POSSIBLE LIMIT LAWS FOR ENTRANCE TIMES OF AN ERGODIC APERIODIC DYNAMICAL SYSTEM

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In memory of Anzelm Iwanik

Abstract. Let $\mathcal{G}$ denote the set of decreasing $G\colon \mathbb{R} \to \mathbb{R}$ with $G \equiv 1$ on $]-\infty, 0]$, and $\int_0^\infty G(t)dt \leq 1$. Let X be a compact metric space, and $T : X \to X$ a continuous map. Let μ denote a T-invariant ergodic probability measure on X, and assume (X, T, μ) to be aperiodic. Let $U \subset X$ be such that $\mu(U) > 0$. Let $\tau_U(x) = \inf\{k \geq 1 : T^k x \in U\}$, and define

$$G_U(t) = \frac{1}{\mu(U)} \mu(\{x \in U : \mu(U)\tau_U(x) > t\}),\ t \in \mathbb{R}.$$

We prove that for μ-a.e. $x \in X$, for any $G \in \mathcal{G}$, there exists a sequence $(U_n)_{n\geq 1}$ of neighbourhoods of x such that $\{x\} = \cap_n U_n$, and

$$G_{U_n} \to G \text{ weakly.}$$

We also construct a uniquely ergodic Toeplitz flow $(\mathcal{O}(x^\infty), S, \mu)$, the orbit closure of a Toeplitz sequence x^∞, such that the above conclusion still holds, with moreover the requirement that each U_n be a cylinder set.

1. Introduction

1.1. Preliminaries and notations.

Throughout X shall denote a compact metric space, $T\colon X \to X$ a continuous map, and μ shall be a Borel T-invariant ergodic probability measure on X.

Given $U \subset X$ with $\mu(U) > 0$, Poincaré's recurrence theorem asserts the following random variable to be μ-a.s. well defined :

$$\tau_U(x) = \inf\{k \geq 1\colon T^k x \in U\}, \quad x \in X.$$

Next Kac's return time theorem [K] reads

$$\mathbb{E}(\mu(U)\tau_U) = \sum_{t\geq 1} t\mu(U \cap \{\tau_u = t\}) = 1,$$

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where the expectation is computed with respect to the induced probability measure on U, $\mu_U := \frac{\mu}{\mu(U)}$.
A remark is that (X,T,μ) is ergodic if and only if Kac's estimation for the expectation is valid for any subset having positive measure.

It is natural to try to understand finer statistical properties of the variable $\mu(U)\tau_U$ on the space $(U,\mathcal{B}(X)\cap U,\mu_U)$, for instance the distribution function $F_{\mu(U)\tau_U}(t)=\mu_U(\{x:\mu(U)\tau_U\le t\})$.

Another approach, rather rapidly developing in the last decade, is to describe asymptotics for entrance times : let $x\in X$ and let $(U_n)_{n\ge1}$ denote a decreasing sequence of neighbourhoods for x, with $\cap_n U_n=\{x\}$. The question then is weither $(F_{\mu(U_n)\tau_{U_n}})_{n\ge1}$ converges weakly or not, and in the case it does, to describe the limit.

The limit has been shown to exist μ-a.s. for suitably chosen $(U_n)_{n\ge1}$ and to be the distribution function of the positive exponential law with parameter 1, in many classes of mixing systems [AG], [BSTV], [C], [CC], [CG], [H1,2], [HSV], [P], [S], [Y]. Non exponential asymptotics have been obtained in [CF] and [DM].

This short note is devoted to the two following goals :

– *describe possible F_Us*;

– *describe possible asymptotics* (a question raised in most papers listed above).

Essentially the second relies on the sufficient achievement of the first. Let us therefore start by describing F_Us. For conveniency, we shall put

$$G_U=1-F_U.$$

Clearly, G_U is

$$\begin{cases}(i)\colon \text{decreasing},\\ (ii)\colon \text{right continuous},\\ (iii)\colon \text{simple},\\ (iv)\colon \text{has discontinuities at } \{t_1\mu(U)<\ldots<t_k\mu(U)<\ldots\},\\ (v)\colon G_U\equiv 1 \text{ on }]-\infty,0],\end{cases}$$

where $\{t_1<t_2<\ldots\}=\{\tau_U(x)\colon x\in U\}$. Since $\mu(U)\tau_U$ is a positive random variable, we can compute its expectation by

$$1=\mathbb{E}(\mu(U)\tau_U)=\int_0^\infty G_U(t)dt, \tag{vi}$$

which is yet another translation of Kac's theorem.

1.2. Results.

We first partially enough describe G_Us.

We call (X,T) **minimal** if any T-invariant closed subset $F\subset X$ is trivial. Within the class of minimal systems, recurrence to open subsets U occurs with **bounded gaps**, that is the set $\{t_1<t_2<\ldots\}$ is finite.

Proposition 1. *Properties* $(i—vi)$ *from the preliminaries, with the requirement that the set of discontinuity points* $\{t_1\alpha < t_2\alpha < \dots\}$ *be finite, are characteristic for* G_U*s arising from minimal systems* (X, T).

Let us now introduce a class $\mathcal{G}$, which will later serve as the set of weak limits of G_Us :

$$\mathcal{G} = \{G: \mathbb{R} \to \mathbb{R}, \text{ decreasing}, \equiv 1 \text{ on }]-\infty, 0], \text{ with } \int_0^\infty G(t)dt \le 1\}.$$

Theorem 1. *For any ergodic aperiodic* (X, T, μ)*, and* μ*-a.e.* $x \in X$*, for any* $G \in \mathcal{G}$*, there exists a sequence* $(U_n)_{n\geq 1}$ *of neighbourhoods of* x*, with* $\cap_n U_n = \{x\}$*, such that*

$$G_{U_n} \to G \textit{ weakly.}$$

Whence all possible asymptotics live in any aperiodic ergodic system.

The key tool in proving Theorem 1, which is, as K. Petersen pointed out to the author, a Van Strassen type theorem for asymptotics, is to understand how G_Us having rational parameters can be constructed within periodic systems.

This result is of course in contrast with exponential asymptotics as quoted to before, but this is due to the fact that the U_ns in Theorem 1 are extremely weired.

Whence it seemed interesting to us to obtain the same result but within a system where the U_ns may be chosen to be "real" cylinder sets. This is possible indeed, and before stating our last result we will introduce a few more notions.

Unique ergodicity of the flow (X, T) means that there is only one T-invariant probability measure μ, it implies ergodicity.

Strict ergodicity means both minimality and unique ergodicity.

Let $\mathcal{A}$ denote a finite set, an **alphabet**.

A **pattern** $U = (u_0, \dots, u_{n-1})$ over $\mathcal{A}$ is a finite string over $\mathcal{A}$.

Its **length** is $|U| = n - 1$.

Given two patterns U and V, their **concatenation** is the obvious pattern, denoted UV, of length $|U| + |V|$.

Let $x \in \Omega^+ := \mathcal{A}^{\mathbb{N}}$ or $\Omega := \mathcal{A}^{\mathbb{Z}}$ be a one or two sided sequence over $\mathcal{A}$.

On Ω^+ or Ω, define $S(x_n) = (x_{n+1})$, the **shift** transformation.

The **orbit closure of** x, $\mathcal{O}(x)$, is the closure of the orbit, $\mathcal{O}(x) := \overline{\{S^n x\}}$.

It is shift invariant, $(\mathcal{O}(x), S)$ is a topological flow, the **flow generated** by x.

A **Toeplitz flow** is the flow generated by a Toeplitz sequence.

A **Toeplitz sequence** [JK], [O], is an element $x \in \Omega$ which is non-periodic, but satisfies :

$$\forall n,\ \exists p(n) > 0: \forall k,\ x_n = x_{n+kp(n)}.$$

A Toeplitz flow is always minimal.

Theorem 2. *There exists a uniquely ergodic Toeplitz sequence $x^\infty \in \{0,1\}^{\mathbb{Z}}$ such that if (X,T,μ) denotes the generated shift dynamical system, and if (V_k) denotes the sequence of cylinders intersecting to x^∞, for any $G \in \mathcal{G}$, there exists a subsequence (U_n) of (V_k), intersecting to x^∞, with the property that*

$$\lim_n G_{U_n} = G, \text{ weakly.}$$

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2. Proof of Proposition 1

•$\underline{\Longrightarrow}$: this is already done in the Preliminaries sub-section where conditions $(i — vi)$ are shown to hold for any G_U.

By minimality the set of recurrence times $\{t_1 < t_2 < \dots\}$ must be finite.

•$\underline{\Longleftarrow}$: let us be given a G having finitely many discontinuities located at $\{t_1\alpha < \dots < t_{k+1}\alpha\}$ for some positive α.

We let $\beta_j = G(t_j\alpha^-) - G(t_j\alpha^+)$, $j = 1, .., k+1$.

We will produce a strictly ergodic $y \in \{0,1\}^{\mathbb{N}}$ such that if $0 = k_0 < k_1 < \dots$ denotes the set of indices n with $y_n = 1$, then

$$(1) \qquad \begin{cases} \lim_N \frac{1}{N}\#\{i\colon k_i < N\} = \alpha, \\ \forall i,\ k_{i+1} - k_i \in \{t_j\}, \\ \lim_N \frac{1}{N}\#\{i\colon k_i < N \text{ and } k_{i+1} - k_i = t_j\} = \alpha\beta_j,\ j = 1, ..., k+1. \end{cases}$$

If this can be achieved, then we will put

$$X = \overline{\{S^n y\colon n \geq 0\}},\ T = S,\ U = 1,$$

where 1 denotes both the symbol 1 or the cylinder set $1 = \{z \in X\colon z_0 = 1\}$. It is then an exercise to check that $G = G_U$.

Construction of y.

We put $I_j = [\sum_{s<j} \beta_s, \sum_{s\leq j} \beta_j[$ (recall $\sum_j \beta_j = 1$).

Let $\mathbb{T} = \mathbb{R} \big/ \mathbb{Z} = [0,1[$ and select θ irrational, such that θ and the $\sum_{j<s} \beta_j$'s are rationally independent.

Define $Rz = z + \theta \mod 1$, $z \in \mathbb{T}$.

The partition (I_j) of $\mathbb{T}$ is generating for the minimal uniquely ergodic rotation R on $\mathbb{T}$.

Apply repeatedly R to produce a sequence $(j_n) \in \{1, \dots, k+1\}^{\mathbb{N}}$, defined by

$$j_n = j \Longleftrightarrow R^n(0) \in I_j.$$

Then construct y by defining the sequence $(k_0 = 0 < k_1 < \dots)$ first :

$$k_n = \sum_{s<n} t_{j_s},\ n \geq 0.$$

Finally y is obtained by putting "1"s at positions k_n and "0"s elsewhere. $\square$

3. Proof of Theorem 1

3.1. Rational Gs and red/green ladders.

We say a G as in Proposition 1 is **rational** if its parameters, i.e. α, β_j, and t_j, are rational.

By Proposition 1, and its proof, such G is a G_U.

However in the case of rationality of parameters, we shall show how to produce G_U such that $G = G_U$ for some pattern U, and some uniquely ergodic periodic system $X = \{S^n x \colon 0 \leq n < P\}$ (with period P).

Set for rational G, $\alpha = p/q$ and $\beta_j = p_j/q$.

As in Proposition 1, we can assume that $t_1 \gg 3$.

Let $N = Mq^2$, and set $n_j = Mpp_j$.

Consider a perfectly periodic Rohlin tower of height N, label its levels numbered from 0 to $N-1$, and assume at the beginning that they all have a green color.

Change to red the color of the levels with numbers

$$0, t_1, 2t_1, \ldots, (n_1-1)t_1, n_1t_1, n_1t_1+t_2, \ldots, n_1t_1+n_2t_2, \ldots, \sum_{j<k} n_jt_j + (n_k-1)t_k.$$

Then let U denote the union of the red levels. It has density, within the tower, equal to α by construction, and red levels are separated by runs of consecutive greens having lengths $t_j - 1$, hence U returns to itself with times $t_1, \ldots, t_k$.

Moreover, the density of reds returning to reds by time t_j is precisely $\alpha\beta_j$ within the tower, for each j. Whence $G = G_U$.

We will keep in mind this idea that a rational G with finitely many discontinuities can be figured by this red/green ladder, a periodic Rohlin tower, with U equal to the union of the red levels, bottom marked red..

We observe that the set of weak limits of such Gs are the elements of $\mathcal{G}$.

3.2. Aperiodicity and Kac towns.

Lemma 1. *The system (X, T, μ) is non-periodic (if and) only if for μ-a.e. $x \in X$, for any $m \geq 1$, there exists $\varepsilon > 0$ such that if $B(x, \varepsilon)$ denotes the open ball centered at x and having radius ε, then*

$$B(x, \varepsilon) \subseteq \{\tau_{B(x,\varepsilon)} > m\}.$$

Proof. Assume that for some $m \geq 1$, if $A_m = \{x : \forall \varepsilon > 0,\ B(x, \varepsilon) \cap \{\tau_{B(x,\varepsilon)} \leq m\}\}$, $\mu(A_m) > 0$. Then pick a subsequence $\varepsilon_q \downarrow 0$ together with some $n \leq m$, such that for each q, there exists y_q with $d(x, y_q) < \varepsilon_q$ ("d" is some distance defining the X-topology), and $d(x, T^n y_q) < \varepsilon_q$. Then get

$$0 \leq d(x, T^n x) \leq \limsup_q d(x, T^n y_q) + \limsup_q d(T^n y_q, T^n x) = 0,$$

because $y_q \to x$ and T^n is continuous. Whence $A_m \subset \{x : \exists n \leq m,\ x = T^n x\}$, hence if $\mu(A_m) > 0$ for some m, (X, T, μ) has a periodic component.

The "if" part is left to the reader. $\square$

The standard way to prove Kac's theorem within an ergodic system is to take some U with $\mu(U) > 0$, to consider that $\mu(\{\tau_U < +\infty\}) = 1$ by ergodicity, and to cut be basin U into sets

$$U \cap \{\tau_U = k\},\ k \geq 1.$$

Then above each $U \cap \{\tau_U = k\}$ having positive measure starts a skyscraper having k floors exactly (bottom one included), along which the action of T goes up until it reaches the top floor where it returns to U some way.

Whence the whole space X is (mod 0) the union of those skyscrapers with basins of the form $U \cap \{\tau_U = k\}$ that cover U. This picture is a **Kac town**.

Our Lemma 1 states that **in an aperiodic ergodic system, a.s., Kac towns based on ε-balls at the point, having arbitrarily large minimal heights, exist**.

3.3. Lighting floors on or off : proof of Theorem 1.

Take a rational G having finitely many discontinuities, as in Proposition 1, and construct a red/green ladder or sheave realizing it.

Choose some $\varepsilon > 0$ arbitrary. Consider a Kac town based on some small neighbourhood of $x \in X$ having positive measure, and let $m \gg 1$ be its minimal height ($\{\tau_U \geq m\}$ on U).

Within each skyscraper in the town, start piling up red/green ladders starting from the bottom floor, assuming a ladders step has a floor height. Don't stop piling unless the ladder piled exceeds the skyscrapers top floor.

Then according to weither a ladders step is red or green, either turn the floors light on, or leave it turned off.

Set W **equals to the union of the lightened floors in Kac's city**.

Then W has measure

$$|\mu(W) - \alpha| \leq \mu(\text{union of floors above ladders}),$$

which can be made less than ε provided m is chosen large enough.

Second, $\{t_1, \ldots, t_k\} \subset \tau_W(W)$, and

$$\mu(\{x \in W : \tau_W(x) \notin \{t_1, \ldots, t_k\}\})$$

can be made smaller than ε provided m is chosen large enough.

Taking a sequence $\varepsilon_q \downarrow 0$, this procedure shows that **any weak limit of a sequence of rational Gs as in Proposition 1 can be attained as a weak limit of G_Ws as constructed above**.

Finally collect rational Gs in a sequence $(G_n)_{n \geq 0}$. Let $G_{n,q} = G_{W_{n,q}}$ be the G_W associated to G_n and ε_q, and collect all together within a sequence $(U_n)_{n \geq 0}$ the union of the sequences $(W_{n,q})$ and $(B(x, \varepsilon_q))$. Then $\cap_n U_n = \{x\}$. $\square$

4. Proof of Theorem 2

4.1. Rational Gs : holes instead of red/green ladders and periodic sequences.

Recall a G as in Proposition 1 is **rational** if its parameters, i.e. α, β_j, and t_j, are rational. Such G is a G_U, by Proposition 1.

However in the case of rationality of parameters, we shall show how to produce G_U such that $G = G_U$ for some pattern U, and some uniquely ergodic periodic system $X = \{S^n x: 0 \leq n < P\}$.

Set for rational G, $\alpha = p/q$ and $\beta_j = p_j/q$.

As in Proposition 1, we can assume that $t_1 \gg 3$.

Let $N = Mq^2$, and set $n_j = Mpp_j$.

Assume throughout that $\vee$ **denotes a symbol not in** $\mathcal{A}$.

Select $U = ab^2$, with $a \neq b$ in $\mathcal{A}$.

Then put

$$J_j = ab^{t_j - 3} \vee a, \;\; 1 \leq j \leq k+1,$$

and set

$$J = \underbrace{J_1 \ldots\ldots J_1}_{n_1 \text{ times}} \ldots \underbrace{J_j \ldots\ldots J_j}_{n_j \text{ times}} \ldots \underbrace{J_{k+1} \ldots\ldots J_{k+1}}_{n_{k+1} \text{ times}} = J_1^{n_1} \ldots J_{k+1}^{n_{k+1}}$$

(notice that $|J| = N$, $|J_j| = t_j$).

Finally set $x = \ldots J \ldots J \ldots JJ \ldots = J^\infty$, with $x[0, |J|[= J$ $(x \in (\mathcal{A} \cup \{\vee\})^{\mathbb{Z}})$.

Then the least period of x is $|J|$, because since $t_1 < \ldots < t_{k+1}$, J can occur in x only at positions $[n|J|, (n+1)|J|[$ for some $n \geq 0$.

The desired frequencies are issued by construction, i.e. $G = G_U$ where $U = \{z \in \{S^n x: n \geq 0\}: z[0, |U|[= U\}$ (once again, we do not distinguish the pattern and the cylinder it defines on the orbit closure).

4.2. Induction.

The set of rational Gs satisfying conditions of Proposition 1 is countable.

We can collect them all along a sequence, $(G^{(n)})_{n \geq 0}$.

•**<u>First step</u>** : let $x^{(0)}$ be as in sub-section 4.1. for $G^{(0)}$, with notations maintained except that we add some superscript "$^{(0)}$".

•**<u>Second step</u>** : $G^{(1)}$ comes with parameters $\alpha^{(1)} = p^{(1)}/q^{(1)}$, $\beta_j^{(1)} = p_j^{(1)}/q^{(1)}$, $t_1^{(1)} < \ldots < t_{k^{(1)}+1}^{(1)}$.

The idea is to construct $x^{(1)} = (J^{(1)})^\infty$ realizing $G^{(1)}$, with $J^{(1)}$ a concatenation of $J^{(0)}$s, among which some have had their holes filled in.

We can assume, reducing $\alpha^{(1)}$ if necessary, and multiplying the $t_j^{(1)}$s consequently, that $t_j^{(1)} = s_j^{(1)} N^{(0)}$, for each j, and $s_1^{(1)} \gg 3$.

Let us put $N^{(1)} = M^{(1)}(q^{(1)})^2$, and $n_j^{(1)} = M^{(1)}p^{(1)}p_j^{(1)}$.

Each $J_j^{(0)}$ has a single hole (letter $\vee$).

For $c \in \mathcal{A}$, and B a pattern over alphabet $\mathcal{A} \cup \{\vee\}$, let ${}^cB$ be the pattern deduced from B by filling in all the holes with a "c".

Set $J_j^{(1)} = {}^aJ^{(0)}({}^bJ^{(0)})^{s_j^{(1)}-3}J^{(0)}\,{}^aJ^{(0)}$, then set

$$J^{(1)} = (J_1^{(1)})^{n_1^{(1)}} \dots (J_{k^{(1)}+1}^{(1)})^{n_{k^{(1)}+1}^{(1)}},$$

which has length $N^{(1)}$.

Then using the fact that $s_1^{(1)} < \dots < s_{k^{(1)}+1}^{(1)}$, and the procedure followed to fill in some holes, together with the fact that $J^{(0)}$ was the minimal periodic pattern of $x^{(0)}$, we deduce that $x^{(1)} = (J^{(1)})^\infty$ has minimal period $N^{(1)} = |J^{(1)}|$.

Next set $U^{(1)} = {}^aJ^{(0)}\,{}^bJ^{(0)}\,{}^bJ^{(0)}$.

Then $G^{(1)} = G_{U^{(1)}}$ on $(\mathcal{O}(x^{(1)}), S)$.

Notice that $G^{(0)}$ remains equal to $G_{U^{(0)}}$ on $(\mathcal{O}(x^{(1)}), S)$.

This is because ab^2 has received no new possible occurence in the process of filling holes that deduces $x^{(1)}$ from $x^{(0)}$.

•<u>**Induction**</u> : let $n \geq 1$ (similar to the passage from step "0" to "1").

We have for superscripts $\ell = 0, \dots, n$ constructed periodic sequences $x^{(\ell)}$, with $x^{(\ell)} = (J^{(\ell)})^\infty$, $|J^{(\ell)}| = N^{(\ell)}$, $x^{(\ell)}$ with minimal period $N^{(\ell)}$, we have for $1 \leq \ell \leq n$ constructed $U^{(\ell)} = {}^aJ^{(\ell)}\,{}^bJ^{(\ell)}\,{}^bJ^{(\ell)}$, and $U^{(0)} = abb$, such that on $(\mathcal{O}(x^{(n)}), S)$, $G_{U^{(\ell)}}$ realizes $G^{(\ell)}$. Then $G^{(n+1)}$ come along with parameters $\alpha^{(n+1)} = p^{(n+1)}/q^{(n+1)}$, $\beta_j^{(n+1)} = p_j^{(n+1)}/q^{(n+1)}$, $t_1^{(n+1)} < \dots < t_{k^{(n+1)}+1}^{(n+1)}$, and we already assume that $t_j^{(n+1)} = s_j^{(n+1)}N^{(n)}$ with $s_j^{(n+1)} \gg 3$.

Then set $N^{(n+1)} = M^{(n+1)}(q^{(n+1)})^2$, $n_j^{(n+1)} = M^{(n+1)}p^{(n+1)}p_j^{(n+1)}$, and $J_j^{(n+1)} = {}^aJ^{(n)}({}^bJ^{(n)})^{s_j^{(n+1)}-3}J^{(n)}\,{}^aJ^{(n)}$, $J^{(n+1)} = (J_1^{(n+1)})^{n_1^{(n+1)}} \dots (J_{k^{(n+1)}+1}^{(n+1)})^{n_{k^{(n+1)}+1}^{(n+1)}}$. Finally set $U^{(n+1)} = {}^aJ^{(n)}\,{}^bJ^{(n)}\,{}^bJ^{(n)}$, $x^{(n+1)} = (J^{(n+1)})^\infty$.

By construction, since $s_1^{(n+1)} < \dots < s_{k^{(n+1)}+1}^{(n+1)}$, we have $x^{(n+1)}$ with least period $N^{(n+1)} = |J^{(n+1)}|$. And the process deducing $x^{(n+1)}$ from $x^{(n)}$ did not add any new occurence of $U^{(0)}, \dots, U^{(n)}$. Whence, $(\mathcal{O}(x^{(n+1)}), S)$ realizes $G^{(n+1)} = G_{U^{(n+1)}}$, it does so simultaneously for $G^{(0)}, \dots, G^{(n)}$.

4.3. Conclusion.

The sequence $(x^{(n)})_{n\geq 0}$ converges pointwise to some $x^{(\infty)} \in \{a, b, \vee\}^{\mathbb{Z}}$.

Since $N^{(n)} = |J^{(n)}| \to \infty$, it follows by construction that in fact $x^{(\infty)}$ has no more holes, i.e. $x^{(\infty)} \in \{a, b\}^{\mathbb{Z}}$.

The periodic pattern $J^{(n)}$ of $x^{(n)}$ has say h_n holes. Then the one of $x^{(n+1)}$ has $h_{n+1} = h_n(\sum_{j=1}^{k^{(n+1)}+1} n_j^{(n+1)}) = h_n p^{(n+1)} M^{(n+1)}(\sum_j p_j^{(n+1)})$. Else the period of $x^{(n+1)}$ is $N^{(n+1)} = \sum_j |J_j^{(n+1)}| n_j^{(n+1)} = M^{(n+1)} p^{(n+1)} N^{(n)}(\sum_j p_j^{(n+1)} s_j^{(n+1)})$.

We deduce that the density of holes in $x^{(n+1)}$ is $\frac{h_{n+1}}{N^{(n+1)}} = \frac{\sum_j p_j^{(n+1)}}{\sum_j p_j^{(n+1)} s_j^{(n+1)}} \leq \frac{1}{s_1^{(n+1)}}$. **We assume that $\frac{1}{\mathbf{s_1^{(n)}}} \to \mathbf{0}$.**

The conclusion is that by [JK], x^∞ is the quasi-uniform limit of the periodic sequences $x^{(n)}$. Hence it follows that $(\mathcal{O}(x), S)$ is a uniquely ergodic (regular) Toeplitz flow.

Further by construction $U^{(n)}$ occurs in x^∞ in positions where it did in $x^{(n)}$. A consequence, by unique ergodicity, is that $G_{U^{(n)}}$ still realizes $G^{(n)}$, for each n, on $(\mathcal{O}(x^\infty), S, \mu)$.

To conclude we need to know that $\cap U^{(n)} = \{x^\infty\}$ in $\mathcal{O}(x^\infty)$. This will hold as soon as $|U^{(n)}| \to \infty$, because x^∞ is Toeplitz (usually cylinder basins have patterns extending to both $-\infty$ and $+\infty$) : indeed $\{x^\infty\}$ is the fiber over zero for the maximal equicontinuous factor of the Toeplitz flow, when constructed as in [W].

Since this factor is some adding machine, for which forward orbits determine the point, x^∞ is completely determined by $x^\infty[0, +\infty[$.

To fit Theorem 2, set $\mathcal{A} = \{0, 1\}$. $\square$

References

[AG] M. Abadi & A. Galves, *Inequalities for the occurence times of rare events in mixing processes. The state of art*, Markov Process. Related Fields **7** (2001), 97–112.

[BSTV] H. Bruin, B. Saussol, S. Troubetzkoy & S. Vaienti, *Return time statistics via inducing*, To appear, Ergod. Th. & Dynam. Sys. (2002).

[C] Z. Coelho, *Asymptotic laws for symbolic dynamical systems*, Proceedings of the Conference held in Temuco, 1997, London Math. Soc. Lecture Note Ser. **279** (2000), Cambridge University Press, 123–165.

[CC] Z. Coelho & P. Collet, *Poisson laws associated to subsystems of finite type in symbolic dynamical systems*, Preprint (2000).

[CF] Z. Coelho & E. de Faria, *Limit laws for entrance times for homeomorphisms of the circle*, Israel Jour. Math. **69** (1990), 235–249..

[CG] P. Collet & A. Galves, *Statistics of close visits to the indifferent fixed point of an interval map*, J. Statist. Phys. **72** (1993), 459–478.

[DM] F. Durand & A. Maass, *Limit laws for entrance times for low complexity Cantor minimal systems*, Nonlinearity **14** (2001), 683-700.

[F] S. Ferenczi, *Systems of finite rank*, Colloq. Math. **73** (1997), 35–65.

[H1] M. Hirata, *Poisson law for Axiom A diffeomorphisms*, Ergod. Th. & Dynam. Sys. **13** (1993), 533–556.

[H2] M. Hirata, *Poisson law for the dynamical systems with the "self-mixing" conditions*, Dynamical systems and chaos **1** (1995), World Sci. Publishing, NJ, 87–96.

[HSV] M. Hirata, B. Saussol & S. Vaienti, *Statistics of return times : a general framework and new applications*, Comm. Math. Phys. **206** (1999), 33-55.

[JK] K. Jacobs & M. Keane, *0-1 sequences of Toeplitz type*, Z. Wahrsch. Verw. Gebiete **13** (1969), 123–131.

[K] M. Kac, *On the notion of recurrence in discrete stochastic processes*, Bull. A.M.S. **53** (1947), 1002–1010.

[O] J.C. Oxtoby, *Ergodic sets*, Bulletin A.M.S. **58** (1952), 116–136.

[P] B. Pitskel, *Poisson limit laws for Markov chains*, Ergod. Th. & Dynam. Sys. **11** (1991), 501–513.

[S] B. Saussol, *Etude statistique de systèmes dynamiques dilatants*, PHD Thesis, Toulon, France (1998).

[W] S. Williams, *Toeplitz minimal flows which are not uniquely ergodic*, Z. Wahrsch. Verw. Gebiete **67** (1984), 95–107.

[Y] L.S. Young, *Recurrence times and rates of mixing*, Israel J. Math. **110** (1999), 153–188.

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