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POSSIBLE-TRANSLATIONS ALGEBRAIZATION FOR PARACONSISTENT LOGICS

Abstract

This note proposes a new notion of algebraizability, which we call *possible-translations algebraic semantics*, based upon the newly developed possible-translations semantics. This semantics is naturally adequate to obtain an algebraic interpretation for paraconsistent logics, and generalizes the well-known method of algebraization by W. Blok and D. Pigozzi. This generalization obtains algebraic semantics up to translations, applicable to several non-classical logics and particularly apt for paraconsistent logics, a philosophically relevant class of logics with growing importance for applications.

Keywords: finitely algebraizable logics, paraconsistent logics, possible-translations semantics.

1. Why algebraizing?

Algebraic methods devoted to study logics based upon correspondence between theorems on logical systems and identities on classes of algebras are a heritage of the spirit of the traditional Polish school initiated by A. Tarski and J. Lukasiewicz, by its turn influenced by the 19th century view of algebra as a realization of logic due to G. Boole.

In intuitive terms, according to this tradition, to algebraize a logic is to forget negligible differences between formulas, and to collect formulas into classes, warranted that the classes form a congruence relation. Then one constructs the *quotient algebra*, based on the equivalent relation that is also a congruence, where the operations of this algebra are induced by the connectives of the logic. When this can be done, we can forget the logic and play with the algebra of classes (or quotient algebra).

To define such an algebra it is sufficient that, when starting from equivalent formulas (that is, from formulas interdeductible by means of the underlying consequence relation), other formulas constructed using them are also equivalent. This procedure, when possible, is known as *Lindenbaum–Tarski algebraization*: in this case, two formulas A and B are said to be equivalent if, only if, one can prove in such deductive system $A \vdash B$ and $B \vdash A$. This deductive equivalence relation, usually denoted by $A \equiv B$, is obviously an equivalence relation (in the presence of the usual properties of consequence operators), and one can naturally form the set of equivalence classes modulo $\equiv$. In some cases, depending upon the language and upon properties of the consequence relation, $A \equiv B$ may coincide with $\vdash A \leftrightarrow B$. In any case, one thus has to show that each equivalence class is closed under the connectives of the logic, that is, if $(A_1 \equiv B_1)$, $(A_2 \equiv B_2)$, $\dots$, $(A_n \equiv B_n)$, then $c^n(A_1, \dots, A_n) \equiv c^n(B_1, \dots, B_n)$, for each n-ary connective c^n. When this holds, the deductive equivalence relation is said to be a *congruence*. For such reasons, this method is informally referred to as congruence algebraization, or method of quotient algebras.

Obtaining quotient algebras is not very problematic if the logic in question is endowed with a replacement theorem, that is, if the intersubstitutivity of provable equivalents (IpE) holds (see Section 5 of [CM02]). This sufficient condition is, however, not necessary. Although the procedure of Lindenbaum–Tarski algebraization imitating what is done in the classical case, can be carried out for several logics besides classical logic (such as intuitionistic logic, some modal logics, and certain finite-valued logics) there are some well-established logical systems to which Lindenbaum–Tarski procedure fails in establishing a non-trivial congruence, for several reasons (though it may be difficult to establish such failure).

The point is that working by analogy with Lindenbaum–Tarski construction does not constitute a definition of algebraizability, because analogy does not give a way to decide when a logic is not algebraizable. Such a definition is precisely what W. J. Blok and D. Pigozzi suggested in [BP89], by substituting congruences by systems of equations, replacing the truth constant by a finite system of equations in one variable and by requiring that the interpretations of the logic consequence relation and the equational consequence relation be inverses of one another. In this way a much more general and wide sense of algebraization is obtained, defining what is now known as Blok–Pigozzi algebraization. For details consult [BP89], and [Cze01]. Other important references including also emphasizing the more general (weaker) concept of protoalgebraic logics are [BP86], [Cze80] and [FJ96].

However, not every logical system can be algebraized even within such wide concepts. A simple counter-example is the implicationless fragment IPC* of the intuitionistic propositional logic: IPC* is neither algebraizable in the sense of Blok–Pigozzi, nor even protoalgebraizable (cf. [BP89], pp. 56). Other systems that are not algebraizable in the Blok–Pigozzi sense are the paraconsistent logics C_n, although these systems are trivially protoalgebraic. Our interest is to propose a new method of algebraization that extends the Blok–Pigozzi method and is able to algebraize paraconsistent systems in general.

Paraconsistent logics are an emergent issue for some new paradigms of computation, specially for problems of semantics of sequential evaluation, querying and repairing contradictory databases, and for more flexible logic programming (paraconsistent logic programming is a specially flourishing area). The fact that paraconsistency affects computer science has strict connections to the fact that several purely humanistic domains also require a paraconsistent approach, as theories containing contradictory bodies of law, contradictory scientific theories and some philosophical systems (see [CCM04]).

The slippery task of algebraizing paraconsistent logics is consequently justified, since this will provide to them an algebraic realization counterpart, in such a way that the paraconsistent entailment relations can be simulated by algebraic entailments through the proposed syntactical interpretations.

It should be noted that deciding whether or not a logic is algebraizable is much more complicated when (IpE) does not hold. It does not mean, in such cases, that the logic cannot be algebraizable, but that almost certainly the Lindenbaum–Tarski approach will not apply, or the resulting algebraization will be of doubtful relevance.

A particularly well-known case is that of the logic **Cila**, cf. [CM02] (**Cila** is another way of defining the logic C_1 of da Costa of [dC74]) for which (IpE) does not hold. This result can also be obtained as a particular case of Theorem 3.51 of [CM02] which characterizes non-realizability of (IpE).

Several attempts (like that of [dC66]) had been pursued to obtain other kinds of congruence or other kinds of algebraization for the logic C_1 (or **Cila**), until C. Mortensen proved in 1980 (cf. [Mor80]) that this was a hopeless enterprise: no non-trivial quotient algebra is possible for **Cila**, or for any logic weaker than **Cila**. Some years later, in 1991, R. A. Lewin, I. F. Mikenberg, and M. G. Schwarze showed (cf. [LMS91]) that C_1 is not even algebraizable in the more general sense of Blok–Pigozzi. Hence, C_1 is not algebraizable, neither in Lindenbaum–Tarski nor in Blok–Pigozzi sense. Moreover, since any deductive extension of an algebraizable logic (in the same language) is also algebraizable (since, when a logic has sufficient deductive machinery to prove that the deductive equivalence relation is a congruence, all other deductive extensions will do) we obtain as a consequence that no such algebraization for any other of the weaker calculi in the infinite hierarchy C_n is possible.

Lacking of congruence algebraization is not a fate of paraconsistent logics, since some logics, as the logic C_1^+ of da Costa, Béziau and Bueno in [dCBB95] (coincident with **Cilo** of the taxonomical classification of [CM02]) can be assigned a non-trivial congruence. Indeed, this can be done by defining two formulas to be equivalent when they are provably equivalent and also provably consistent: $(A \equiv B)$ iff $\vdash ((A \leftrightarrow B) \wedge (\circ A \wedge \circ B))$.

Various extensions of **Cila** can be shown to have non-trivial quotient algebras, as the ones proposed by C. Mortensen in [Mor89]. But how to deal with recalcitrant cases as **Cila** itself and its subsystems, and with other similar logics hard to algebraize?

It is worth noting that many paraconsistent logics are Blok–Pigozzi algebraizable: for example the maximal three-valued paraconsistent logic P_1, as a deductive extension of C_1, is immune to both the arguments of Mortensen in [Mor80] and of Lewin, Mikenberg and Schwarze in [LMS91] concerning impossibility of algebraization. Indeed, it can be shown that P_1 is algebraizable in Blok–Pigozzi's sense (as proved by Lewin, Mikenberg and Schwarze in [LMS90]). Their argument can be extended to a large family of 8,192 three-valued maximal paraconsistent logics (cf. [CM02], Fact 3.82) which are all shown to be Blok–Pigozzi algebraizable.

For the purposes of clarifying the notions of algebrization here introduced it is convenient to briefly review the bases of Blok and Pigozzi's method.

The basic idea of their algebraic semantics, in formal terms, is to change the notion of "equivalent formula" (used to separate equivalence classes) by a finite set of formulas with two variables, usually denoted by $\Delta(\varphi,\psi)$, and to substitute the notion of equation with truth constant by a finite set of defining equations, usually denoted by $\epsilon_i(p_i) \approx \delta_i(p_i)$. More demanding, an equivalent algebraic semantics requires, moreover, that the interpretations of the logic consequence relation and the equational consequence relation be invertible.

A deductive system $\mathcal{S}$ is defined to be a structure $\mathcal{S} = \langle For, \vdash \rangle$, where $\vdash$ is a consequence relation over For, i.e., a relation $\vdash \subseteq (\wp(For) \times For))$ between theories and formulas of For, where $\wp(For)$ denotes the power set of For. The consequence relation of a given logic is often defined by its axioms and rules, or else from some semantical interpretation associated to this logic.

The relation $\vdash$ is required to follow certain specific requirements which are the following:

1. If $A \in \Gamma$ then $\Gamma \vdash A$ (reflexivity)
2. If $\Gamma \vdash A$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash A$ (monotonicity)
3. If $\Gamma \vdash A$ and $\Delta, A \vdash B$ then $\Gamma, \Delta \vdash B$ (transitivity)
4. $\Gamma \vdash A$ implies that $\Delta \vdash A$ for some finite $\Delta \subseteq \Gamma$ (finitariness)
5. $\Gamma \vdash A$ implies $\widehat{\sigma}(\Gamma) \vdash \widehat{\sigma}(A)$ for every substitution σ (structurality)

Let L be a propositional language and $\mathcal{K}$ any class of L-algebras (a quasi-variety). Let $\models_{\mathcal{K}}$ be the relation that holds between a set of equations Γ and a single equation $\varphi \approx \psi$, in symbols, $\Gamma \models_{\mathcal{K}} \varphi \approx \psi$, if every interpretation of $\varphi \approx \psi$ in a member of $\mathcal{K}$ holds provided each equation in Γ holds under the same interpretation. Thus, $\Gamma \models_{\mathcal{K}} \varphi \approx \psi$ iff for each algebra $\mathcal{A} \in \mathcal{K}$ and every interpretation a of the variables of $\Gamma \cup \{\varphi \approx \psi\}$ as elements of A, if $\xi^{\mathcal{A}}(a) = \eta^{\mathcal{A}}(a)$ for every $\xi \approx \eta \in \Gamma$ then $\varphi^{A}(a) = \psi^{A}(a)$. In this case we say that $\varphi \approx \psi$ is a $\mathcal{K}$-*consequence* of Γ. The relation $\models_{\mathcal{K}}$ is called the (semantical) *equational consequence relation* determined by $\mathcal{K}$.

Definition 1.1. *Let $\mathcal{S} = \langle L, \vdash_{\mathcal{S}} \rangle$ be a deductive system and $\mathcal{K}$ a class of algebras. $\mathcal{K}$ is called an algebraic semantics for $\mathcal{S}$ if $\vdash_{\mathcal{S}}$ can be interpreted in $\models_{\mathcal{K}}$ in the following sense: there exists a finite systems $\delta_i(p) \approx \epsilon_i(p)$, for $i < n$, of so-called defining equations with a single variable p such that, for all $\Gamma \cup \{\phi\} \subseteq For_L$ and each $j < n$:*

(i) $\Gamma \vdash_{\mathcal{S}} \varphi$ *iff* $\{\delta_i[\gamma/p] \approx \epsilon_i[\gamma/p] : i < n, \gamma \in \Gamma\} \models_K \delta_j[\varphi/p] \approx \epsilon_j[\varphi/p]$

Definition 1.2. *Let $\mathcal{S} = \langle L, \vdash_{\mathcal{S}} \rangle$ be a deductive system and $\mathcal{K}$ an algebraic semantics for $\mathcal{S}$ with defining equations $\delta_i(p) \approx \epsilon_i(p)$; $\mathcal{K}$ is called an equivalent algebraic semantics for $\mathcal{S}$ if, moreover, the following holds: there exists a finite system of (primitive or defined) connectives $\Delta_j(p, q)$ for $j < m$ of so-called equivalence formulas such that, for $i < n, j < m$:*

(ii) $\varphi \approx \psi \dashv\models_{\mathcal{K}} \delta_i(\Delta_j(\varphi, \psi)) \approx \epsilon_i(\Delta_j(\varphi, \psi))$

In order to obtain negative results in the Blok–Pigozzi algebraization, a useful tool is the Leibniz operator. The Leibniz operator Ω defines binary relations $\Omega_{\mathcal{A}}(F)$ on the domain of an algebra $\mathcal{A}$ by:

$\Omega_{\mathcal{A}}(F) = \{\langle a, b\rangle : \varphi^{\mathcal{A}}(a, c_0 \cdots, c_{k-1}) \in F$ iff $\varphi^{\mathcal{A}}(b, c_0, \cdots, c_{k-1}) \in F$, for every formula $\varphi(p, q_0 \cdots, q_{k-1})$ of $\mathcal{S}$ and every $c_0, \cdots c_{k-1} \in A\}$.

It can be shown that a deductive system is Blok–Pigozzi algebraizable if, and only if, there exists a quasi-variety $\mathcal{K}$ such that, for every algebra $\mathcal{A}$, the Leibniz operator $\Omega_{\mathcal{A}}$ induces an isomorphism between the lattice of $\mathcal{S}$-filters and the lattice of compatible $\mathcal{K}$-congruences (for details see [BP89] p. 43).

As a consequence, if such an operator fails to define an isomorphism between the lattice of $\mathcal{S}$-filters and the lattice of compatible $\mathcal{K}$-congruences, then the deductive system $\mathcal{S}$ is not algebraizable. This was the basis of the combinatorial argument used by Lewin, Mikenberg and Schwarze in [LMS91] and that was extended in [CM02], Theorem 3.83, to show that even the stronger logic **Cibaw** is not algebraizable. Of course, no weaker logic extended by **Cibaw** will be algebraizable (as it is the case of **Cil**, **Cila** and all C_n, in particular).

So there is no hope, neither in the classical Lindenbaum–Tarski nor in the new sense of algebraizability of Blok–Pigozzi, to algebraize any logic weaker than **Cibaw**.

However, some years before the proposal by Blok and Pigozzi, an algebraic counterpart to some of these non-algebraizable C-systems was proposed and investigated by W. A Carnielli and L. P de Alcantara (cf. [CdA84]) and subsequently by J. Seoane and de Alcantara (cf. [SdA91]). This was also further developed in categorial terms by V. Vasyukov in [Vas00]. A variety called da Costa algebras for a fragment of the paraconsistent logic C_1 was introduced in [CdA84], and a Stone-like representation theorem was proven, showing that every da Costa algebra is isomorphic to a paraconsistent algebra of sets. This defines a (non-equivalent) algebraic semantics for **Cila**, offering the first response to the question of finding an algebraic interpretation for paraconsistent logics. This proposal indeed preceded the notions of Blok–Pigozzi's algebraization and protoalgebraic logics, and for all effects was the more conclusive proposal to the general problem of algebraizing general paraconsistent logics up to now.

The conclusion is that Blok and Pigozzi's approach, even if it is so wide as to accommodate new algebraizations for some systems, and so sharp as to be capable of showing that some logics are not algebriazable, may not be appropriate for the subtleties of paraconsistency. We offer a step forward, proposing a new form of algebraization up to translations, inspired in the paradigm of possible-translations semantics. This is what we explain below.

2. Possible-translations semantics

Given logics $\mathcal{L} = \langle For, \vdash_{\mathcal{L}} \rangle$ and $\mathcal{L}' = \langle For', \vdash_{\mathcal{L}'} \rangle$, a *translation* from $\mathcal{L}$ into $\mathcal{L}'$ is a mapping between their sets of formulas which preserves derivability, that is, if A is provable in $\mathcal{L}$ from premises Γ (i.e., $\Gamma \vdash_{\mathcal{L}} A$) and t is a translation from $\mathcal{L}$ into $\mathcal{L}'$, then $t(A) = A'$ should be provable in $\mathcal{L}'$ from premises $\Gamma' = \{t(B) : B \in \Gamma\}$ (i.e., if $\Gamma \vdash_{\mathcal{L}} A$ then $\Gamma' \vdash_{\mathcal{L}'} A'$). When "if ... then" is changed to "iff", the translation is said to be *conservative*.

The concept of possible-translations semantics was introduced already in 1990 (cf. [Car90]), and reworked later on. (cf. [Car00], see also [Mar99] and [CCM04]). In intuitive terms, this semantics works similarly to the act of deciphering a hieroglyphic Rosetta Stone: the idea is to project a strange, not known, "hieroglyphic" logic by means of translations into a collection of simpler (usually many-valued) systems, and use all together the forcing relations of such logics in order to obtain a sound and complete semantical interpretation for the initial unknown system.

In less colorful terms, the basic idea is that, starting from a logic $\mathcal{L}$, it can be splitten with the help of a collection of simpler logics, seen as factors, under a certain collection of translations. By suitably combining such translations a new semantics emerges, providing a sound and complete interpretation for the initial logic $\mathcal{L}$.

In formal terms, given a logic $\mathcal{L} = \langle For_{\mathcal{L}}, \vdash_{\mathcal{L}} \rangle$ with a known syntax, and for which we intend to give an interpretation, consider a collection T of translations whose common domain is the set of formulas of $\mathcal{L}$. Each function $t \in T$ will have as image the wffs in a logic $\mathcal{S}_t$ (which, supposedly, has an acceptable semantics). A *possible-translations structure* for $\mathcal{L}$ is a pair $\mathbf{PT} = \langle \{\mathcal{S}_t\}_{t\in T}, T \rangle$, where T is an adequate collection of translations and $\{\mathcal{S}_t\}_{t\in T}$ is a collection of logics. A *possible-translations interpretation* for a formula A in $\mathcal{L}$ will be given by the collection of all translations $t(A)$, each $t(A)$ in $\mathcal{S}_t$. If all logics in $\{\mathcal{S}_t\}_{t\in T}$ are characterized (i.e., are sound and complete) with respect to their semantics, we have a *possible-translations semantics*. In this case, given a set of formulas $\Gamma \cup A$ in $For(\mathcal{L})$ and a particular translation t in T, we define the *local forcing relation for* $\mathcal{L}$, $\models^t_{\mathbf{PT}}$, as:

$$\Gamma \models^t_{\mathbf{PT}} A \text{ iff } t(\Gamma) \models_{\mathcal{S}_t} t(A), \text{ where } \models_{\mathcal{S}_t} \text{ is the forcing relation in } \mathcal{S}_t.$$

The *global forcing relation for* $\mathcal{L}$, $\models_{\mathbf{PT}}$, is defined by:

$$\Gamma \models_{\mathbf{PT}} A \text{ iff } \Gamma \models^t_{\mathbf{PT}} A, \text{ for every } t \text{ in } T.$$

We say that the logic $\mathcal{L}$ is *sound and complete with respect to the possible-translations structure* $\mathbf{PT}$ when, for every $\Gamma \cup A$ in $For_{\mathcal{L}}$:

$$\Gamma \vdash_{\mathcal{L}} A \text{ iff } \Gamma \models_{\mathbf{PT}} A.$$

It is shown in [Car00] (cf. also [Mar99]) that, despite the fact that the paraconsistent systems C_n cannot be characterized by finite-valued semantics, the possible-translations semantics do obtain sound and completeness for systems C_n with respect to the three-valued logic LCD:

THEOREM 2.1. *For each logic* C_n*,*

$$\Gamma \vdash_{C_n} A \text{ iff } \Gamma \models_{\mathbf{PT^n_3}} A, \text{ for every } \Gamma \cup A \text{ in } For_{C_n}.$$

where $\mathbf{PT^n_3}$ *is a possible-translations structure based upon an adequate collection of translations* T_n *.*

The concept of possible-translations semantics is very general, and in a certain sense any logic can be interpreted in a possible-translations environment.

Natural restrictions on the set of translations, or restrictions in the nature of the factors, offer a fine control for obtaining relevant semantical models. In this direction, particular cases of possible-translations semantics known as society semantics (introduced in [CLM99]), dyadic semantics (treated in [CCM05]) and non-deterministic semantics (in the sense of [AL]) are already successfully employed in the literature.

3. A closer regard to the three-valued logics that characterize C_n

It turns out that the mentioned characterization of the paraconsistent hierarchy C_n by means of possible-translations semanticsis is quite natural and elegant, since LCD coincides with the paraconsistent maximal logic J_3 introduced in [DdC70], which by its turn coincides with the system **CLuNs** (see [Bat89]), with the system **LFI1** (cf. [CMdA00]), and quite surprisingly, with the system Φ_v introduced years before in [Sch60] (Chapter II.7) for proof-theoretical purposes. The matrices of J_3 are the following:

$\vee^J$	1	$1/2$	0
1	1	1	1
$1/2$	1	$1/2$	$1/2$
0	1	$1/2$	0

	∇_J
1	1
$1/2$	1
0	0

	$\neg_J$
1	0
$1/2$	$1/2$
0	1

where $\{1, 1/2\}$ are the distinguished truth-values.

By identifying $\top$ to 1, $\top^-$ to $1/2$ and F to 0, we can easily define all connectives of J_3 in LCD using only the connectives $\{\neg_L, \neg_C, \wedge_3\}$, as computed in [Mar99]. Conversely, all connectives of LCD can be defined in J_3.

Since LCD and J_3 are interdefinable and have the same distinguished truth-values, they are deductively equivalent. Moreover, all other connectives $\{\wedge_1, \wedge_2, \vee_1, \vee_2, \vee_3, \to_1, \to_2, \to_3\}$ of LCD can be defined starting from $\{\neg_L, \neg_C, \wedge_3\}$.

It is also possible to define inside J_3 the table $\to^L$ for the implication of the three-valued Łukasiewicz logic L_3:

$$A \to^L B \stackrel{\text{def}}{=} ((\nabla^J(\neg^J A)) \vee^J B) \wedge^J ((\nabla^J B) \vee^J (\neg^J A)).$$

giving the matrix:

$\to^L$	1	1/2	0
1	1	1/2	0
1/2	1	1	1/2
0	1	1	1

As the negation $\neg^L$ of L_3 is the same as $\neg^J$, we can define the basic connectives of J_3 from $\to^L$ and $\neg^L$:

$$A \vee^J B \stackrel{\text{def}}{=} (A \to^L B) \to^L B;$$
$$\nabla^J A \stackrel{\text{def}}{=} (\neg^L A) \to^L A.$$

These definitions show that it is possible to write J_3 and L_3 in the same language. Although J_3 and L_3 define the same matrices, they are not equivalent, because L_3 has only 1 as its distinguished truth-value; this allows to check, for example, that $\vdash_{J_3} (\neg p \to p) \to p$ while $\nvdash_{L_3} (\neg p \to p) \to p$. In [BP] it is shown that J_3 and L_3 are algebraizable in the same algebraic quasi-variety, by using the the notion of invertible strong translations[1].

Moreover, additionally to this conservative translation from J_3 into L_3, a finer result is proven in [BP]: it is shown in that paper that J_3 and L_3 are *strongly* conservatively translated in the sense that there exist invertible strong translations between that J_3 and L_3. Indeed, for any formula A in the common language of J_3 and L_3, $A \dashv\models_{J_3} \Diamond\Box A$ $A \dashv\models_{L_3} \Box\Diamond A$ hold, where $\rho(A) = \Box A$ is a translation from L_3 into J_3 defined as $\Box A \stackrel{\text{def}}{=} \neg^L \Diamond \neg^L A$ and $\tau(A) = \Diamond A$ is a translation from J_3 into L_3 defined as $\Diamond A \stackrel{\text{def}}{=} \neg^L A \to^L \neg^L A$. That is, the translations τ and ρ are invertible strong translations; the tables for $\Diamond$ and $\Box$ are the following:

	$\Diamond$
1	1
1/2	1
0	0

	$\Box$
1	1
1/2	0
0	0

These facts permit to conclude that J_3 and L_3 are algebraizable by the same equivalent algebraic semantics, namely, by the three-valued MV-algebras (or Łukasiewicz-Moisil algebras) as shown in [BP].

[1] In [BP] the authors use the term "equivalent" instead of "invertible strong translations".

We now suggest an extension of the Blok–Pigozzi algebraizability criteria which, in particular, offers a solution for the desired algebraizzation of the paraconsistent systems C_n.

4. The possible-translations algebraic semantics

The approach suggested here means to provide algebraic semantics for recalcitrant logics by means of an analogue of the possible-translations structure. Suppose we have in hands a certain logic $\mathcal{L}$ which cannot be algebraizable in the Blok–Pigozzi sense but has a possible-translations structure associated to it, by means of a collection T of translations whose common domain is the collection of formulas of $\mathcal{L}$. Suppose also that the components of such possible-translations structure are algebraizable. Each function t, thus, for $t \in |T|$, will have as image the wffs of a calculus $\mathcal{S}_t$ with consequence relation $\vdash_t$ and with equivalent algebraic semantics in an algebraic variety $\mathcal{K}_t$ (with equational consequence relation $\models_{\mathcal{K}_t}$). The algebraic interpretation for a formula A in $\mathcal{L}$ will then be given by the collection of all algebraic interpretations for $t(A)$ in K_t.

Let $\mathcal{L}$ be a complete logic with respect to the possible-translations structure **PT**. A possible-translations algebraic structure for $\mathcal{L}$ is a triple:

$$\mathbf{PA} = \langle \{\mathcal{S}_t\}_{t\in T}, \{\mathcal{K}_t\}_{t\in T}, T\rangle$$

such that:

1. $\mathbf{PT} = \langle \{\mathcal{S}_t\}_{t\in T}, T\rangle$ is a possible-translations structure for $\mathcal{L}$;
2. For each $t \in T, \mathcal{K}_t$ is an equivalent algebraic semantics for $\mathcal{S}_t$.

In such case, we say that $\mathcal{L}$ is characterized by a *possible-translations algebraic semantics* $\mathbf{PA} = \langle \{\mathcal{S}_t\}_{t\in T}, \{\mathcal{K}_t\}_{t\in T}, T\rangle$ and that $\{\mathcal{K}_t\}_{t\in T}$ is a *possible-translations algebraic semantics* for $\mathcal{L}$ (up to the translations T).

Consequently, the following immediate characterization holds:

Theorem 4.1. *Let $\mathcal{L}$ be a logic characterized by a possible-translations algebraic semantics $\mathbf{PA} = \langle \{\mathcal{S}_t\}_{t\in T}, \{\mathcal{K}_t\}_{t\in T}, T\rangle$; then the consequence relation $\vdash_{\mathcal{L}}$ holds iff for every translation t in T, $\mathcal{K}_t$ is an equivalent algebraic semantics interpreting $\vdash_{\mathcal{S}_t}$.*

PROOF. Given a set of formulas $\Gamma \cup \{A\}$ in $For_{\mathcal{L}}$ and a particular translation t in T the following holds:

(a) By the definition of local forcing relation, $\Gamma \models^t_{\mathbf{PT}} A$ iff $t(\Gamma) \models_{\mathcal{S}_t} t(A)$;

(b) Since each calculus $\mathcal{S}_t$ is characterized (i.e, is sound and complete) by its own semantics, we also have: $t(\Gamma) \models_{\mathcal{S}_t} t(A)$ iff $t(\Gamma) \vdash_{\mathcal{S}_t} t(A)$;

(c) Since $\mathcal{S}_t$ has an equivalent algebraic semantics, the conditions of Definition 1.1 and Definition 1.2 are satisfied;

(d) Considering that, by hypothesis, $\mathcal{L}$ is complete with respect to the possible-translations structure $\mathbf{PT}$, we obtain that the consequence relation $\vdash_{\mathcal{L}}$ holds iff the translated consequence relation $\vdash_{\mathcal{S}_t}$ holds for each translation t in T, iff, for each translation t in T, $\mathcal{K}_t$ is an equivalent algebraic semantics interpreting $\vdash_{\mathcal{S}_t}$.

□

We can now state the main result of this paper:

THEOREM 4.2. *For each n, the variety of three-valued reducts of MV-algebras (or Łukasiewicz-Moisil algebras) is a possible-translations algebraic semantics for* C_n.

PROOF. From Theorem 4.1 and Theorem 2.1. □

The arguments above show that the paraconsistent logics C_n can be algebraized by means of an extended concept of algebraizability here defined, the possible-translations algebraic semantics. This new notion of algebraizability generalizes the one of Blok–Pigozzi, in the precise sense that a Blok–Pigozzi algebraization for a logic $\mathcal{L}$, where $\mathcal{K}$ is an algebraic semantics for $\mathcal{L}$, coincides with a possible-translations algebraic semantics $\mathbf{PA} = \langle \{\mathcal{S}\}, \{\mathcal{K}\}, T \rangle$ where T is a singleton containing just the identity translation.

Although there are other generalizations of the Blok–Pigozzi definition, as the protoalgebraic logics (cf. [BP86]), the possible-translations algebraic semantics offers a non-trivial alternative at least for paraconsistent logics, in view of the fact that every logic C_n is trivially protoalgebraic (see Section 1).

The concept of possible-translations algebraic semantics offers an alternative to the quest of obtaining equivalent algebraic counterparts not only to paraconsistent logics, but also to other logics which are complete with respect to possible-translations semantics. A categorial approach to possible-translations algebraic semantics was discussed in [BCC04], where it is shown that, given a possible-translation semantics for a logic L, there exist conservative translations from L into a product Π of some family of logics, and vice-versa.

Moreover, if the cardinality of the (finite) sets of defining formulas and equivalence formulas involved in the possible-translations algebraic semantics is bound (note that we may have an infinite collection of such finite sets), the product Π is not only algebraizable but there also exists a conservative translation from L into Π. This is precisely the case for the possible-translations algebraic semantics for C_n discussed here, which can thus be shown (in such categorial setting) to be algebraizable up to just one translation.

Possible-translations algebraic semantics are not confined to paraconsistency: other logics, as for instance Łukasiewicz and Bochvar three-valued logics (cf. [FC03]), can be shown to be sound and complete with respect to society semantics, a particular case of possible-translations semantics (see end of Section 2). Thus, in principle, possible-translations algebraic semantics can also be assigned to such logics, and it would be instructive to compare this new algebraic semantics with the traditional algebraization.

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