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POSSIBILITY OR IMPOSSIBILITY OF THE METRIZABILITY OF CONE METRIC SPACES VIA RENORMING THE BANACH SPACES

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Abstract. In this paper we modify the proof of main theorem on the paper 'Metrizability of Cone Metric Spaces Via Renorming the Banach Spaces' ([3]), and also we state and confirm that we can convert non-normal cone to normal with constant one. Therefore the metrizability of cone metric spaces via renorming the Banach spaces is possible. We also present an famous example herein.

1. Introduction

In 2007 Huang and Zhang [7], generalized the concept of a metric space, by introducing cone metric space, and obtained some fixed point theorems for mappings satisfying certain contractive conditions. The study of fixed point theorems in such spaces known as cone metric spaces was taken up by some other mathematicians. But a basic question remained unanswered: *"Are those spaces a real generalization of metric spaces?"* Recently this question has been investigated in the author's paper [2] and in other papers [1, 4, 5, 6, 9]. The authors showed that cone metric spaces are metrizable and defined the equivalent metric using a variety of approaches. However there was another question *"Do maps satisfy an analogous contractive condition in the equivalent metric to that satisfied for the cone metric?"* Various authors answered this affirmatively for a number of contractive conditions but it is impossible to answer the question in general.

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In this paper, we modify the proof of main theorem in the paper 'Metrizability of Cone Metric Spaces Via Renorming the Banach Spaces' ([3]), since modification is according to an counterexample which find by Z. Kadelburg and S. Radenović in the paper which is presented in [8], also we state that we can convert non-normal cone to normal with constant one.

2. Preliminaries

Let E be a real Banach space. A nonempty convex closed subset $P \subset E$ is called a cone in E if it satisfies:

(i) $P \neq \{0\}$,
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in P$ imply that $ax + by \in P$,
(iii) $x \in P$ and $-x \in P$ imply that $x = 0$.

Let E be a real Banach space. A nonempty convex closed subset $P \subset E$ is called a cone in E if it satisfies:

(i) $P \neq \{0\}$,
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in P$ imply that $ax + by \in P$,
(iii) $x \in P$ and $-x \in P$ imply that $x = 0$.

The space E can be partially ordered by the cone $P \subset E$; by $x \preceq y$ if and only if $y - x \in P$. Also we write $x \prec\prec y$ if $y - x \in P^o$, where P^o denotes the interior of P.
A cone P is called normal if there exists a constant $K > 0$ such that $0 \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$.
In the sequel we suppose that E is a real Banach space, P is a cone in E with nonempty interior *i.e.*, $P^o \neq \emptyset$ and $\preceq$ is the partial ordering with respect to P.

Definition 2.1. ([7]) Let X be a nonempty set. Assume that the mapping $D : X \times X \to E$ satisfies

(i) $0 \preceq D(x, y)$ for all $x, y \in X$ and $D(x, y) = 0$ iff $x = y$,
(ii) $D(x, y) = D(y, x)$ for all $x, y \in X$,
(iii) $D(x, y) \preceq D(x, z) + D(z, y)$ for all $x, y, z \in X$.

Then D is called a cone metric on X, and (X, D) is called a cone metric space.

The following theorem state that we can convert every normal cone with $K > 1$ to normal cone with $K = 1$.

Theorem 2.2. *Let $(E, \|.\|)$ be a real Banach space with a normal cone P with $K > 1$. Then there exists a norm on E such that P is a normal cone with constant $K = 1$ with respect to this norm.*

Proof. Define $|||.|||: E \to [0, \infty)$ by

$$|||x||| := \inf\{\|u\| : x \preceq u\} + \inf\{\|v\| : v \preceq x\},$$

for all $x \in E$. We shall show that $|||.|||$ is a norm on E. Firstly, by definition of $|||.|||$ it is clear that if $x = 0$, then $|||x||| = 0$. If $|||x||| = 0$, then

$$\exists \ u_n, v_n \in E \quad \text{such that} \quad v_n \preceq x \preceq u_n,$$

where $u_n \to 0$ and $v_n \to 0$ as $n \to \infty$. Since P is a normal cone then we get $x = 0$.
Also

$$\begin{aligned} |||-x||| &= \inf\{\|u\| : -x \preceq u\} + \inf\{\|v\| : v \preceq -x\} \\ &= \inf\{\|u\| : -u \preceq x\} + \inf\{\|v\| : x \preceq -v\} \\ &= \inf\{\|v'\| : v' \preceq x\} + \inf\{\|u'\| : x \preceq u'\} \\ &= |||x|||. \end{aligned}$$

For $\lambda > 0$,

$$\begin{aligned} |||\lambda x||| &= \inf\{\|u\| : \lambda x \preceq u\} + \inf\{\|v\| : v \preceq \lambda x\} \\ &= \inf\left\{\lambda\|\frac{1}{\lambda}u\| : x \preceq \frac{1}{\lambda}u\right\} + \inf\left\{\lambda\|\frac{1}{\lambda}v\| : \frac{1}{\lambda}v \preceq x\right\} \\ &= \lambda|||x|||. \end{aligned}$$

Therefore $|||\lambda x||| = |\lambda||||x|||$ for all $x \in E$ and $\lambda \in \mathbb{R}$.

To prove the triangle inequality of $|||.|||$, let $x, y \in E$

$$\forall\, \varepsilon > 0, \ \exists\, u_1, v_1 \quad s.t. \quad v_1 \preceq x \preceq u_1, \quad \|u_1\| + \|v_1\| - \varepsilon < |||x|||,$$

$$\forall\, \varepsilon > 0, \ \exists\, u_2, v_2 \quad s.t. \quad v_2 \preceq y \preceq u_2, \quad \|u_2\| + \|v_2\| - \varepsilon < |||y|||.$$

Therefore $v_1 + v_2 \preceq x + y \preceq u_1 + u_2$. Hence

$$|||x + y||| \leq \|v_1 + v_2\| + \|u_1 + u_2\| \leq |||x||| + |||y||| + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we obtain

$$|||x + y||| \leq |||x||| + |||y|||.$$

So $|||.|||$ is a norm on E. Now we show that P is normal cone with $K = 1$ with respect $|||.|||$. Let $0 \preceq x \preceq y$. So $\inf\{\|v\| : v \preceq x\} = \inf\{\|v\| : v \preceq y\} = 0$, hence $|||x||| \leq |||y|||$. □

3. Main Results

Theorem 3.1. *Let* $(E, \|.\|)$ *be a real Banach space with a positive cone* P*. Then there exists a norm on* E *such that* P *is a normal cone with constant* $K = 1$ *with respect to this norm.*

Proof. We make $m(.) : E \to [0, \infty)$ with $m(.) \neq 0$ by the following conditions:

$$\begin{aligned} x = 0 \quad &\Rightarrow \quad m(x) = 0, \\ (m(x) = \|x\| \text{ or } m(x) = 0) \quad &\Rightarrow \quad x = 0, \\ m(\lambda x) \quad &= \quad |\lambda| m(x), \\ \|x\| - \|y\| \quad &\leq \quad m(x) - m(y) \text{ when } 0 \preceq x \preceq y, \quad (3.1) \\ \|x + y\| - \|x\| - \|y\| \quad &\leq \quad m(x + y) - m(x) - m(y), \quad (3.2) \end{aligned}$$

for all $x, y \in E$ and $\lambda \in \mathbb{R}$. By (3.1) if we choose $x = 0$, then

$$m(x) \leq \|x\|, \quad \forall\, x \in P. \quad (3.3)$$

Now put $n(x) = \|x\| - m(x)$ for every $x \in E$. Hence $n(.)$ is a norm. And also for every $x, y \in E$ with $0 \preceq x \preceq y$ we get $n(x) \leq n(y)$. □

Remark 3.2. The $m(.)$ exists. It is enough that we consider $m(x) = \|x\|$.

Example 3.3. Let $E := C^1_{\mathbb{R}}([0, 1])$ with $\|x\| := \|x\|_\infty + \|x'\|_\infty$. Hence $P := \{x \in E : x \geq 0\}$ is a non-normal cone. Now if we consider $m(.)$ by $m(x) := \|x'\|_\infty$, then

$$n(x) := \|x\| - m(x) = \|x\|_\infty,$$

and P is normal cone with $K = 1$.

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