

POSSIBLE EMITTANCE INCREASE THROUGH FILAMENTATION DUE TO SPACE CHARGE  
IN CONTINUOUS BEAMS

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Summary

The study of oscillations which can take place in a beam under space charge conditions has been extended to other distributions than the one described by Kapchinsky and Vladimirsky (K.V.) in 1959. Two sets of equations similar to the ones they established can be written provided axis and emittance values are replaced by rms expressions applicable to any type of distribution. In case of a non K.V. distribution, the comparison of these sets gives differential equations relating the rms emittances to correcting terms which depend on the charge distribution inside the beam. Such equations can be used to interpret some rms emittance increases observed either with a mismatch in a smooth focusing channel or with a matched beam in an AG focusing system.

Introduction

Experimental measurements of output beam emittance in linear accelerators have shown in the CERN Proton synchrotron injector that there is always an increase in emittance values from input to output for large intensities (P. Lapostolle et al [1968] C. Taylor [1970]). Numerical studies making use of computer simulation (M. Promé et al [1970]) have more recently given results which agree reasonably well with the experiments and have shown such an increase. The interpretation of this effect is however not clear yet.

In order to simplify the problem a two-dimensional (x, y) case, corresponding to a supposedly continuous beam has been simulated on a computer (Tanguy [1970]) and thoroughly investigated (P. Lapostolle [1970]). The results obtained and the theoretical work it led to are described in the following paragraphs.

Computer Results

Since a complete family of stationary charge distributions are known theoretically (P. Lapostolle [1969]) they have first been put in the program to check its accuracy. Oscillations have later on been studied.

Oscillations can be of two kinds: envelope oscillations (or oscillations of the overall beam dimensions) and oscillations of the density distribution inside the beam.

In order to define unambiguously beam dimensions in the computer simulation and to include the possibility of studying other types of distribution (Gaussian like for instance), rms values have been introduced, as follows:

$$\left\{ \begin{array}{l} a^2 = 4 \overline{x^2} \qquad b = 4 \overline{y^2} \\ EM_x^2 = 16 (\overline{x^2 x'^2} - \overline{xx'})^2, \quad EM_y^2 = 16 (\overline{y^2 y'^2} - \overline{yy'})^2 \end{array} \right. \quad (1)$$

Fig. 1 and 2 show how  $\overline{x^2}$ ,  $\overline{y^2}$  and  $EM_x$ ,  $EM_y$  vary during symmetrical and antisymmetrical oscillations. Though treated at a low energy this case can be scaled; at 50 MeV it would correspond approximately to the normal injection current into the CERN synchrotron. It is seen that over about 40 betatron oscillations, the rms beam emittance increases by a factor of 2 to 3 for an initial mismatch of about 2.4 (maximum beam size/matched beam size). During this process while the oscillations are damped, the arithmetic average of  $\overline{x^2}$  or  $\overline{y^2}$  remains constant. The emittance increase is accompanied by a small oscillation at the same frequency as the envelope which can be seen in the computer runs to be related to density oscillations inside the beam; these density oscillations are induced by envelope oscillations and the increase of the average emittance depends on the relative phase between envelope oscillations and emittance oscillations.

A similar increase is also found in the computer simulation of an AG focused matched beam: as soon as the wriggling factor is not very small one observes an rms emittance increase which can be very large at large intensities or low energies and decreases asymptotically to zero when the space charge is reduced, in a way which could not be obtained with the program mentioned above due to lack of accuracy (in particular the possible existence of a threshold could not be decided upon).

A new computer program making use of more modern and faster techniques (R. Le Bail [1970]) has now been prepared to extend the previous work.

First Theoretical Results

The observations described above led to the prediction of one interesting theorem: any stationary distribution (which remains unchanged during the motion of the individual particles) satisfies the couple of equations (P. Lapostolle [1970]).

$$\begin{aligned} \omega_x^2 a - \frac{EM_x^2}{a^3} - \frac{2A}{a+b} &= 0 \\ \omega_y^2 b - \frac{EM_y^2}{b^3} - \frac{2A}{a+b} &= 0 \end{aligned} \quad (2)$$

where  $\omega_x$  and  $\omega_y$  specify the focusing forces (betatron angular frequencies:  $\omega = 2\pi/\lambda$ ) and

$$A = \frac{2 r_0 I/e}{c \beta^3 \gamma^3} \quad (3)$$

I is the current in the beam,  $r_0$  the electromagnetic radius of the particles of charge e; c,  $\beta$  and  $\gamma$  have their usual meaning in relativity.

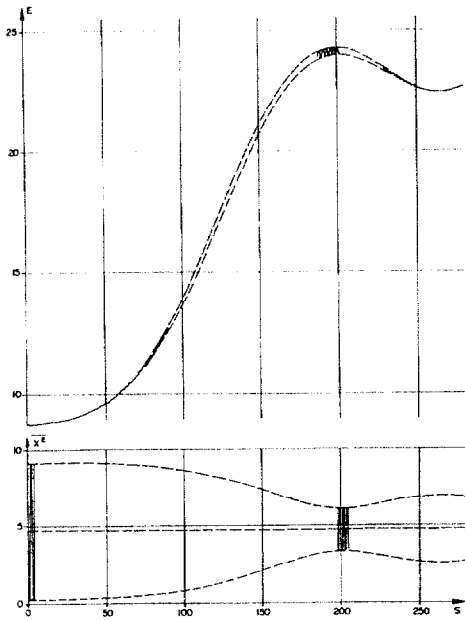


Fig. 1

Evolution of the rms emittances and mean square amplitudes in a smooth focusing channel for a mismatched beam with low space charge.

Fig. 1 - Symmetrical Oscillations  
Fig. 2 - Antisymmetrical Oscillations.

Intensity 1 mA  
Energy 750 keV  
 $\lambda = 4,72$  m  
EM in mrad mm  
 $x^2$  in mm<sup>2</sup>  
s in m

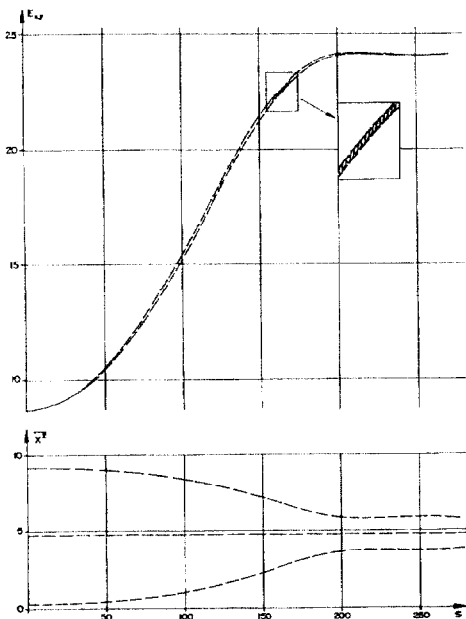


Fig. 2

Such a theorem has been proved since then by R. Gluckstern [1970 a] for the case of circular symmetry and by F. Sacherer [1970] for the case of elliptical symmetry.

While the new computer program was prepared more theoretical work was performed extending the previous theorem.

Generalized Kapchinsky Vladimirsky Equations  
(P. Lapostolle [1971])

The pioneer work in the field of space charge was presented in 1959 by Kapchinsky and Vladimirsky [1959] who treated the case of a uniform density distribution. They gave the equations of a and b, semi axes of the elliptic cross section the beam has in their treatment.

Let us now consider a general case where

$$\begin{cases} x'' + \omega_x^2 x - E_x = 0 \\ y'' + \omega_y^2 y - E_y = 0 \end{cases} \quad (4)$$

(where the derivative is with respect to s, the longitudinal coordinate) are the equations of motion,  $E_x$  and  $E_y$  being the space charge force terms. The beam is assumed to have  $Ox_s$  and  $Oy_s$  for planes of symmetry.

A first set of K.V. equations is obtained by multiplying the first equ. by x and the second by y and taking the averages:

Using the property (see Annex)

$$\overline{x E_x} + \overline{y E_y} = A/2 \quad (5)$$

and putting

$$\frac{\overline{x E_x}}{a} - \frac{\overline{y E_y}}{b} = A \frac{a+b}{4ab} \epsilon_1 \quad (6)$$

one gets

$$\begin{cases} a'' + \omega_x^2 a - \frac{EM^2}{a^3} - \frac{2A}{a+b} - \frac{A}{a} \epsilon_1 = 0 \\ b'' + \omega_y^2 b - \frac{EM^2}{b^3} - \frac{2A}{a+b} + \frac{A}{b} \epsilon_1 = 0 \end{cases} \quad (7)$$

$\epsilon_1$  is zero when there is an additional symmetry with respect to the bissectrices; it is zero for an elliptical distribution; is very close to zero in all studied cases (between 10,004 for a rectangular distribution...). It might however become appreciable in the presence of an outer conductor of flat elliptic shape at short distance.

For a stationary state, neglecting  $\epsilon_1$ , one finds equ. (2).

A second set of K.V. equations is obtained by multiplying the first equ. (4) by  $x'$  and the second by  $y'$  and taking the averages: this leads to the usual total energy equations; seeing that

$$\overline{x' E_x} + \overline{y' E_y} = 2 \frac{dW_{ce}}{ds} \quad (8)$$

where  $W_{ce}$  is the space charge energy and making use of the properties (see Annex)

$$W_{ce} = \frac{A}{2} \overline{\text{Log}(X+Y)} + C^t \quad (9)$$

$$W_{ce} = \frac{A}{4} [\text{Log}(a+b) + \epsilon_2] + C^t \quad (10)$$

## Conclusions

one can write

$$\begin{cases} a'' + \omega_x^2 a - \frac{EM_x^2}{a^3} - \frac{2A}{a+b} + \frac{EM_x EM'_x}{a^2 a'} - \frac{2A}{a'} (\epsilon'_2)_x = 0 \\ b'' + \omega_y^2 b - \frac{EM_y^2}{b^3} - \frac{2A}{a+b} + \frac{EM_y EM'_y}{b^2 b'} - \frac{2A}{b'} (\epsilon'_2)_y = 0 \end{cases} \quad (11)$$

$\epsilon_2$  is found in most practical distributions to be comprised between 0,10 and 0,05; one has put

$$\begin{cases} \overline{x'E_x} = Aa'/2(a+b) + A(\epsilon'_2)_x/2 \\ \overline{y'E_y} = Ab'/2(a+b) + A(\epsilon'_2)_y/2 \end{cases} \quad (12)$$

### Applications of K.V. Equations

Making the difference between (7) and (11) one gets:

$$\begin{cases} \frac{EM_x EM'_x}{a^2} = 2A(\epsilon'_2)_x - A \frac{a'}{a} \epsilon_1 \\ \frac{EM_y EM'_y}{b^2} = 2A(\epsilon'_2)_y + A \frac{b'}{b} \epsilon_1 \end{cases} \quad (13)$$

$\epsilon_1$  can probably be neglected in most cases; the sum of  $(\epsilon'_2)_x + (\epsilon'_2)_y$  is small but not negligible.

In case of a round beam and symmetrical oscillations, one finds

$$EM_x EM'_x = a^2 A \epsilon_1 \quad (14)$$

In the absence of envelope oscillations one then has  $\Delta EM_x = 2a^2 A \Delta \epsilon_1$  which is well observed in the computations. If however a oscillates together with  $\epsilon_2$  (and EM) according to the relative phase of these oscillations one can obtain more or less secular change of the rms emittance.

Similar results can be obtained for anti-symmetrical oscillations with the additional possibility in this case of an exchange between  $EM_x^2$  and  $EM_y^2$  since it is the sum of  $(\epsilon'_2)_x$  and  $(\epsilon'_2)_y$  which is small while each term may become appreciable. Such an exchange is also observed in numerical computations.

The final stationary state towards which the beam yields asymptotically in Fig. 1 and 2 can be found by equating final and initial energy obtained from (11); one finds the property of quasi conservation of the arithmetic average of  $a^2$  and  $b^2$ . A similar computation cannot be done in the case of an AG focusing since the energy is not necessarily conserved: when going from a focusing to a defocusing space or vice versa there is a change in focusing energy by coupling with longitudinal energy.

A main purpose of this paper is to present a new formalism for studying some non-linear effects of the space charge field. The ordinary set of K.V. generalized equations or the energy set, assuming that the value of the emittances and of their short term variations are known predict accurately envelope oscillations. The new set of equations obtained by difference relates emittance variations to the variations of the charge distribution inside the beam. The problem would then be to study the oscillations of these distributions.

R. Gluckstern [1970 b] has introduced the concept of oscillation modes in a beam. Such modes include what we called envelope oscillations as well as density oscillations. A general study of such modes (so far limited to the K.V. distribution for a round beam) and of their excitation by envelope oscillation (because the equilibrium distribution is changing with the beam size or shape) would shed light on the possible variations of  $\epsilon_2$  and hence perhaps explain the observed long term rms emittance growths. Very much has still to be done on such a study.

A statistical approach (where rms values play an essential role) considering the space charge potential as a stochastic function (with adequate correlating function) might also offer another insight into the problem.

The final aim would be to understand the observed growth of emittances in high intensity linacs and to find out whether such a process always exists, only at a lower extent for low intensities and large energies, or whether there is a threshold below which nothing happens.

It is worth noticing that in the case of an applied non-linear field, equations similar to (7), (11) and (13) can be written; then  $\epsilon_1$  and  $(\epsilon'_2)$  are easily expressed as a sum of moments in  $x^k y^{n-k}$  and  $x^{k-1} y^{n-k-1}$  or  $x^k y^{n-k-1}$ . For the space charge field, the image effects which would be produced by far enough boundaries can be expressed in the form of products of moments; such a form has not yet been found for the self field.

### Remark

Generalized K.V. equations (7), (11) and equation (13) also apply when so called coherent oscillations exist in a beam. It can be seen, in this case, that in free space  $\epsilon_1$  and  $(\epsilon'_2)$  relative to coherent oscillations are equal to zero in such a way that there is no coupling between incoherent and coherent oscillations, these last ones being always on the same frequency whatever the intensity is. In the case where image effects are present, on the contrary,  $\epsilon_1$  and  $(\epsilon'_2)$  are not zero and coherent oscillations can produce an emittance growth.

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## Annex

Considering the relation

$$\frac{1}{2} \operatorname{div} E^2 r = (\vec{E} \cdot \vec{r}) \operatorname{div} E - E \cdot \operatorname{rot} r \times E$$

it is easy to see that for a cylindrical beam having two planes of symmetry, the second term of the rhs is an odd function of x and y such that its integral over a symmetrical contour is nil.

One then has

$$\iint \vec{E} \cdot \vec{r} \rho \, dx \, dy = Q^2 / 4\pi \epsilon_0$$

where Q is the total charge per unit length of the beam. The space charge energy  $W_{ce}$  then satisfies

$$\begin{aligned} W_{ce} &= \frac{\epsilon_0}{2} \iint E^2 \, dx \, dy = \frac{1}{2} \iint v_{ce} \rho \, dx \, dy \\ &= \frac{Q^2}{4\pi \epsilon_0} \operatorname{Log}(0,99R) - \iint E \, r \, \rho \operatorname{Log} \frac{|x|+|y|}{2} \, dx \, dy \end{aligned}$$

which can be written

$$W_{ce} = \frac{Q^2}{4\pi \epsilon_0} \overline{\overline{\operatorname{Log} R}} - \frac{Q^2}{4\pi \epsilon_0} \overline{\overline{\operatorname{Log} \frac{X+Y}{2}}}$$

where the integrals are performed over the cross section of a very large cylinder of radius R or of any other equipotential surface external to the charges (the double bar indicates an average over the cross section weighed by the function  $E \, r \, \rho$ ).

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