follows that for all bounded functions F_i the bounded operator

$$\varrho_i(F_i(\hat{\Phi}(f_{a_i})) := E_i F_i(\hat{\Phi}(f_{a_i}) | \mathscr{H}$$

belongs to $\mathscr{P}(W^c)^w$. Since the vector $\Phi(f_{a_2})\ldots\Phi(f_{a_n})\Omega$ belongs to the domain of $\hat{\Phi}(f_{a_i})$, we can approximate $\Phi(f_{a_1})\ldots\Phi(f_{a_n})\Omega$ by

$$\varrho_1(F_1(\hat{\Phi}(f_{a_1}))\Phi(f_{a_2})\ldots\Phi(f_{a_n})\Omega$$

with a suitable bounded function F_1. Now $\varrho_1(F_1(\hat{\Phi}(f_{a_1}))$ is a bounded operator so we can iterate this procedure. Thus we can for every $\varepsilon>0$ find bounded functions $F_1,\ldots,F_n$ such that

$$\|\Phi(f_{a_1})\ldots\Phi(f_{a_n})\Omega-\varrho_1(F_1(\hat{\Phi}(f_{a_1}))\ldots\varrho_n(F_n(\hat{\Phi}(f_{a_n}))\Omega\|\leqq\varepsilon.$$

Since $\mathscr{P}(W^c)^w$ is an algebra by Lemma 1.4 it follows that $\mathscr{P}(W^c)^w\Omega$ is dense in $\mathscr{H}$.

3.2. Remark. Condition (3) can be replaced by the apparently weaker condition:

(3′) There exists a test function f with a strictly nonvanishing Fourier transform and support in the interior of a wedge W such that $\mathscr{W}$ is centrally positive with respect to f when restricted to the subalgebra generated by f and the *test functions f_L, where L denotes a Poincaré transformation such that* $\operatorname{supp} f_L \subset W^c$.

The equivalence of (3′) and (3) follows from the transitivity of relative locality [6]: The net

$$R\mapsto\mathscr{B}(R):=\{\Phi(f_L)\,|\,L \text{ Poincaré transformation, } \operatorname{supp} f_L\subset R^c\}^w$$

is relatively local to the (unbounded) net generated by the operators $\Phi(f_L)$. This net in turn has Ω as a cyclic vector and is relatively local to the unbounded net generated by all field operators. It follows that the net $\mathscr{B}$ is relatively local to the field; in particular we have

$$\mathscr{B}(W)\subset\mathscr{P}(W^c)^w,$$

and hence

$$\{\Phi(f_L)\,|\operatorname{supp} f_L\subset W^c\}^w=\mathscr{P}(W^c)^w.$$

Condition (3′) should be compared with Theorems 5.5 and 5.6 in [14]; from the point of view of [14] we have shown that $\Phi(f)$ is "intrinsically local" if and only if the positivity condition stated in (3′) is fulfilled.

4. Conclusions

We have shown that a fairly simple positivity property of the Wightman functions is a *necessary and sufficient condition for the existence of an associated local net of* von Neumann algebras on the Hilbert space of the field, provided the weak commutants $\mathscr{P}(K)^w$ are known to be algebras. In particular the criterion applies if the field satisfy a generalized H-bound, or if $\mathscr{P}(K)$ is an essentially self adjoint operator family in the sense of [21]. The main role of the condition on the weak commutants is to guarantee that the local von Neumann algebras operate on the original Hilbert space of the field. In fact, if one takes the point of view that the construction of local nets from fields is a noncommutative moment problem, it is natural to generalize Definition 1.1 and allow extensions of the field operators that lead out of the Hilbert space. If $\mathscr{P}(K)^w$ is an algebra for all K, however, this is only an apparent generalization: As in the proof of Theorem 3.1, the existence of local self adjoint extensions $\hat{\Phi}(f)$ of the field operators implies that $\mathscr{P}(K^c)^w\Omega$ is dense in the original Hilbert space, even if $\hat{\Phi}(f)$ operates in a larger space. By Theorem 1.3 this implies the existence of a local net in the sense of Definition 1.1.

It is obvious from the proof of Theorem 3.1 that condition (2) is fulfilled if the field operators have local self adjoint extensions, regardless whether the extensions operate in the original Hilbert space or not. It is not clear, however, that this condition is sufficient for such extensions if the weak commutants $\mathscr{P}(K)^w$ are not algebras. There exist stronger positivity conditions [1, 10, 29] that *are* sufficient in all cases, but their precise relation to Theorem 3.1 has still to be worked out.

As a final remark we point out that positivity conditions as in Theorem 3.1 have the nice feature of being stable under limits. It is for example easy to see in this way that all Wick polynomials of free fields satisfy condition (3) of Theorem 3.1: If $\Phi = :P(\Phi_0):$ with P a polynomial and Φ_0 a free field, one can for every real test function f approximate $\Phi(f)$ by a hermitian polynomial in $\Phi_0(f_1), \ldots, \Phi_0(f_n)$, where the supports of the test functions $f_1, \ldots, f_n$ are close to the support of f. Such hermitian polynomials in free field operators can by [14, Theorem 3.3] be extended to self adjoint operators without significantly enlarging the localization domain. Hence the positivity condition is fulfilled for the approximating operators, and passing to the limit one obtains it for the field Φ.

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