Possible-translations semantics for some weak classically-based paraconsistent logics

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Abstract

This note provides interpretation by way of possible-translations semantics for a group of fundamental paraconsistent logics extending the positive fragment of classical propositional logic. The logics PI, C_{min}, **mbC**, **bC**, **mCi** and **Ci**, among others, are all initially presented through their bivaluation semantics and sequent versions and then split by way of possible-translations semantics —the set of 3-valued matrices of the ingredient logics is put forward, together with the set of admissible translating mappings, in each case. Precise statements and all non-obvious details of proofs are supplied. Other details are left to the reader.

Key words: Possible-translations semantics, paraconsistent logics.

1 Languages, bivaluations, and sequents

Let $\mathcal{P} = \{p_1, p_2, \ldots, p_m, \ldots\}$ be a denumerable set of sentential letters, and consider the sets of formulas

$$\begin{aligned} \mathcal{S}_0 &:= p \mid (\varphi \wedge \psi) \mid (\varphi \vee \psi) \mid (\varphi \supset \psi), \\ \mathcal{S}_1 &:= \mathcal{S}_0 \mid {\sim}\varphi, \\ \mathcal{S}_2 &:= \mathcal{S}_1 \mid \circ\varphi, \\ \mathcal{S}_3 &:= \mathcal{S}_2 \mid \bullet\varphi, \end{aligned}$$

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where p ranges over $\mathcal{P}$, and $\wedge$ ('conjunction'), $\vee$ ('disjunction'), $\supset$ ('implication'), $\sim$ ('negation'), $\circ$ ('consistency'), $\bullet$ ('inconsistency') are connective symbols. As usual, the binary connective $\equiv$ ('bi-implication') is defined by considering $\varphi \equiv \psi$ as an abbreviation for $(\varphi \supset \psi) \wedge (\psi \supset \varphi)$. Outermost parentheses are omitted whenever there is no risk of confusion.

A mapping $b : \mathcal{S}_i \longrightarrow \{0, 1\}$ is called a bivaluation over $\mathcal{S}_i$. One can easily write some possible axioms governing the set of admissible bivaluations:

(b1.1) $b(\varphi \wedge \psi) = 1 \;\Rightarrow\; b(\varphi) = 1$ and $b(\psi) = 1$
(b1.1$^{\mathbf{r}}$) $b(\varphi \wedge \psi) = 0 \;\Rightarrow\; b(\varphi) = 0$ or $b(\psi) = 0$
(b1.2) $b(\varphi \vee \psi) = 1 \;\Rightarrow\; b(\varphi) = 1$ or $b(\psi) = 1$
(b1.2$^{\mathbf{r}}$) $b(\varphi \vee \psi) = 0 \;\Rightarrow\; b(\varphi) = 0$ and $b(\psi) = 0$
(b1.3) $b(\varphi \supset \psi) = 1 \;\Rightarrow\;$ if $b(\varphi) = 1$ then $b(\psi) = 1$
(b1.3$^{\mathbf{r}}$) $b(\varphi \supset \psi) = 0 \;\Rightarrow\; b(\varphi) = 1$ and $b(\psi) = 0$
(b2) $b({\sim}\varphi) = 0 \;\Rightarrow\; b(\varphi) = 1$
(b3) $b(\circ\varphi) = 1 \;\Rightarrow\; b(\varphi) = 0$ or $b({\sim}\varphi) = 0$
(b3$^{\mathbf{r}}$) $b(\circ\varphi) = 0 \;\Rightarrow\; b(\varphi) = 1$ and $b({\sim}\varphi) = 1$
(b4) $b({\sim}\circ\varphi) = 1 \;\Rightarrow\; b(\varphi) = 1$ and $b({\sim}\varphi) = 1$
(b5.n) $b(\circ{\sim}^n\circ\varphi) = 1$, given $n \in \mathbb{N}$
(b6) $b({\sim}{\sim}\varphi) = 1 \;\Rightarrow\; b(\varphi) = 1$
(b6$^{\mathbf{r}}$) $b({\sim}{\sim}\varphi) = 0 \;\Rightarrow\; b(\varphi) = 0$

where ${\sim}^0\varphi \stackrel{\text{def}}{=} \varphi$ and ${\sim}^{n+1}\varphi \stackrel{\text{def}}{=} {\sim}^n{\sim}\varphi$.

The converse of (b4) clearly follows from (b2) and (b3), and the latter two axioms are to be respected by most logics we will consider below. Moreover, the reader will surely have noticed the difference between (b4) and (b3$^{\mathbf{r}}$), the converse of (b3):

Fact 1.1 In the presence of (b2), axiom (b3$^{\mathbf{r}}$) can be derived from (b4). The axiom (b4) can be derived from (b3$^{\mathbf{r}}$) in the presence of (b3) and (b5.0).

All the above axioms are in 'dyadic form' (cf. [10]). In that case, there is a canonical method for transforming all of them into appropriate sequent rules, as devised in [9]. This results in the following:

(s1.1) $\varphi \wedge \psi \vdash \varphi$ and $\varphi \wedge \psi \vdash \psi$
(s1.1$^{\mathbf{r}}$) $\varphi, \psi \vdash \varphi \wedge \psi$
(s1.2) $\varphi \vee \psi \vdash \varphi, \psi$
(s1.2$^{\mathbf{r}}$) $\varphi \vdash \varphi \vee \psi$ and $\psi \vdash \varphi \vee \psi$
(s1.3) $\varphi \supset \psi, \varphi \vdash \psi$
(s1.3$^{\mathbf{r}}$) $\vdash \varphi, \varphi \supset \psi$ and $\psi \vdash \varphi \supset \psi$
(s2) $\vdash \varphi, {\sim}\varphi$
(s3) $\circ\varphi, \varphi, {\sim}\varphi \vdash$
(s3$^{\mathbf{r}}$) $\vdash \circ\varphi, \varphi$ and $\vdash \circ\varphi, {\sim}\varphi$
(s4) ${\sim}\circ\varphi \vdash \varphi$ and ${\sim}\circ\varphi \vdash {\sim}\varphi$
(s5.n) $\vdash \circ{\sim}^n\circ\varphi$, given $n \in \mathbb{N}$
(s6) ${\sim}{\sim}\varphi \vdash \varphi$
(s6$^{\mathbf{r}}$) $\varphi \vdash {\sim}{\sim}\varphi$

For the sake of legibility, the side contexts of the above rules were dropped. Any subset of those rules, together with reflexivity, weakening, cut, and the usual structural rules, determines a specific sequent system. We will write $\Gamma \dashv\vdash \Delta$ as an abbreviation for $\Gamma \vdash \Delta$ and $\Delta \vdash \Gamma$.

The following is a straightforward byproduct of the above:

Fact 1.2 Rule (s5.0) is derivable with the help of (s2), (s3) and (s4). Rules (s5.n), for $n \in \mathbb{N}$, are all derivable in the presence of (s3), (s4), (s5.0) and (s6).

2 Some fundamental paraconsistent logics

Let CL^+ denote the positive fragment of classical propositional logic, built over the set of formulas $\mathcal{S}_0$, axiomatized by way of the rules (s1.X) and interpreted through the set of all bivaluations respecting the axioms (b1.X).

The very weak paraconsistent logic PI (cf. [7]) is built over $\mathcal{S}_1$ simply by adding (s2) to the rules of CL^+ or (b2) to its bivaluational axioms. The full classical propositional logic, CL, could be obtained now from PI over $\mathcal{S}_1$ by adding

(b2$^{\mathbf{r}}$) $\quad b({\sim}\varphi) = 1 \ \Rightarrow\ b(\varphi) = 0$

to the bivaluational axioms of PI, or, equivalently, by adding

(s2$^{\mathbf{r}}$) $\quad \varphi, {\sim}\varphi \vdash$

to PI's sequent rules. The bivaluational axioms (b2) and (b2$^{\mathbf{r}}$) are thus sufficient for interpreting classical negation in isolation from the other connectives, and the sequent rules (s2) and (s2$^{\mathbf{r}}$) can be seen as the pure characterizing rules of classical negation.

A fundamental logic of formal inconsistency (cf. [18]) called **mbC** is built next over $\mathcal{S}_2$ by adding (s3) to the rules of PI or, equivalently, by adding (b3) to its bivaluational axioms. A 0-ary connective $\bot$ ('bottom'), characterized semantically by setting $b(\bot) = 0$, can be defined in **mbC** if one takes $\bot \stackrel{\text{def}}{=} \circ\psi \wedge (\psi \wedge {\sim}\psi)$, for any formula ψ. As a byproduct:

Fact 2.1 A classical negation $\neg$ can be defined in **mbC** by setting $\neg\varphi \stackrel{\text{def}}{=} \varphi \supset \bot$.

The logic **mbC**, as presented above, had only a primitive consistency connective $\circ$ but no primitive connective for inconsistency. The latter can nonetheless be defined in **mbC** if one just sets $\bullet\varphi \stackrel{\text{def}}{=} {\sim}\circ\varphi$. This way one could in fact rebuild **mbC** over $\mathcal{S}_3$, if that be the case.

An important extension of **mbC** is the logic **mCi**, again built over $\mathcal{S}_2$, but now by adding (s4) and (s5.n), $n \in \mathbb{N}$, to the rules of **mbC**, or (b4) and (b5.n), $n \in \mathbb{N}$, to its bivaluational axioms. The fundamental characteristic of **mCi** is the classical behavior of its consistency connective $\circ$ with respect to the negation $\sim$:

Fact 2.2 In **mCi**:
(i) $b(\sim\circ\alpha) = b(\neg\circ\alpha)$,
(ii) $b(\sim^n\circ\alpha) = 1 \;\Leftrightarrow\; b(\sim^{n+1}\circ\alpha) = 0$.

As a particular consequence, the above mentioned inconsistency connective $\bullet$, in **mCi**, will be perfectly dual to the consistency connective $\circ$. Indeed:

Fact 2.3 In **mCi**, $\circ\alpha \dashv\vdash \sim\bullet\alpha$.

Let $\psi[p]$ denote a formula ψ having p as one of its atomic components, and let $\psi[p/\gamma]$ denote the formula obtained from ψ by uniformly substituting all occurrences of p by the formula γ. Given a pair of formulas α and β, we say that they are *logically indistinguishable* if for every formula $\varphi[p]$ we have that $\varphi[p/\alpha] \dashv\vdash \varphi[p/\beta]$. Algebraically, this will mean that α and β will have the 'same reference', and belong thus to the same congruence class. In terms of bivaluation semantics, this will mean that $b(\varphi[p/\alpha]) = b(\varphi[p/\beta])$, for any formula φ. By the very definition of $\bullet$ we know that the formulas $\bullet\alpha$ and $\sim\circ\alpha$ are logically indistinguishable. However, in spite of the equivalence between the formulas $\circ\alpha$ and $\sim\bullet\alpha$ mentioned in the last fact, such formulas are not logically indistinguishable inside the logics studied in the present paper. We will use our possible-translations tool to check this feature in Example 5.15, further on.

The logics PIf, **bC** and **Ci** extend, respectively, the logics PI, **mbC** and **mCi**, by the addition of the bivaluational axiom (b6) or, equivalently, of the sequent rule (s6). The logic PIf appears in ch.4 of [20] and then at [15] under the appellation C_{min}. Both **bC** and **Ci**, as well as an enormous number of their extensions, are studied in close detail at [18]. The logic **mCi** is suggested at the final section of the latter paper, but axiomatized here for the first time. This logic, together with **mbC**, constitute the most fundamental logics explored in [13]. Inaccuracies in the axiomatization (as introduced in [18]) and in the bivaluation semantics (as presented in [16, 17]) of the logic **Ci** are also fixed at [13].

On a similar vein, the logics $PIfe$, **bCe** and **Cie** can here be introduced as extensions of the previous logics obtained by the further addition of the bivaluational axiom ($\mathrm{b6}^{\mathbf{r}}$) or, equivalently, of the sequent rule ($\mathrm{s6}^{\mathbf{r}}$). In the light of the results from the preceding facts, it might seem natural that **mCi**, **Ci**, and **Cie** should from this point on be built instead directly over the extended set of formulas $\mathcal{S}_3$, where $\bullet$ could be introduced by a definition using $\sim$ and $\circ$, as above.

To summarize the 9 previously mentioned paraconsistent logics:

PI formulas: $\mathcal{S}_1$
sequent rules: (s1.X) and (s2)
axioms on bivaluations: (b1.X) and (b2)

mbC formulas: $\mathcal{S}_2$
sequent rules: as in PI, plus (s3)
axioms on bivaluations: as in PI, plus (b3)

mCi formulas: $\mathcal{S}_3$
sequent rules: as in **mbC**, plus (s4) and (s5.n), $n \in \mathbb{N}$
axioms on bivaluations: as in **mbC**, plus (b4) and (b5.n), $n \in \mathbb{N}$

PIf formulas: $\mathcal{S}_1$
sequent rules: as in PI, plus (s6)
axioms on bivaluations: as in PI, plus (b6)
(a.k.a. C_{min})

bC formulas: $\mathcal{S}_2$
sequent rules: as in **mbC**, plus (s6)
axioms on bivaluations: as in **mbC**, plus (b6)

Ci formulas: $\mathcal{S}_3$
sequent rules: as in **bC**, plus (s4) and (s5.0)
axioms on bivaluations: as in **bC**, plus (b4) and (b5.0)

$PIfe$ formulas: $\mathcal{S}_1$
sequent rules: as in PIf, plus (s6$^{\mathbf{r}}$)
axioms on bivaluations: as in PIf, plus (b6$^{\mathbf{r}}$)

bCe formulas: $\mathcal{S}_2$
sequent rules: as in **bC**, plus (s6$^{\mathbf{r}}$)
axioms on bivaluations: as in **bC**, plus (b6$^{\mathbf{r}}$)

Cie formulas: $\mathcal{S}_3$
sequent rules: as in **Ci**, plus (s6$^{\mathbf{r}}$)
axioms on bivaluations: as in **Ci**, plus (b6$^{\mathbf{r}}$)

The simplification in the rules and axioms of **Ci**, as compared to those of **mCi**, is sanctioned by the results in Fact 1.2.

For a quick scan, one can find in Figure 1 a schematic illustration displaying the relationships between the above logics. An arrow $\mathcal{L}1 \longrightarrow \mathcal{L}2$ indicates that the logic $\mathcal{L}1$ is (properly) extended by the logic $\mathcal{L}2$.

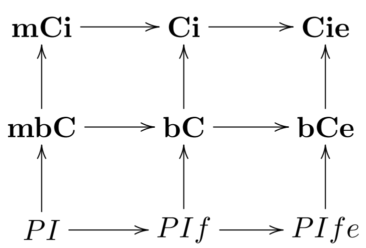

Figure 1: Some fundamental paraconsistent logics.

3 Bivalued entailment, modalities and matrices

Fixed any of the logics presented in the above section, let biv be its set of admissible bivaluations. Given $b \in \mathsf{biv}$, let $\Gamma \vDash_b \Delta$ hold good, for given sets of formulas Γ and Δ, iff $(\exists\gamma \in \Gamma)b(\gamma) = 0$ or $(\exists\delta \in \Delta)b(\delta) = 1$. The canonical *entailment* relation $\vDash_{\mathsf{biv}}$ is defined as usual: $\Gamma \vDash_{\mathsf{biv}} \Delta$ iff $\Gamma \vDash_b \Delta$ for every $b \in \mathsf{biv}$. Moreover, given a set of sequent rules seq, let $\vdash_{\mathsf{seq}}$ denote the *derivability* relation defined by its canonical notion of (multiple-conclusion) proof-from-premises. Entailment and derivability relations are examples of *consequence* relations. Given any consequence relation $\rhd$ associated to a logic $\mathcal{L}$, we will write $\Gamma \not\rhd \Delta$ to say that the inference $\Gamma \rhd \Delta$ fails according to $\mathcal{L}$, and we will write $\Gamma \lhd\rhd \Delta$ to say that both $\Gamma \rhd \Delta$ and $\Delta \rhd \Gamma$ hold good in $\mathcal{L}$.

Can the 9 above paraconsistent logics be given semantics that are more informative than their respective bivaluation semantics? Good question. It should be remarked for instance that those logics cannot be endowed with usual modal-like semantics. Indeed, all of them fail the *replacement property*, a property that is typical of normal modal systems:

Theorem 3.1 In any of the logics from Figure 1, $\dashv\vdash$ does not constitute a congruence relation over the set of formulas, that is, there are formulas α and β such that $\alpha \dashv\vdash \beta$, but $\sim\!\alpha \nvdash \sim\!\beta$.

Proof: Consider the 3-valued matrices of the logic **LFI1**, at Table 2, where F is the only undesignated truth-value.

$\wedge$	T	t	F
T	T	t	F
t	t	t	F
F	F	F	F

$\vee$	T	t	F
T	T	T	T
t	T	t	t
F	T	t	F

$\supset$	T	t	F
T	T	t	F
t	T	t	F
F	T	T	T

	$\sim$	$\circ$
T	F	T
t	t	F
F	T	T

Figure 2: Matrices of the logic **LFI1**.

It is easy to check that **LFI1** (properly) extends all the above paraconsistent logics —it constitutes in fact a maximally paraconsistent extension of those logics (cf. [20, 19]). Nevertheless, in **LFI1**, while tautologies such as $(p \vee \sim\!p)$ and $(q \vee \sim\!q)$ are equivalent, the formulas $\sim\!(p \vee \sim\!p)$ and $\sim\!(q \vee \sim\!q)$ are not equivalent: To see that, consider any 3-valued valuation such that the atomic sentence p receives the value t while q receives a different value. □

Note 3.2 (A seeming paradox) The logic of formal inconsistency **mbC** (and any of its non-trivial paraconsistent extensions) can be seen both as a conservative extension and as a deductive fragment of classical logic, CL. Indeed, for the first assertion, recall the set of formulas $\mathcal{S}_0$ of positive classical logic (Section 1), and consider now the sets of formulas:

$$\mathcal{S}_4 := \mathcal{S}_0 \mid \neg\varphi,$$
$$\mathcal{S}_5 := \mathcal{S}_4 \mid {\sim}\varphi \mid \circ\varphi.$$

Interpret the connectives from $\mathcal{S}_4$ as in CL, using the bivaluational axioms (b1.X) and (b2.X) (where $\neg$ takes the place of $\sim$). Interpret the new connectives in $\mathcal{S}_5$ as in **mbC**, using the bivaluational axioms (b2) and (b3). It is clear that this last move provides just a new way of presenting **mbC**. Indeed, as we have seen in Fact 2.1, $\neg$ can be defined from the original presentation of **mbC**. Consider again the matrices of **LFI1**, from Table 2, a logic that deductively extends **mbC**. The classical negation $\neg$ in **LFI1**, defined as above, would be such that $v(\neg\varphi) = T$ if $v(\varphi) = F$, and $v(\neg\varphi) = F$ otherwise. It is easy to see, in that case, that the matrices of $\sim$ and $\circ$, the new connectives of $\mathcal{S}_5$ cannot be defined, in **LFI1**, from the matrices of the connectives in $\mathcal{S}_4$. If you recall now that CL is a maximal logic, then you have concluded the proof that **mbC** can be seen as a (proper) conservative extension of CL. For the second assertion, consider CL to be written in the language of $\mathcal{S}_5$. Recall that classical logic is presupposed consistent, and interpret the connective $\circ$ accordingly, by taking as axiom $b(\circ\varphi) = 1$. Based on the received idea that there is just 'one true classical negation', interpret both $\neg$ and $\sim$ using axioms (b2) and (b2$^{\mathbf{r}}$). In that case **mbC** is clearly characterized as a (proper) deductive fragment of CL. Notice that this is, however, a very peculiar fragment of CL —it is a fragment into which all classical reasoning can be internalized by way of a definitional translation.

Note 3.3 (More on internalizing stronger logics) Not only can **mbC** faithfully internalize classical logic, but it can also internalize the reasoning of other logics of formal inconsistency that are deductively stronger than itself. To see that, consider now the following sets of formulas:

$$\mathcal{S}_6 := \mathcal{S}_0 \mid \bot,$$
$$\mathcal{S}_7 := \mathcal{S}_6 \mid {\sim}\varphi,$$
$$\mathcal{S}_8 := \mathcal{S}_7 \mid \circ\varphi.$$

Interpret the 0-ary connective ('bottom') from $\mathcal{S}_6$ by taking as axiom $b(\bot) = 0$, and interpret the new connectives from $\mathcal{S}_7$ and $\mathcal{S}_8$ as in **mbC**. Again, this provides just another presentation for **mbC**, as we have seen in Section 1 that $\bot$ is definable in this logic. On the other hand, a new consistency connective strictly stronger than $\circ$ can be defined using the connectives from $\mathcal{S}_7$. Indeed, as in [18], consider a connective $\tilde{\circ}$ defined by setting $\tilde{\circ}\varphi \stackrel{\text{def}}{=} (\varphi \supset \bot) \vee ({\sim}\varphi \supset \bot)$ (or, equivalently, $\tilde{\circ}\varphi \stackrel{\text{def}}{=} \neg\varphi \vee \neg{\sim}\varphi$). This connective is naturally characterizable by axiom (b3) and its converse (b3$^{\mathbf{r}}$), while the original consistency connective of **mbC** was characterized by axiom (b3) alone. If you recall Fact 1.1 you will notice that the last definition determines a logic of formal inconsistency that lies right in between **mbC** and **mCi**. As a matter of fact, this approach provides one way of presenting the logic **CLuN**, the preferred logic of adaptive logicians (cf. [8]), often used as the lower limit logic of their inconsistency-adaptive systems. Though the first presentations of **CLuN** made this logic coincide with PI, it has been more recently presented as a conservative extension of PI obtained by adding a bottom connective to the language of the latter, as in $\mathcal{S}_7$ above. If one writes the whole thing in the language of $\mathcal{S}_8$, using the above defined consistency connective, **CLuN** is very naturally recast thus as a logic of formal inconsistency that lies in between **mbC** and **mCi**.

Problem 3.4 Is there a definitional translation of **mCi** into **mbC**? Can the logic **mbC** faithfully internalize in some way the reasoning of **mCi**?

Note 3.5 (Other logics extending mbC but not mCi)
Besides **CLuN**, there are many other interesting logics of formal inconsistency that extend **mbC** but do not go through **mCi**. There is even a large class of such logics that satisfies the full replacement property. I have shown in [21, 23], in fact, that any non-degenerate normal modal logic can be easily recast as a logic of formal inconsistency extending **CLuN** (and thus extending **mbC**), but not **mCi**.

Before the diversion provided by the above set of notes, we had seen in Theorem 3.1 that the 9 paraconsistent logics from the last section cannot be endowed with usual modal-like semantics. The reader might now be wondering whether those logics would still stand some chance of being *truth-functional*, should they turn out themselves to be characterizable by way of some convenient set of finite-valued matrices (just like their extension **LFI1**). But some negative results about that possibility can also be promptly checked as follows. The following theorem and its corollary correct a result suggested in [1]:

Theorem 3.6 No sequent of the form $\vdash \sim^i\varphi \equiv \sim^j\varphi$ is derivable, for non-negative $i \neq j$, in logics from the first two columns of Figure 1.

Proof: Consider a set of infinite-valued matrices that take the natural numbers $\mathbb{N}$ as truth-values, where 0 is the only undesignated truth-value. Define the matrices for the connectives as follows:

$$v(\varphi \wedge \psi) = \begin{cases} 1, & \text{if } v(\varphi) > 0 \text{ and } v(\psi) > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$v(\varphi \vee \psi) = \begin{cases} 1, & \text{if } v(\varphi) > 0 \text{ or } v(\psi) > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$v(\varphi \supset \psi) = \begin{cases} 0, & \text{if } v(\varphi) > 0 \text{ and } v(\psi) = 0 \\ 1, & \text{otherwise} \end{cases}$$

$$v(\sim\varphi) = \begin{cases} 1, & \text{if } v(\varphi) = 0 \\ v(\varphi) - 1, & \text{otherwise} \end{cases} \qquad v(\circ\varphi) = \begin{cases} 0, & \text{if } v(\varphi) > 1 \\ 1, & \text{otherwise} \end{cases}$$

It is easy to check that all the sequent rules from Section 1 are validated by the above matrices, with the sole exception of (s6$^{\mathbf{r}}$). At the same time, the above matrices can also easily be seen to invalidate all sequents of the form $\vdash \sim^i\varphi \equiv \sim^j\varphi$, for non-negative $i \neq j$. □

Corollary 3.7 (Uncharacterizability by finite matrices, *version* I) None of the logics from the first two columns of Figure 1 (i.e., the fragments of **Ci**) is finite-valued.

Proof: Would any of these logics be characterized by matrices with only m truth-values, then we would have, by the Pigeonhole Principle, some $i < j \leq (i + m^m)$ such that $v(\neg^i p) = v(\neg^j p)$, for all v. This would in turn validate some sequent of the form $\vdash \sim^i\varphi \equiv \sim^j\varphi$, for $i < j$. □

The following theorem and its corollary correct a result suggested in [11]:

Theorem 3.8 Let δ_{ij}, for $i, j \neq 0$, denote the formula $\circ p_i \wedge p_i \wedge \sim p_j$, and let δ^n denote the disjunctive formula $\bigvee_{1 \leq i < j \leq n}(\delta_{ij} \supset p_{n+1})$, for $n > 0$. No sequent of the form $\vdash \delta^n$ is derivable in the logics from the first two lines of Figure 1.

Proof: Take now the truth-values from the set $\mathbb{N} \cup \{\omega\}$, where ω is the only undesignated truth-value. Define the matrices for the connectives as follows:

$$v(\varphi \wedge \psi) = \mathsf{max}(v(\varphi), v(\psi)) \qquad v(\varphi \vee \psi) = \mathsf{min}(v(\varphi), v(\psi))$$

$$v(\varphi \supset \psi) = \begin{cases} \omega, & \text{if } v(\varphi) \in \mathbb{N} \text{ and } v(\psi) = \omega \\ v(\psi), & \text{if } v(\varphi) = \omega \text{ and } v(\psi) \in \mathbb{N} \\ 0, & \text{if } v(\varphi) = \omega = v(\psi) \\ \mathsf{max}(v(\varphi), v(\psi)), & \text{otherwise} \end{cases}$$

$$v(\sim\varphi) = \begin{cases} \omega, & \text{if } v(\varphi) = 0 \\ 0, & \text{if } v(\varphi) = \omega \\ v(\varphi), & \text{otherwise} \end{cases} \qquad v(\circ\varphi) = \begin{cases} 0, & \text{if } v(\varphi) \in \{0, \omega\} \\ \omega, & \text{otherwise} \end{cases}$$

It is easy to check that all the sequent rules from Section 1 are validated by the above matrices. At the same time, the above matrices can be seen to invalidate all sequents of the form $\vdash \delta^n$. Indeed, just consider a model such that $v(p_i) = i$, for $i \leq n$, and $v(p_{n+1}) = \omega$. □

Corollary 3.9 (Uncharacterizability by finite matrices, *version* II) None of the logics from the first two lines of Figure 1 (i.e., extensions of **mbC**) is finite-valued.

Proof: Notice, again using the Pigeonhole Principle, that the formula δ^n is validated by any set of m-valued matrices that is adequate for the logics extending **mbC** (use (s3) and (s1.2.2)) and such that $m < n$. □

One logic from Figure 1, however, was not covered by the previous results. So, the following is here left open:

Problem 3.10 Find a proof that $PIfe$ is not characterizable by finite matrices.

4 Interpretations through possible translations

We will see in this section that all the previous paraconsistent logics can still be given adequate interpretations in terms of *combinations* of 3-valued logics, by way of specific possible-translations semantics (PTS). Consider the 3-valued matrices of $\mathcal{M}$, at Table 3), where F is the only undesignated truth-value.

$\wedge$	T	t	F
T	t	t	F
t	t	t	F
F	F	F	F

$\vee$	T	t	F
T	t	t	t
t	t	t	t
F	t	t	F

$\supset$	T	t	F
T	t	t	F
t	t	t	F
F	t	t	t

	$\sim_1$	$\sim_2$	$\sim_3$
T	F	F	F
t	F	t	t
F	T	t	T

	$\circ_1$	$\circ_2$	$\circ_3$
T	T	t	F
t	F	F	F
F	T	t	F

Figure 3: Matrices of $\mathcal{M}$.

Given a 3-valued assignment $a : \mathcal{P} \longrightarrow \{T, t, F\}$, let w be its unique homomorphic extension into the whole language of $\mathcal{M}$, and let $\Gamma \vDash_w \Delta$ hold good, for given sets of formulas Γ and Δ, iff $(\exists\gamma \in \Gamma)w(\gamma) = F$ or $(\exists\delta \in \Delta)w(\delta) \in \{T, t\}$. Then, the canonical (multiple-conclusion) entailment relation $\vDash_{\mathcal{M}}$ determined by the above 3-valued matrices is set by taking $\Gamma \vDash_{\mathcal{M}} \Delta$ iff $\Gamma \vDash_w \Delta$ for every interpretation $w \in \mathcal{M}$.

Consider next the following possible restrictions over the set of admissible translating mappings $* : \mathcal{S}_i \longrightarrow \mathcal{M}$:

(tr0) $p^* = p$, for $p \in \mathcal{P}$
(tr1) $(\varphi \bowtie \psi)^* = (\varphi^* \bowtie \psi^*)$, for $\bowtie \in \{\wedge, \vee, \supset\}$
(tr2.1) $(\sim\varphi)^* \in \{\sim_1\varphi^*, \sim_2\varphi^*\}$
(tr2.2) $(\sim\varphi)^* \in \{\sim_1\varphi^*, \sim_3\varphi^*\}$
(tr2.3) $(\sim^{n+1}\circ\varphi)^* = \sim_1(\sim^n\circ\varphi)^*$

(tr3.1)	$(\circ\varphi)^* \in \{\circ_2\varphi^*, \circ_3\varphi^*, \circ_2(\sim\varphi)^*, \circ_3(\sim\varphi)^*\}$
(tr3.2)	$(\circ\varphi)^* \in \{\circ_1\varphi^*, \circ_1(\sim\varphi)^*\}$
(tr3.3)	if $(\sim\varphi)^* = \sim_1\varphi^*$ then $(\circ\varphi)^* = \circ_1(\sim\varphi)^*$
(tr4)	if $(\sim\varphi)^* = \sim_3\varphi^*$ then $(\sim\sim\varphi)^* = \sim_3(\sim\varphi)^*$

One can now select appropriate sets of restrictions in order to split each of the paraconsistent logics from the last section by way of PTS:

Logic	Restrictions over the translating mappings
PI	(tr0), (tr1), (tr2.1)
mbC	(tr0), (tr1), (tr2.1), (tr3.1)
mCi	(tr0), (tr1), (tr2.1), (tr2.3), (tr3.2)
PIf	(tr0), (tr1), (tr2.2)
bC	(tr0), (tr1), (tr2.2), (tr3.1)
Ci	(tr0), (tr1), (tr2.2), (tr3.2), (tr3.3)
$PIfe$	(tr0), (tr1), (tr2.2), (tr4)
bCe	(tr0), (tr1), (tr2.2), (tr3.1), (tr4)
Cie	(tr0), (tr1), (tr2.2), (tr3.2), (tr3.3), (tr4)

Let Tr denote some set of translating mappings defined according to an appropriate subset of the previously mentioned restrictions. Define a pt-*model* as a pair $\langle w, *\rangle$, where $* \in \mathsf{Tr}$ and $w \in \mathcal{M}$, and let $\Gamma \Vdash^*_w \Delta$ hold good, for given sets of formulas Γ and Δ, iff $\Gamma^* \vDash_w \Delta^*$. A pt-*consequence relation* $\Vdash_{\mathsf{pt}}$ is then set by taking $\Gamma \Vdash_{\mathsf{pt}} \Delta$ iff $\Gamma \Vdash^*_w \Delta$ for every pt-model $\langle w, *\rangle$ allowed by Tr. Equivalently, in the cases presently under consideration, $\Gamma \Vdash_{\mathsf{pt}} \Delta$ also means, more simply, that $\Gamma^* \vDash_{\mathcal{M}} \Delta^*$, for every admissible translation $* \in \mathsf{Tr}$.

Note 4.1 (The development of PTS) A logic $\mathcal{L}$ is said to have a *possible-translations semantics* when it can be given an adequate interpretation in terms of pt-models, for some appropriate set of translating mappings. Each translation can then be seen as a sort of interpretation scenario for $\mathcal{L}$. This intuition is good enough for the purposes of the present paper, but the possible-translations tool is in fact more general than that. For a generous and clear formal definition of this sort of structures, check [22]. For other more specific and carefully explained examples, check [20, 15, 12]. The interested reader will notice that the PTS offered for **Ci** above is distinct from the one presented in [16]. Possible-translations semantics were first introduced in [11], restricted to the splitting of a logic into finite-valued truth-functional scenarios. The embryo was then frozen for a period, and in between 1997 and 1998 it was publicized under the denomination 'non-deterministic semantics', in [14], and in several talks by Carnielli and a few by myself. Noticing that the non-deterministic element was but a particular accessory of the more general picture, from 1999 on the semantics returned to its earlier denomination.

Note 4.2 (PTS and non-deterministic semantics) PTS are related to (but are more general than) the *non-deterministic semantics* (NDS) proposed by Avron & Lev (cf. [5]) in ways that are still to be more carefully explained. On what concerns the logics studied in the present paper, it should be noticed that [4] proposes a 2-valued NDS for PI, and [2] also offers an 3-valued NDS for PIf which is strikingly similar to the PTS presented for this logic above (and that comes from [20, 15]). More recently, [3] offers 3-valued NDS for the logics **mbC**, **bC**, **bCe**. Roughly speaking, one could say that *dynamic* NDS are based on clauses having the same format of (tr0)–(tr2.2), and *static* NDS additionally impose constraints having the format of (tr2.3) or (tr4) for each of the involved connectives. There is a mechanical way, thus, to move from a given NDS to an equivalent PTS. Further discussion of that issue shall be postponed to a future work.

We now have a number of quite diverse consequence relations associated to each of the above logics. Of course we want to keep this fauna under control —in the best of all possible worlds we want to be able to prove that all those consequence relations deliver just the same the result, for each given logic, that is, we want to prove that:

$$\vdash_{\mathsf{seq}} = \vDash_{\mathsf{biv}} = \Vdash_{\mathsf{pt}}$$

That is matter for the next, and final, section.

5 Adequacy of each of the newly proposed PTS

As mentioned in Section 1, the technology that solves the first part of our problem is well-known, and its outcome will here be taken for granted: $\vdash_{\mathsf{seq}} = \vDash_{\mathsf{biv}}$.

Now, to check soundness of each of the paraconsistent logics in section 2 with respect to its specific PTS in section 4, one has two alternatives from the start. The first is to prove it directly from the axiomatizations in section 1 and the appropriate sets of translating mappings:

Theorem 5.1 (Soundness) $\vdash_{\mathsf{seq}} \subseteq \Vdash_{\mathsf{pt}}$.

Proof: Just translate each sequent axiom in all possible ways allowed by Tr and check that these translations are validated by $\mathcal{M}$. □

The second alternative is to prove that each pt-model is bisimulated by some appropriate bivaluation:

Theorem 5.2 (Convenience)

$$(\forall w \in \mathcal{M})(\forall * \in \mathsf{Tr})(\exists b \in \mathsf{biv})\ \vDash_b \alpha \ \Leftrightarrow\ \Vdash^*_w \alpha.$$

Proof: Just set $b(\alpha) = 0$ iff $w(\alpha^*) = F$. Then check that the axioms in biv are all respected, in each case. ☐

Corollary 5.3 (Soundness again) $\vDash_{\mathsf{biv}} \subseteq \Vdash_{\mathsf{pt}}$.

Now for completeness. Given that the evaluation of the consistency connective, $\circ$, in the way we have defined it, takes into account the evaluation of the negation connective, $\sim$, it will be helpful, when doing some of the next proofs by induction on the complexity of the formulas, to make use of the following non-canonical measure of complexity, **mc**:

(mc0) $\mathbf{mc}(p) = 0$, for $p \in \mathcal{P}$
(mc1) $\mathbf{mc}(\varphi \bowtie \psi) = \mathbf{mc}(\varphi) + \mathbf{mc}(\psi) + 1$, for $\bowtie \in \{\wedge, \vee, \supset\}$
(mc2) $\mathbf{mc}(\sim\varphi) = \mathbf{mc}(\varphi) + 1$
(mc3) $\mathbf{mc}(\circ\varphi) = \mathbf{mc}(\sim\varphi) + 1$

With such apparatus in hands, we can start looking for a proof that each particular bivaluation is bisimulated by some appropriate pt-model:

Theorem 5.4 (Representability)

$$(\forall b \in \mathsf{biv})(\exists w \in \mathcal{M})(\exists * \in \mathsf{Tr}) \;\; \Vdash^*_w \alpha \;\Leftrightarrow\; \vDash_b \alpha.$$

From what it would easily follow that:

Corollary 5.5 (Completeness) $\vDash_{\mathsf{biv}} \supseteq \Vdash_{\mathsf{pt}}$.

With respect to the above mentioned representability result, still to be proven, the safest strategy at this point seems to be that of checking it for each of our paraconsistent logics on its own turn, refining the statements and proofs to better suit each case. So, here we go:

Theorem 5.6 (PI-representability)

$$\begin{gathered}(\forall b \in \mathsf{biv})(\exists w \in \mathcal{M})(\exists * \in \mathsf{Tr})\\ w(\alpha^*) = t \;\Leftrightarrow\; b(\alpha) = 1, \text{ and}\\ w(\alpha^*) = F \;\Leftrightarrow\; b(\alpha) = 0.\end{gathered}$$

Proof: To take care of w, set, for $p \in \mathcal{P}$:

(rw) $a(p) = F$ if $b(p) = 0$, and
$a(p) = t$ otherwise

and extend a into w homomorphically, according to the strictures of $\mathcal{M}$.

On what concerns $*$, set:

(rt0) $p^* = p$, for $p \in \mathcal{P}$
(rt1) $(\varphi \bowtie \psi)^* = (\varphi^* \bowtie \psi^*)$, for $\bowtie \in \{\wedge, \vee, \supset\}$
(rt2) $(\sim\varphi)^* = \sim_1\varphi^*$, if $b(\sim\varphi) = 0$
$(\sim\varphi)^* = \sim_2\varphi^*$, otherwise

The main statement above can now easily be checked by induction on the complexity measure **mc**. ☐

Theorem 5.7 (mbC-representability)

$$(\forall b \in \mathsf{biv})(\exists w \in \mathcal{M})(\exists * \in \mathsf{Tr})$$
$$w(\alpha^*) = T \;\Rightarrow\; b(\sim\alpha) = 0, \text{ and}$$
$$w(\alpha^*) = F \;\Leftrightarrow\; b(\alpha) = 0.$$

Proof: To take care of w, set, for $p \in \mathcal{P}$:

(rw) $a(p) = F$ if $b(p) = 0$,
$a(p) = T$ if $b(\sim p) = 0$, and
$a(p) = t$ otherwise

and extend a into w homomorphically, according to the strictures of $\mathcal{M}$.

On what concerns $*$, set:

(rt0) $p^* = p$, for $p \in \mathcal{P}$
(rt1) $(\varphi \bowtie \psi)^* = (\varphi^* \bowtie \psi^*)$, for $\bowtie \in \{\wedge, \vee, \supset\}$
(rt2) $(\sim\varphi)^* = \sim_1\varphi^*$, if $b(\sim\varphi) = 0$ or $b(\varphi) = 0 = b(\sim\sim\varphi)$
$(\sim\varphi)^* = \sim_2\varphi^*$, otherwise
(rt3) $(\circ\varphi)^* = \circ_3\varphi^*$, if $b(\circ\varphi) = 0$
$(\circ\varphi)^* = \circ_2(\sim\varphi)^*$, if $b(\circ\varphi) = 1$ and $b(\sim\varphi) = 0$
$(\circ\varphi)^* = \circ_2\varphi^*$, otherwise

Check now the result by induction on **mc**. Notice from (rt3) how the non-standard clause (mc3) of the previously defined non-canonical measure of complexity finally proves to be useful. □

Theorem 5.8 (mCi-representability)

$$(\forall b \in \mathsf{biv})(\exists w \in \mathcal{M})(\exists * \in \mathsf{Tr})$$
$$w(\alpha^*) = T \;\Rightarrow\; b(\sim\alpha) = 0, \text{ and}$$
$$w(\alpha^*) = F \;\Leftrightarrow\; b(\alpha) = 0.$$

Proof: Do as in parts (rt0)–(rt2) of Theorem 5.7, but now set:

(rt3) $(\circ\varphi)^* = \circ_1(\sim\varphi)^*$, if $b(\sim\varphi) = 0$
$(\circ\varphi)^* = \circ_1\varphi^*$, otherwise
(rt4) $(\sim^{n+1}\circ\varphi)^* = \sim_1(\sim^n\circ\varphi)^*$

Check the result by induction on **mc**. □

Theorem 5.9 (PIf-representability)

$$(\forall b \in \mathsf{biv})(\exists w \in \mathcal{M})(\exists * \in \mathsf{Tr})$$
$$w(\alpha^*) = T \;\Rightarrow\; b(\sim\alpha) = 0, \text{ and}$$
$$w(\alpha^*) = F \;\Leftrightarrow\; b(\alpha) = 0.$$

Proof: Do as in Theorem 5.6, except that in now setting:

(rt2) $(\sim\varphi)^* = \sim_3\varphi^*$, if $b(\varphi) = 1 = b(\sim\varphi)$
$(\sim\varphi)^* = \sim_1\varphi^*$, otherwise

Check the result by induction on **mc**.

(A slightly different proof of this fact —check clause (rw)— can be found in the ch.4 of [20] and in [15] —bear in mind though that this logic PIf shows up there under the name C_{min}.) □

Theorem 5.10 (bC-representability)

$$(\forall b \in \mathsf{biv})(\exists w \in \mathcal{M})(\exists * \in \mathsf{Tr})$$
$$w(\alpha^*) = T \;\Rightarrow\; b(\sim\alpha) = 0, \text{ and}$$
$$w(\alpha^*) = F \;\Leftrightarrow\; b(\alpha) = 0.$$

Proof: Do as in Theorem 5.7, except that in now setting (rt2) as in Theorem 5.9. Check the result by induction on **mc**. □

Theorem 5.11 (Ci-representability)

$$(\forall b \in \mathsf{biv})(\exists w \in \mathcal{M})(\exists * \in \mathsf{Tr})$$
$$w(\alpha^*) = T \;\Rightarrow\; b(\sim\alpha) = 0, \text{ and}$$
$$w(\alpha^*) = F \;\Leftrightarrow\; b(\alpha) = 0.$$

Proof: Do as in Theorem 5.10, except that in now setting:

(rt3) $(\circ\varphi)^* = \circ_1(\sim\varphi)^*$, if $b(\circ\varphi) = 1$
$(\circ\varphi)^* = \circ_1\varphi^*$, otherwise

Check the result by induction on **mc**.
(Notice that the PTS offered for **Ci** in the paper [16] uses different interpretations for the consistency connective and is based on a stricter set of restrictions over the set Tr. The present semantics seems, in a sense, to be more in accordance with the classical behavior of $\circ$ with respect to $\sim$.) □

Theorem 5.12 ($PIfe$-representability)

$$(\forall b \in \mathsf{biv})(\exists w \in \mathcal{M})(\exists * \in \mathsf{Tr})$$
$$w(\alpha^*) = T \;\Rightarrow\; b(\sim\alpha) = 0, \text{ and}$$
$$w(\alpha^*) = F \;\Leftrightarrow\; b(\alpha) = 0.$$

Proof: Do as in Theorem 5.9, except that in now setting the extra requirement:

(rt4) if $(\sim\varphi)^* = \sim_3\varphi^*$ then $(\sim\sim\varphi)^* = \sim_3(\sim\varphi)^*$

Check the result by induction on **mc**.
(The practical difference in this proof with respect to the previous ones is that one will not only have a base case of induction for the atomic sentences and a complex case for each of the connectives, but one will also explicitly have to take into consideration the extra case of complex formulas preceded by at least two negation symbols.) □

Theorem 5.13 (bCe-representability)

$$(\forall b \in \mathsf{biv})(\exists w \in \mathcal{M})(\exists * \in \mathsf{Tr})$$
$$w(\alpha^*) = T \ \Rightarrow\ b(\sim\alpha) = 0, \text{ and}$$
$$w(\alpha^*) = F \ \Leftrightarrow\ b(\alpha) = 0.$$

Proof: Do as in Theorem 5.10, except that in now setting (rt4) as in Theorem 5.12. Check the result by induction on **mc**. □

Theorem 5.14 (Cie-representability)

$$(\forall b \in \mathsf{biv})(\exists w \in \mathcal{M})(\exists * \in \mathsf{Tr})$$
$$w(\alpha^*) = T \ \Rightarrow\ b(\sim\alpha) = 0, \text{ and}$$
$$w(\alpha^*) = F \ \Leftrightarrow\ b(\alpha) = 0.$$

Proof: Do as in Theorem 5.8, except that in now setting (rt4) as in Theorem 5.12. Check the result by induction on **mc**. □

Example 5.15 We could now use the above defined PTS to check that, in **Cie** (thus, also in **Ci**, **bC**, **mCi**, **CLuN** or **mbC**), the formulas $\circ\alpha$ and $\sim\bullet\alpha$ are logically *distinguishable* even if equivalent, as announced in Section 2. Indeed, by the definition of $\bullet$, the formula $\sim\bullet\alpha$ is logically indistinguishable from the formula $\sim\sim\circ\alpha$. Yet, given a formula φ of the form $\sim p$ and a formula ψ of the form $\varphi[p/(p \wedge p)]$, it is easy to see that, in spite of the equivalence between $\varphi[p/\circ p]$ and $\varphi[p/\sim\sim\circ p]$ in logics as weak as **mCi**, formulas such as $\psi[p/\circ p]$ and $\psi[p/\sim\sim\circ p]$ are not equivalent in **Cie**. To check that, select some **Cie**-admissible translating mapping such that $(\circ p)^* = \circ_1\sim_1 p$, $(\sim(\circ p \wedge \circ p))^* = \sim_1(\circ p \wedge \circ p)^*$ and $(\sim(\sim\sim\circ p \wedge \sim\sim\circ p))^* = \sim_3(\sim\sim\circ p \wedge \sim\sim\circ p)^*$, and then select a 3-valued model $w \in \mathcal{M}$ for which $w(p) = t$.

Note 5.16 (Dualizing the above constructions) One might now start everything all over again, back from Section 1, and easily dualize all results for paracomplete counterparts of all the above paraconsistent logics. To such an effect, one only needs to explore the symmetry of the present multiple-conclusion environment, exchange each bivaluational axiom (si) and each sequent rule (si) for their converses (b$i^{\mathbf{r}}$) and (s$i^{\mathbf{r}}$), and exchange the consistency connective for a completeness, or determinedness, connective (as in [21]), and so on and so forth. The case of the dual of PIf was already explored in ch.4 of [20] and in [15], under the appellation D_{min}.

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