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Post-Newtonian Equations of Motion in the Flat Universe

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Using the (3+1) formalism, we derive the post-Newtonian (PN) equations of motion in a flat universe. To derive the equations of motion, we must carefully consider two points, one being the choice of the density in the Newtonian order (ρ_N) and the other the choice of the gauge condition. In choosing ρ_N, we require that the density fluctuation $\rho_N-\rho_0$ agrees with a gauge invariant quantity in the linear approximation theory. As a gauge condition, we propose the cosmological post-Newtonian (CPN) slice condition with the pseudo transverse-traceless gauge condition, by which the evolution of the geometric variables derived in the PN approximation in the early stage of universe agrees with that of the gauge invariant quantities in the linear approximation. In the derived equations of motion, the force is calculated from six potentials which satisfy the Poisson equations. Hence, our formalism can be easily applied to numerical simulations in which the standard technique (e.g., particle-mesh method) is used. We apply the PN formula to the one-dimensional (1D) Zel'dovich solution to demonstrate that our strategy works well, and also to determine the effect of the PN forces on the evolution of the large-scale structure. It is found that the behavior of the density fluctuation and metric quantities in the early stage obtained by the present formalism agrees with that of the gauge invariant quantities in the linear approximation, although they do not always agree within the previous formalism due to the appearance of spurious gauge modes. We also discuss the evolution of the non-linear density fluctuation with very large scale, which may be affected by the PN correction in the last stage of the evolution.

§ 1. Introduction

The big-bang cosmological model is now considered to be successful. In this scenario, the large-scale structure of the universe has grown from the small density fluctuations at an early time in a homogeneous and isotropic background.[1] Its evolution depends not only on cosmological parameters such as the Hubble parameter H, the density parameter Ω, the fraction of the baryon Ω_b, and the cosmological constant Λ, but also on the initial spectrum of the density fluctuation. This means that we have the possibility to understand our universe well if we clarify the evolution of the large-scale structure of the universe theoretically and compare the theoretical prediction with the observation of the large-scale structure.[2),3)] Therefore, theoretical investigations of the evolution of the large-scale structure are among the most important subjects in cosmology.

When we investigate the evolution of the large-scale structure of the universe, we usually assume that the Newtonian theory in cosmology is correct, and use it.[1] In this theory, equations of motion are derived assuming that the scale of the non-linear density fluctuation is much smaller than the horizon scale H^{-1} of the universe. In most cases, this assumption is correct because the size of our Galaxy and the cluster of galaxies are, respectively, about $10^{-5}H^{-1}$ and $10^{-3}H^{-1}\ll H^{-1}$. However, we may ask about the very large-scale structure. Because of limited observations of very large scales,[3] we have only a few observational results which suggest the existence of non-linear density fluctuations of very large scale, beyond $100h^{-1}$Mpc scale,[4),5)] but in this decade, we will know whether such fluctuations really exist or not by means of galaxy survey projects such as the SDSS (Sloan Digital Sky Survey) project.[6)] For a non-linear density fluctuation with very large scale, it is not clear at all whether the application of the Newtonian theory is appropriate. To understand the evolution of the large-scale structure of the universe, it is important to clarify up to what scale we may use the Newtonian theory as a sufficiently accurate theory. For this purpose, in this paper, we consider the post-Newtonian (PN) equations of motion in the flat universe and analyze them.

As for the PN equations of motion in cosmology, there have been several works.[7),8)] Futamase[7)] derived the PN equation for the scale factor to analyze the correction of the expansion rate due to the inhomogeneity in the universe. In his formalism, he did not derive the PN equations of motion consistently, so his formalism cannot be used to calculate the evolution of the density fluctuations in the PN order. On the other hand, Tomita[8)] has derived the PN equations of motion for the N-body system consistently. However, his equations of motion are very complicated, and it seems difficult to study physical effects of the PN correction analytically. Even for numerical simulations, they do not seem practical because we need to perform many direct summations in calculating the force term of his equations of motion. This is very time consuming.[9)] To see qualitative effects of the PN correction, it is desirable that the equations of motion can be treated analytically, and to perform the numerical simulation, we should write the equations to which we can apply a standard numerical method. In this way, we need appropriate PN equations of motion, which are easily analyzed by both analytical and numerical calculations.

This paper is organized as follows. In § 2, we derive the PN equations of motion by using the (3+1) formalism in general relativity. In deriving the equations in the PN approximation, we must specify 1) the density in the Newtonian order (ρ_N) and 2) gauge conditions. Their choice is very important in eliminating unphysical gauge modes, an shown below. In choosing ρ_N, we require that the density fluctuation $\rho_N - \rho_0$ becomes gauge invariant when the linear approximation theory holds ($a \ll 1$).[1)] As for gauge conditions, we suggest guiding principles, and propose the pseudo TT gauge and cosmological post-Newtonian slice for an aprropriate set of the gauge conditions. In § 3, we apply the derived PN formula to the 1D Zel'dovich solution in order to test the gauge conditions as well as to determine the PN effects on the evolution of the large-scale structure. We illustrate that the gauge conditions we propose in this paper work well. We also discuss the PN effects on the evolution of the large-scale structure. We show that the PN effect is not important at all for ordinary large-scale structure $<100h^{-1}$Mpc, although it may become important for the very large-scale structure in the late stage of the evolution. Section 4 is devoted to a summary. Throughout this paper, we use the units $c=G=1$. Greek and Latin indices take 0, 1, 2, 3 and 1, 2, 3, respectively, and h denotes the Hubble parameter in units of 100 km/sec/Mpc.

§ 2. The (3+1) formalism and post-Newtonian approximation

2.1. *The (3+1) formalism for Einstein equation*

For the sake of convenience, we use the (3+1) formalism to perform the post-Newtonian (PN) approximation.[10] In the (3+1) formalism, the metric is split as

$$g_{\mu\nu}=\gamma_{\mu\nu}-n_\mu n_\nu\,,$$
$$n_\mu=(-\alpha,\mathbf{0})\,,$$
$$n^\mu=\left(\frac{1}{\alpha},-\frac{\beta^i}{\alpha}\right), \tag{2·1}$$

where α, β^i and γ_{ij} are the lapse function, shift vector and metric on a 3D hypersurface respectively. Then the line element is written as

$$ds^2=-(\alpha^2-\beta_i\beta^i)dt^2+2\beta_i dt dx^i+\gamma_{ij}dx^i dx^j\,. \tag{2·2}$$

Using the (3+1) formalism, the Einstein equation, $G_{\mu\nu}=8\pi T_{\mu\nu}-\Lambda g_{\mu\nu}$, is split into the constraint equations and the evolution equations. The former set of equations constitutes the so-called Hamiltonian and momentum constraints. These become

$$R-K_{ij}K^{ij}+K^2=16\pi E+2\Lambda\,, \tag{2·3}$$

$$D_iK^i{}_j-D_jK=8\pi J_j\,, \tag{2·4}$$

where K_{ij}, K, R and D_i are the extrinsic curvature, the trace part of K_{ij}, the scalar curvature of a 3D hypersurface and the covariant derivative with respect to γ_{ij}, respectively. E and J_j are defined as

$$E=T_{\mu\nu}n^\mu n^\nu\,,$$
$$J_j=-T_{\mu\nu}n^\mu\gamma^\nu{}_j\,. \tag{2·5}$$

Evolution equations for the metric and extrinsic curvature become

$$\frac{\partial}{\partial t}\gamma_{ij}=-2\alpha K_{ij}+D_i\beta_j+D_j\beta_i\,, \tag{2·6}$$

$$\frac{\partial}{\partial t}K_{ij}=\alpha(R_{ij}+KK_{ij}-2K_{il}K^l{}_j)-D_iD_j\alpha-\alpha\Lambda\gamma_{ij}$$
$$+(D_j\beta^m)K_{mi}+(D_i\beta^m)K_{mj}+\beta^m D_m K_{ij}-8\pi\alpha\left(S_{ij}+\frac{1}{2}\gamma_{ij}(E-S^l{}_l)\right), \tag{2·7}$$

$$\frac{\partial}{\partial t}\gamma=2\gamma(-\alpha K+D_i\beta^i)\,, \tag{2·8}$$

$$\frac{\partial}{\partial t}K=\alpha(R+K^2)-D^iD_i\alpha+\beta^jD_jK+4\pi\alpha(S^l{}_l-3E)-3\alpha\Lambda\,, \tag{2·9}$$

where R_{ij}, γ and S_{ij} are, respectively, the Ricci tensor with respect to γ_{ij}, determinant of γ_{ij} and

$$S_{ij}=T_{kl}\gamma^k{}_i\gamma^l{}_j\,. \tag{2·10}$$

To distinguish among the expansion part, wave part and non-wave part in the 3 D metric, we use $\tilde{\gamma}_{ij}=a(t)^{-2}\psi^{-4}\gamma_{ij}$ instead of γ_{ij}, where $a(t)$ is assumed to denote the scale factor of the flat universe which depends only on t. In the following, we only consider the flat universe and use Cartesian coordinates for simplicity. Then we can define $\det(\tilde{\gamma}_{ij})=1$ and $\gamma=a^6\psi^{12}$. We also define $\tilde{A}_{ij}$ as

$$\tilde{A}_{ij}\equiv a^{-2}\psi^{-4}A_{ij}\equiv a^{-2}\psi^{-4}\Big(K_{ij}-\frac{1}{3}\gamma_{ij}K\Big)\,. \tag{2·11}$$

We should note that in our notation, indices of $\tilde{A}_{ij}$ are raised and lowered by $\tilde{\gamma}_{ij}$, so that the relations $\tilde{A}^i{}_j=A^i{}_j$ and $\tilde{A}^{ij}=a^2\psi^4A^{ij}$ hold. Using these variables, the evolution equations (2·6)~(2·9) can be rewritten as

$$\frac{d}{dt}\tilde{\gamma}_{ij}=-2\alpha\tilde{A}_{ij}+\tilde{\gamma}_{il}\frac{\partial\beta^l}{\partial x^j}+\tilde{\gamma}_{jl}\frac{\partial\beta^l}{\partial x^i}-\frac{2}{3}\tilde{\gamma}_{ij}\frac{\partial\beta^l}{\partial x^l}\,, \tag{2·12}$$

$$\frac{d}{dt}\tilde{A}_{ij}=\frac{1}{a^2\psi^4}\Big[\alpha\Big(R_{ij}-\frac{1}{3}\gamma_{ij}R\Big)-\Big(D_iD_j\alpha-\frac{1}{3}\gamma_{ij}D^kD_k\alpha\Big)\Big]+\alpha(K\tilde{A}_{ij}-2\tilde{A}_{il}\tilde{A}^l{}_j)$$
$$+\frac{\partial\beta^m}{\partial x^i}\tilde{A}_{mj}+\frac{\partial\beta^m}{\partial x^j}\tilde{A}_{mi}-\frac{2}{3}\frac{\partial\beta^m}{\partial x^m}\tilde{A}_{ij}-8\pi\frac{\alpha}{a^2\psi^4}\Big(S_{ij}-\frac{1}{3}\gamma_{ij}S^l{}_l\Big)\,, \tag{2·13}$$

$$\frac{d}{dt}\psi+\frac{\dot{a}}{2a}\psi=\frac{\psi}{6}\Big(-\alpha K+\frac{\partial\beta^i}{\partial x^i}\Big)\,, \tag{2·14}$$

$$\frac{d}{dt}K=\alpha\Big(\tilde{A}_{ij}\tilde{A}^{ij}+\frac{1}{3}K^2\Big)-D^iD_i\alpha+4\pi\alpha(S^i{}_i+E)-\alpha\Lambda\,, \tag{2·15}$$

where

$$\frac{d}{dt}=\frac{\partial}{\partial t}-\beta^i\frac{\partial}{\partial x^i}\qquad\text{and}\qquad\dot{a}=\frac{\partial a}{\partial t}\Big)_{x^i}\,. \tag{2·16}$$

The constraint equations are also written as

$$\tilde{\Delta}\psi=\frac{1}{8}\tilde{R}\psi-2\pi E\psi^5a^2-\frac{\psi^5a^2}{8}\Big(\tilde{A}_{ij}\tilde{A}^{ij}-\frac{2}{3}K^2+2\Lambda\Big)\,, \tag{2·17}$$

$$\tilde{D}_i(\psi^6\tilde{A}^i{}_j)-\frac{2}{3}\psi^6\tilde{D}_jK=8\pi\psi^6J_j\,, \tag{2·18}$$

where $\tilde{\Delta}$, $\tilde{D}_i$ and $\tilde{R}$ are the Laplacian, the covariant derivative and the scalar curvature with respect to $\tilde{\gamma}_{ij}$, respectively.

We split R_{ij} in Eq. (2·13) into two parts as

$$R_{ij}=\tilde{R}_{ij}+R^{\psi}_{ij}\,, \tag{2·19}$$

where $\tilde{R}_{ij}$ is the Ricci tensor with respect to $\tilde{\gamma}_{ij}$, and

$$R^{\psi}_{ij}=-\frac{2}{\psi}\tilde{D}_i\tilde{D}_j\psi-\frac{2}{\psi}\tilde{\gamma}_{ij}\tilde{D}^k\tilde{D}_k\psi+\frac{6}{\psi^2}(\tilde{D}_i\psi)(\tilde{D}_j\psi)-\frac{2}{\psi^2}\tilde{\gamma}_{ij}(\tilde{D}_k\psi)(\tilde{D}^k\psi)\,. \tag{2·20}$$

Making use of the property $\det(\tilde{\gamma}_{ij})=1$, $\tilde{R}_{ij}$ is written as

$$\tilde{R}_{ij}=\frac{1}{2}[\tilde{\gamma}^{kl}(\tilde{\gamma}_{lj,ik}+\tilde{\gamma}_{li,jk}-\tilde{\gamma}_{ij,lk})+\tilde{\gamma}^{kl}{}_{,k}(\tilde{\gamma}_{lj,i}+\tilde{\gamma}_{li,j}-\tilde{\gamma}_{ij,l})]-\tilde{\Gamma}^{l}_{kj}\tilde{\Gamma}^{k}_{li}, \qquad (2\cdot21)$$

where $_{,i}$ denotes $\partial/\partial x^i$ and $\tilde{\Gamma}^k_{ij}$ is the Christoffel symbol with respect to $\tilde{\gamma}_{ij}$. We split $\tilde{\gamma}_{ij}$ and $\tilde{\gamma}^{ij}$ as $\delta_{ij}+h_{ij}$ and $\delta^{ij}+f^{ij}$, where δ_{ij} denotes the flat geometry, and rewrite $\tilde{R}_{ij}$ as

$$\tilde{R}_{ij}=\frac{1}{2}[-h_{ij,kk}+h_{jl,li}+h_{il,lj}+f^{kl}{}_{,k}(h_{lj,i}+h_{li,j}-h_{ij,l})$$
$$+f^{kl}(h_{lj,ik}+h_{li,jk}-h_{ij,lk})]-\tilde{\Gamma}^{l}_{kj}\tilde{\Gamma}^{k}_{li}. \qquad (2\cdot22)$$

Now, let us consider the gauge conditions. As for the spatial gauge, we adopt the transverse gauge as

$$h_{ij,j}=0. \qquad (2\cdot23)$$

This condition is guaranteed if $h_{ij,j}=0$ at $t=0$ and β^i always satisfies

$$-\beta^k_{,j}\tilde{\gamma}_{ij,k}=\left(-2\alpha\tilde{A}_{ij}+\tilde{\gamma}_{il}\beta^l{}_{,j}+\tilde{\gamma}_{jl}\beta^l{}_{,i}-\frac{2}{3}\tilde{\gamma}_{ij}\beta^l{}_{,l}\right)_{,j}. \qquad (2\cdot24)$$

By means of the transverse gauge, we can erase the vector part in h_{ij} and guarantee that h_{ij} only contains the tensor mode in the PN order.[10] Thus, we need only take into account the linear term in h_{ij} because we will perfom the PN approximation. In this case, Eq. (2·22) becomes

$$\tilde{R}_{ij}=-\frac{1}{2}\Delta_{\rm flat}h_{ij}, \qquad (2\cdot25)$$

where $\Delta_{\rm flat}$ is the Laplacian with respect to δ_{ij}. In the linear order in h_{ij}, the traceless property $h_{ii}=0$ is also guaranteed, so that we may call this gauge condition the pseudo TT (transverse-traceless) gauge. Note that $\tilde{R}=0$ is guaranteed in the pseudo TT gauge condition.

As for the slice condition for α, we first set

$$K=-3H(t)+Q, \qquad (2\cdot26)$$

where

$$H(t)^2=\left(\frac{\dot{a}(t)}{a(t)}\right)^2=\frac{8\pi\rho_0(t)}{3}+\frac{\Lambda}{3}. \qquad (2\cdot27)$$

$H(t)$ and $\rho_0(t)$ are the Hubble parameter and the homogeneous density of the flat universe, respectively. For simplicity, we define ρ_0 as an averaged value of Da^{-3}, where D denotes the conserved mass density (see below Eq. (2·56)). Q is an arbitrary function which is determined in fixing the slice condition. Here, in determining Q, we should note that the coordinate time in Eq. (2·27) is not the proper time, but rather a coordinate time in the PN approximation. This means that Eq. (2·27) has different meanings in different slice conditions, and we should carefully consider a natural extension of the homogeneous expansion law to that for the PN approximation. If we choose an inappropriate slice condition in the PN approximation, the undesirable gauge mode will appear in geometric variables (see § 2.2). Fortunately, we know that we may choose $Q=0$ in the Newtonian approximation, so that for the present case, Q should be determined in the PN order. We will specify this in the next section.

If we fix Q, Eq. (2·15) becomes an equation to determine α as:

$$D_iD^i\alpha=4\pi\alpha(S_i{}^i+E)+\alpha(\tilde{A}_{ij}\tilde{A}^{ij}+8\pi\rho_0)-12\pi\rho_0-\frac{dQ}{dt}-2\alpha\frac{\dot{a}}{a}Q+\alpha\frac{Q^2}{3}. \quad (2\cdot28)$$

Using the gauge conditions, Eq. (2·13) is rewritten as

$$\frac{d}{dt}\tilde{A}_{ij}+3\alpha H\tilde{A}_{ij}=\frac{1}{\psi^4a^2}\Big[-\frac{1}{2}\Delta_{\rm flat}h_{ij}+\alpha\Big(R^{\psi}_{ij}-\frac{1}{3}\gamma_{ij}R^{\psi}_{kl}\tilde{\gamma}^{kl}\Big)$$
$$-\Big(D_iD_j\alpha-\frac{1}{3}\gamma_{ij}D^kD_k\alpha\Big)\Big]+\alpha\{Q\tilde{A}_{ij}-2\tilde{A}_{il}\tilde{A}^l{}_j\}$$
$$+\beta^m{}_{,i}\tilde{A}_{mj}+\beta^m{}_{,j}\tilde{A}_{mi}-\frac{2}{3}\beta^m{}_{,m}\tilde{A}_{ij}-8\pi\frac{\alpha}{\psi^4a^2}\Big(S_{ij}-\frac{1}{3}\gamma_{ij}S^l{}_l\Big), \quad (2\cdot29)$$

where we use the pseudo TT gauge condition as well as the linear approximation for h_{ij} in the above equation. Combining Eqs. (2·12) and (2·29), we obtain the equation of gravitational waves for h_{ij} in the flat universe. In the wave equation for h_{ij}, however, the source term for h_{ij} appears first in the second PN order.[10] Since we only consider the first PN correction in the following, we neglect h_{ij} in this paper.

We note that the above equations do not contain the Λ term explicitly, except for the equation to determine the expansion law, (2·27). Therefore, the above formula can be used for the cases $\Lambda=0$ and $\Lambda\neq0$ only if we control Eq. (2·27).

Finally, we give the equations for matter. Since in this paper we consider the evolution of the density fluctuation after decoupling of matter and radiation, we adopt pressure-free dust as matter. (We also consider N-body systems in Appendix B.) The energy momentum tensor for the dust is written as

$$T^{\mu\nu}=\rho u^\mu u^\nu, \quad (2\cdot30)$$

where u^μ and ρ are the four velocity and the density, respectively. ρ obeys the continuity equation

$$\nabla_\mu(\rho u^\mu)=0, \quad (2\cdot31)$$

where ∇_μ is the covariant derivative with respect to $g_{\mu\nu}$. The explicit form becomes

$$\frac{\partial D}{\partial t}+\frac{\partial(Dv^i)}{\partial x^i}=0, \quad (2\cdot32)$$

where $D=\rho\alpha\psi^6a^3u^0$ is the so-called conserved density. The equations of motion are derived from

$$\nabla_\mu T_i{}^\mu=0. \quad (2\cdot33)$$

The explicit form becomes

$$\frac{\partial S_i}{\partial t}+\frac{\partial(S_i v^j)}{\partial x^j}=-\alpha\alpha_{,i}S^0+S_j\beta^j{}_{,i}-\frac{1}{2S^0}S_jS_k\gamma^{jk}{}_{,i}\,, \tag{2·34}$$

where

$$S_i=Du_i=\alpha\psi^6a^3\rho u^0u_i(=\psi^6a^3J_i)\,,$$

$$S^0=Du^0=\alpha\psi^6a^3\rho(u^0)^2\left(=\frac{E\psi^6a^3}{\alpha}\right),$$

$$v^i\equiv\frac{u^i}{u^0}=-\beta^i+\frac{\gamma^{ij}S_j}{S^0}\,. \tag{2·35}$$

2.2. *Cosmological post-Newtonian approximation*

In this section, we consider the first PN approximation of the above equations. First of all, we review the PN expansion of the variables in the expanding universe. In the expanding universe, we can introduce three non-dimensional parameters which are independent of each other in general:[7]

$$\epsilon\equiv\frac{av}{c}\,,$$

$$\kappa\equiv\frac{al}{L}\sim\frac{Hal}{c}\,,$$

$$\chi\equiv\left(\frac{\rho-\rho_0}{\rho_0}\right)^{1/2}, \tag{2·36}$$

where v, L and l denote the peculiar velocity, the horizon scale and the characteristic length scale of the density fluctuation in the comoving frame, respectively. In the cosmological PN formalism, we usually consider the velocity field v^i to be generated by the density fluctuation $\rho-\rho_0$. In this case, the following relation holds:

$$\chi\sim\frac{\epsilon}{\kappa}\,. \tag{2·37}$$

Also, we must impose the condition $al<L$ in the PN approximation, so that $\kappa<1$. Hence, we have two independent small parameters, ϵ and $\kappa\sim\epsilon/\chi$. Note that there is no limitation on χ, so there is no imposed relation between ϵ and κ. For the galactic scale, χ becomes $\sim10^3$ at the present time, so we should impose the condition $\epsilon\gg\kappa$. In this paper, we mainly consider the large-scale structure $\gtrsim 100h^{-1}$Mpc. In this case, it is expected that χ increases from $\ll1$ to ~1 and may exceed unity throughout the evolution of the density fluctuation. In such systems, the relation between ϵ and κ is not simple, so that, in the following, we derive the first PN equations of motion by means of c^{-1} expansion. Note that since ϵ, $\kappa=O(c^{-1})$, any contributions from both ϵ and κ are included in the derived equation, and it can be used for any system formally if ϵ and κ are less than unity. This is because in any case, the leading term among 1PN terms is greater than the higher PN terms, i.e., the equations derived here provide the PN correction always valid up to 1PN order.

For later convenience, we also mention the linear approximation.[1] In the linear approximation, we assume $\epsilon\ll\kappa$ and χ, $\epsilon\ll1$, and any limitations on κ are not imposed only if the relation $\epsilon \ll \kappa$ is satisfied. In the case $a \ll 1$, the linear approximation becomes very good because $\kappa \gg \epsilon \sim 0$, and $\chi \ll 1$ in the early stage. Hence, we usually adopt the linear approximation in the early stage and after χ becomes large (i.e., $\epsilon \lesssim \kappa$), the Newtonian approximation is used. In the PN approximation, the relation $\chi \sim \epsilon/\kappa$ holds, so that if $\epsilon \ll \kappa$ is satisfied, $\chi \ll 1$ is naturally guaranteed. When we consider the linear limit in the system where the PN approximation holds, we only need to take the limit $\chi \ll 1$ or $\epsilon \ll 1$, keeping $\kappa < 1$. In Fig. 1, we describe the conceptual figure about the cosmological PN approximation and linear approximation.

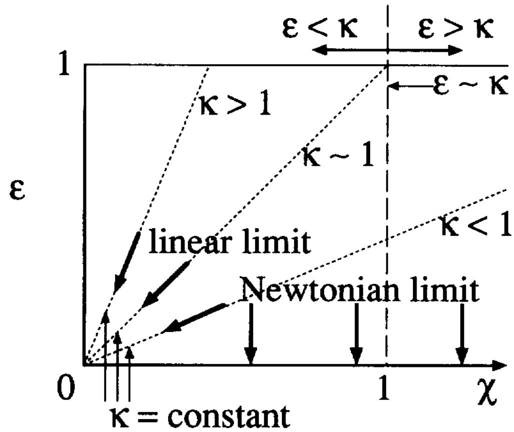

Fig. 1. Conceptual figure of the post-Newtonian (PN) approximation. The horizontal and vertical lines denote χ and ϵ, respectively. In the PN approximation, the relation $\chi \sim \epsilon/\kappa$ holds, and the constraints $\epsilon, \kappa < 1$ is needed. However any limitations for χ are not imposed. On the other hand, $\chi, \epsilon \ll 1$ and $\kappa \gg \epsilon$ must be satisfied in the linear approximation. If we try to take the linear limit in the system where the PN approximation holds, i.e., $\chi \sim \epsilon/\kappa$ and ϵ, $\kappa < 1$, we only need to take the limit $\chi \ll 1$ or $\epsilon \ll 1$ keeping $\kappa < 1$.

Extending Chandrasekhar's description of the PN expansion in the asymptotic flat space-time[13] to the cosmological PN approximation, the four velocity is expanded as

$$
\begin{aligned}
u^0 &= 1 + \left\{\frac{1}{2}a^2v^2 + U\right\} + O(c^{-4}),\\
u_0 &= -\left[1 + \left\{\frac{1}{2}a^2v^2 - U\right\} + O(c^{-4})\right],\\
u^i &= v^i\left[1 + \left\{\frac{1}{2}a^2v^2 + U\right\} + O(c^{-4})\right],\\
u_i &= a^2\left[v^i + \left\{\beta_i^{(3)} + v^i\left(\frac{a^2v^2}{2} + 3U\right)\right\} + O(c^{-5})\right],
\end{aligned} \qquad (2\cdot38)
$$

where terms in the bracket { } denote each first PN term, and U and $\beta_i^{(3)}$ are the Newtonian potential and the first PN term of β^i, respectively (see below). v^i is equal to that defined in Eq. (2·35), and $v^2 = v^i v^i$. Note that in the above, the expansion is in c^{-1}, so that $O(c^{-4})$ denotes $O(\epsilon^{4-n}\kappa^n)$, where $n = 0, 1, 2$. All geometric variables relevant to the present paper are expanded. We have:

$$
\begin{aligned}
a &= 1 - U + a^{(4)} + \cdots,\\
\psi &= 1 + \psi^{(2)} + \cdots,\\
\beta^i &= \beta_i^{(3)} + \cdots,\\
h_{ij} &= h_{ij}^{(4)} + \cdots,\\
Q &= Q^{(3)} + \cdots,
\end{aligned}
$$

$$\tilde{A}_{ij}=\tilde{A}^{(3)}_{ij}+\cdots, \tag{2·39}$$

where subscripts (n) denote the PN order (c^{-n}). The PN expansion of the relation $u^{\mu}u_{\mu}=-1$ becomes

$$(au^0)^2=1+\gamma^{ij}u_iu_j$$
$$=1+a^2v^2+O(c^{-4}). \tag{2·40}$$

Before writing the equation for the metric variables, we need to point out that we have a freedom in choosing the density in the Newtonian order, ρ_N.[14] This is because we have no reason to consider ρ as the density in the Newtonian order. To choose ρ_N, we only require that in the Newtonian limit, it reduces to ρ. For example in the linear approximation theory, a gauge invariant quantity is more important than $\rho-\rho_0$.[11),12)] As we will discuss in Appendix A, if we choose other quantities instead of the gauge invariant quantity, it is not guaranteed to be a *physical* density fluctuation in the linear approximation theory. Thus, we adopt ρ_N in order that $\rho_N-\rho_0$ becomes a gauge invariant quantity for $a\ll 1$. Although in the linear approximation theory, there exists an arbitrary combination of the gauge invariant quantities,[12] we here choose $\rho_N=\rho(1-2U)$*) as the density in the Newtonian order. (The reason for this choice is described in Appendix A.) According to this choice, we can naturally extract a growing density fluctuation in the case $a\ll 1$. Then, by using Eq. (2·28), we find that U should satisfy

$$\Delta_{\rm flat}U=-4\pi(\rho_N-\rho_0)a^2+4\pi\delta\rho_{NL}(t)a^2. \tag{2·41}$$

Here, $\delta\rho_{NL}(t)$ is a non-linear and PN term which is added in order to guarantee that the average of the right-hand side of Eq. (2·41) becomes zero, i.e., $\langle\rho_N-\rho_0-\delta\rho_{NL}\rangle=0$ (as for the averaging, see Eq. (2·56) below). From the Hamiltonian constraint, we also find

$$\psi^{(2)}=\frac{U}{2}. \tag{2·42}$$

As for $\alpha^{(4)}$, we write it as

$$\alpha^{(4)}=\frac{U^2}{2}+X. \tag{2·43}$$

Then, from Eq. (2·28), equation for X becomes

$$\Delta_{\rm flat}X=4\pi a^2(2\rho_Na^2v^2-\rho_0U+4(\rho_N-\rho_0)U)-a^2\left(\dot{Q}^{(3)}+2\frac{\dot{a}}{a}Q^{(3)}\right)+4\pi a^2\delta\rho_{NL}. \tag{2·44}$$

From this equation, we find that X is a gauge-dependent term, because $Q^{(3)}$ determines the slice condition for the first PN order. Since an inappropriate choice of the slice condition may lead to the appearance of an unphysical mode,[11),12)] in determining the slice, we must specify some guiding principles by which we can choose an appropriate slice naturally. Here we impose two principles; 1) the Newtonian limit can be

*) We may choose $\rho_N=\rho-2U\rho_0$.

naturally taken (i.e., ϵ, $\kappa \ll 1$), and 2) in the linear stage (i.e., $\epsilon \ll \kappa$ and $\chi \ll 1$), the behavior of the geometric quantities should agree with that of the gauge invariant quantities in the linear approximation. Condition 1) is trivially satisfied in any $Q^{(3)}$. As for condition 2), we attempt to take the linear approximation of Eq. (2·44). We should consider X as appearing only in the non-linear order, so that Eq. (2·44) becomes

$$\dot{Q}^{(3)}+2\frac{\dot{a}}{a}Q^{(3)}=-4\pi\rho_0 U+\dot{Q}^{(3)}_{NL}+2\frac{\dot{a}}{a}Q^{(3)}_{NL}\,, \tag{2·45}$$

where $Q^{(3)}_{NL}$ is the non-linear part of $Q^{(3)}$. This is the equation for $Q^{(3)}$. In this case, the equation for X becomes*)

$$\Delta_{\text{flat}}X=8\pi a^2(\rho_N a^2 v^2+2(\rho_N-\rho_0)U)+4\pi a^2\delta\rho_{NL}-a^2\left(\dot{Q}^{(3)}_{NL}+2\frac{\dot{a}}{a}Q^{(3)}_{NL}\right). \tag{2·46}$$

In the case $\Lambda=0$, if $\dot{U}$ happens to be a non-linear term, we may also use another simple slice as

$$Q^{(3)}=-3HU+Q^{(3)}_{NL} \tag{2·47}$$

and

$$\Delta_{\text{flat}}X=8\pi a^2(\rho_N a^2 v^2+2(\rho_N-\rho_0)U)+a^2(3H\dot{U}-\dot{Q}^{(3)}_{NL}-2HQ^{(3)}_{NL})\,. \tag{2·48}$$

From this point, we refer to these slices as cosmological post-Newtonian (CPN) slices, because they are natural extensions of the PN slice for the case $\rho_0=0$,[13] and in the case of the linear limit, these slices reduce to Newtonian slices (zero-shear hypersurface slice).[12] In the next section, we will apply the latter CPN slice condition as well as the constant mean curvature slice $Q=Q^{(3)}=0$ to the PN version of the 1D Zel'dovich solution[16] as simple examples. The importance of the choice of $Q^{(3)}$ is shown.

From Eq. (2·24), the relation between $\tilde{A}^{(3)}_{ij}$ and $\beta_i{}^{(3)}$ becomes

$$-2\tilde{A}^{(3)}_{ij}+\beta^{(3)}_{i,j}+\beta^{(3)}_{j,i}-\frac{2}{3}\delta_{ij}\beta^{(3)}_{l,l}=0\,. \tag{2·49}$$

$\tilde{A}^{(3)}_{ij}$ must also satisfy the momentum constraint. Since $\tilde{A}^{(3)}_{ij}$ does not contain the TT part and contains only the longitudinal part, it can be written as

$$\tilde{A}^{(3)}_{ij}=W^{(3)}_{i,j}+W^{(3)}_{j,i}-\frac{2}{3}\delta_{ij}W^{(3)}_{k,k}\,, \tag{2·50}$$

where $W_i{}^{(3)}$ is a vector on the 3D hypersurface and satisfies the momentum constraint in the first PN order as

$$\Delta_{\text{flat}}W_i{}^{(3)}+\frac{1}{3}W^{(3)}_{j,j,i}-\frac{2}{3}Q^{(3)}_{,i}=8\pi\rho_N a^2 v^i\,. \tag{2·51}$$

From Eqs. (2·49) and (2·50), the relation

*) Here, $Q^{(3)}_{NL}$ should be used to guarantee that the average of the right-hand side of Eq. (2·46) becomes zero.

$$\beta_i^{(3)} = 2W_i^{(3)} \tag{2·52}$$

holds, and in the lowest PN order, Eq. (2·14) becomes

$$3(\dot{U} + HU) + Q^{(3)} = \beta^{(3)}_{l,l}\,, \tag{2·53}$$

where

$$\dot{U} = \left.\frac{\partial U}{\partial t}\right)_{x^i}.$$

Hence, Eq. (2·51) is rewritten as

$$\Delta_{\text{flat}}\beta_i^{(3)} = 16\pi\rho_N a^2 v^i + Q^{(3)}_{,i} - (HU_{,i} + \dot{U}_{,i})\,. \tag{2·54}$$

This is the equation for the vector potential in the first PN formalism.

In actual numerical simulations, the term $\dot{U}$ in Eq. (2·54) may make it difficult to maintain numerical accuracy,[15] if we simply perform the time differentiation of U. To avoid this, we need to solve the following equation to estimate $HU + \dot{U}$,

$$\Delta_{\text{flat}}(HU + \dot{U}) = 4\pi a^2(\rho_N v^i)_{,i}\,, \tag{2·55}$$

which is derived from Eqs. (2·53) and (2·54).

Now, we consider the equations of motion. In the first PN approximation, the continuity equation becomes

$$\frac{\partial D}{\partial t} + \frac{\partial (Dv^i)}{\partial x^i} = 0\,, \tag{2·56}$$

where $D = \rho_N a^3(1 + 5U + a^2v^2/2)$. If we average Eq. (2·56) in a large volume, V, we obtain $\langle D\rangle = \rho_0 a^3 = \text{constant}$, where $\langle D\rangle$ is the averaged value of D over V.*) The equations of motion become

$$\frac{\partial S_i}{\partial t} + \frac{\partial (S_i v^j)}{\partial x^j} = Da^2\left[\frac{1}{a^2}(U_{,i} - UU_{,i} - X_{,i}) + v^j\beta^{(3)}_{j,i} + \frac{3}{2}v^2 U_{,i}\right] \equiv DF_i\,, \tag{2·57}$$

where F_i denotes the total force term. For the later convenience, we split this term into Newtonian and PN parts as $F_i^N + F_i^{PN}$, which are, respectively,

$$F_i^N = U_{,i}\,,$$

$$F_i^{PN} = -UU_{,i} - X_{,i} + a^2 v^j \beta^{(3)}_{j,i} + \frac{3}{2}a^2 v^2 U_{,i}\,. \tag{2·58}$$

Equation (2·57) shows that we need to solve six Poisson equations for U, X, $HU + \dot{U}$, and $\beta_i^{(3)}$ (Eqs. (2·41), (2·46), (2·54) and (2·55)) and one ordinary differential equation (2·45) to determine the force term in the equations of motion.

Using the relation, $S_i = Du_i$, Eq. (2·57) can be rewritten as

$$\left(\frac{\partial}{\partial t} + v^j\frac{\partial}{\partial x^j}\right)u_i = F_i\,, \tag{2·59}$$

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*) In the case of a periodic system, we use the scale of the periodicity as V for averaging, and in the case of the non-periodic system, we use the horizon scale. In the former case, $\langle D\rangle$ is exactly constant. In the latter case, it is not exactly constant, but can be regarded as a constant within a sufficient accuracy.

where v^i is calculated from

$$v^i = -\beta_i^{(3)} + \left(1 - \frac{a^2 v^2}{2} - 3U\right) a^{-2} u_i \,. \tag{2·60}$$

Therefore, we can apply the formalism not only to the simulation of dust fluid (solving Eqs. (2·56), (2·57), and Poisson equations by the Eulerian method), but also to the simulation of N-body systems (solving Eq. (2·59) by the Lagrangian method and Poisson equations by the Eulerian one). We note that in the case of the simulation of the N-body system, we adopt an energy momentum tensor which is different from that in Eq. (2·30). (See Appendix B.)

Finally, we point out the following important point: Although the geometric quantities depend on a gauge variable $Q^{(3)}$, the evolution of the density perturbation and the velocity field do not. Let us demonstrate this point. Substituting the relation

$$u_i = a^2 v^i \left(1 + \frac{a^2 v^2}{2} + 3U\right) + a^2 \beta_i^{(3)} \,, \tag{2·61}$$

into Eq. (2·59), the equations of motion become

$$\left(\frac{\partial}{\partial t} + v^j \frac{\partial}{\partial x^j}\right)\left\{a^2 v^i \left(1 + \frac{a^2 v^2}{2} + 3U\right)\right\} = F_i - \left(\frac{\partial}{\partial t} + v^j \frac{\partial}{\partial x^j}\right) a^2 \beta_i^{(3)} \equiv \tilde{F}_i \,. \tag{2·62}$$

$\tilde{F}_i$ is rewritten as

$$\tilde{F}_i = U_{,i} - UU_{,i} - X_{,i} - \frac{\partial (a^2 \beta_i^{(3)})}{\partial t} + \frac{3}{2} a^2 v^2 U_{,i} \,. \tag{2·63}$$

Here, X and $\beta_i^{(3)}$ depend on $Q^{(3)}$. However, parts of the solution (which we denote as X^Q and β_i^Q) for X and $\beta_i^{(3)}$ which depend on $Q^{(3)}$ become

$$X^Q = -\Delta_{\text{flat}}^{-1}\left(\frac{\partial (a^2 Q^{(3)})}{\partial t}\right),$$
$$\beta_i^Q = \Delta_{\text{flat}}^{-1}(Q_i^{(3)}) \,. \tag{2·64}$$

Hence,

$$X_{,i}^Q + \frac{\partial (a^2 \beta_i^Q)}{\partial t} = 0 \,. \tag{2·65}$$

Thus, $\tilde{F}_i$ does not depend on $Q^{(3)}$, and neither do the density fluctuation and the velocity field. The reason is that the gauge fixing for the density fluctuation is achieved by the choice of ρ_N, not by $Q^{(3)}$. $Q^{(3)}$ only affects the evolution of geometric quantities.

§ 3. Post-Newtonian correction to the 1D Zel'dovich solution

In this section, to see the effect of the PN correction, we make use of the Zel'dovich solution in the Einstein-de Sitter universe,[16),17)] which describes the evolution of the 1D density fluctuation in a Newtonian cosmology. The Zel'dovich solution is described as

$$x=q+B(t)S_1(q)_{,q}\,,$$
$$v^x=\dot{B}(t)S_1(q)_{,q}\,,$$
$$\rho_N=\rho_0(1+B(t)S_1(q)_{,qq})^{-1}\,, \tag{3·1}$$

where $S_1(q)$ and $B(t)$ are the functions depending only on q and t, respectively. We assume that $S_1(q)_{,q}$ is a non-dimensional and monochromatic function of order unity, e.g., $\sin kq$, where k is a wave number of the Fourier spectrum of the density fluctuation in the comoving frame. (We also define S_1 to be monochromatic, e.g., as $-k^{-1}\cos kq$.) By means of the Euler equation, U is calculated as

$$U=(a^2\ddot{B}+2a\dot{a}\dot{B})\left(S_1(q)+C_1+\frac{B}{2}(S_{1,q}^2+C_2)\right), \tag{3·2}$$

where C_1 and C_2 are constants which are concerned with the transformation of the time coordinate. Hereafter, we set $C_1=C_2=0$ for simplicity. Substituting Eq. (3·2) into the cosmological Poisson equation (2·41), the equation for B becomes

$$\ddot{B}+2\frac{\dot{a}}{a}\dot{B}=4\pi\rho_0 B\,. \tag{3·3}$$

Using the relation $a=(t/t_0)^{2/3}$, where t_0 is the present time, solutions for B become

$$B(t)\propto a(t)\quad \text{and/or}\quad B(t)\propto a(t)^{-3/2}\,. \tag{3·4}$$

In the following, we only consider the growing mode and rewrite $B(t)$ as $B(t)=B_0a(t)$ $=b_0k^{-1}a(t)$, where b_0 is a non-dimensional constant which approximately determines the epoch when the first caustic ($\rho\to\infty$) is formed. Note that due to the choice $a(t_0)$ $=1$, we may regard $2\pi k^{-1}$ as the present scale of the density fluctuation. We also mention that the applicability of the Zel'dovich solution is restricted; it is applicable only before a caustic is formed, where $1+BS_{1,qq}=0$.[16),17)]

Before proceeding further, we review features of the Zel'dovich solution. Since the time scale of the growth of the density fluctuation is about H^{-1} in the Zel'dovich solution, the peculiar velocity av^x has an order of $a^2k^{-1}H\sim a^{1/2}(kt_0)^{-1}\sim a\kappa$. Thus, the Zel'dovich solution describes a system with $\epsilon\sim a\kappa$.

Next, let us constrain the parameter of the Zel'dovich solution. Since we assume that the length scale of the density fluctuation is less than the horizon scale, the relation

$$a(t)\frac{2\pi}{k}<H^{-1}\,, \tag{3·5}$$

must hold throughout the whole time. Using $a(t)=(t/t_0)^{2/3}$ and $H=2/(3t)$, Eq. (3·5) becomes

$$kt_0>\frac{4\pi}{3}\left(\frac{t_0}{t}\right)^{1/3}=\frac{4\pi}{3}a^{-1/2}\,. \tag{3·6}$$

The most strict constraint is imposed at an initial time. If, for example, we set the initial condition at a redshift $1+z\sim100$, Eq. (3·5) becomes

$$kt_0 > \frac{40\pi}{3}\left(\frac{1+z}{100}\right)^{1/2} \gg 1\,. \tag{3·7}$$

Then, the Zel'dovich solution can be applied to the density fluctuation such that

$$\epsilon_0 \sim \kappa_0 \sim \frac{2\pi}{kt_0} \lesssim 0.1\,, \tag{3·8}$$

where ϵ_0 and κ_0 denote the present values.

Now, let us calculate the PN correction. We first consider the constant mean curvature slice $Q=0$. Using the Zel'dovich solution, Eq. (2·54) for $\beta_i=(\beta_x,0,0)$ becomes*)

$$\Delta\beta_x = \frac{9}{2}B_0a^3H^3\frac{S_{1,q}}{1+B_0aS_{1,qq}}\,, \tag{3·9}$$

where we make use of

$$\left.\frac{\partial U}{\partial x}\right)_t = \frac{3}{2}B_0a^3H^2S_{1,q}\,,$$

$$\left.\frac{\partial}{\partial t}\right)_x\left.\frac{\partial}{\partial x}\right)_t U = -\frac{3}{2}B_0{}^2a^4H^3\frac{S_{1,q}S_{1,qq}}{1+B_0aS_{1,qq}}\,, \tag{3·10}$$

and

$$\left.\frac{\partial}{\partial x}\right)_t = \frac{1}{1+B_0aS_{1,qq}}\left.\frac{\partial}{\partial q}\right)_t\,,$$

$$\left.\frac{\partial}{\partial t}\right)_x = \left.\frac{\partial}{\partial t}\right)_q - \frac{B_0\dot{a}S_{1,q}}{1+B_0aS_{1,qq}}\left.\frac{\partial}{\partial q}\right)_t\,. \tag{3·11}$$

Assuming that the system has a periodicity with respect to q, i.e., $\beta_x(q+q_m)=\beta_x(q)$ and $X(q+q_m)=X(q)$, where q_m is a constant (e.g., $2\pi k^{-1}$), the equation is integrated as

$$\beta_{x,x} = \frac{9}{2}B_0a^3H^3S_1\,. \tag{3·12}$$

Equation (2·46) for X becomes

$$\Delta X = \frac{3}{2}B_0a^6H^4\left[2B_0S_{1,q}^2 - \frac{3}{2a}(1+5B_0aS_{1,qq})\left(S_1+\frac{1}{2}B_0aS_{1,q}^2\right)\right]\frac{1}{1+B_0aS_{1,qq}}\,, \tag{3·13}$$

and it is integrated as

$$X_{,x} = \frac{3}{2}B_0a^6H^4\left[-\frac{3}{2a}\int S_1dq + B_0\left(\frac{35}{4}\int S_{1,q}^2dq - \frac{15}{2}S_1S_{1,q} + C_x\right) - \frac{5}{4}B_0{}^2aS_{1,q}^3\right]\,, \tag{3·14}$$

where C_x is a constant introduced to guarantee the periodicity of X. Gathering the

*) We omit the subscript (3) of $\beta_i^{(3)}$. Also, in this section, Δ denotes the flat Laplacian.

derivatives of the potential, Eqs. (3·10), (3·12) and (3·14), and substituting them into Eq. (2·58), the total PN force term is evaluated as

$$F_x^{PN}=\frac{3}{2}B_0a^6H^4\left[\frac{3}{2a}\int S_1dq+B_0\left(-\frac{35}{4}\int S_{1,q}^2dq+9S_1S_{1,q}-C_x\right)+2B_0^2aS_{1,q}^3\right]. \tag{3·15}$$

Taking into account $S_1\sim k^{-1}$, $S_{1,qq}\sim k$ and $a^6H^4\sim t_0^{-4}$, the order of the magnitude of F_x^{PN} becomes $\sim k^{-3}t_0^{-4}$. On the other hand, that of F_x^N is $\sim k^{-1}t_0^{-2}$. So that, the PN correction force is of order $\kappa_0^2\sim(kt_0)^{-2}$ smaller than that of the Newtonian force.

However, as we conjectured in § 2, X and F_x^{PN} involve a spurious gauge mode which is proportional to a^{-1}. (The reason why it is a spurious mode is that the early evolution ($a\ll1$) of the metric quantities is described by the linear perturbation theory and that in this case, the behavior of X should agree with that of U. See Appendix A.) As we show in § 2, this unwanted behavior does not affect the behavior of the density fluctuation, so if we are only interested in the evolution of the density fluctuation, one may think that the choice of $Q^{(3)}$ is not important. However, in the numerical simulation, this is crucial. $X_{,x}$ and $F_x^{PN}\propto a^{-1}$ in the early phase, and F_x^{PN} may surpass F_x^N for $a\ll1$. In such a case, u_x^{PN} is also dominated by the spurious gauge mode. In an ideal calculation, a cancellation between the spurious modes in β_x and u_x^{PN} occurs, and the spurious mode in v_{PN}^x is excluded. However, this is not always guaranteed in the numerical simulation if a truncation error is generated. Thus, to see physical modes correctly we should not use the constant mean curvature slice in the numerical simulation.

Then, we try to perform the same calculations using a CPN slice. In the case of the Zel'dovich solution, $\dot{U}$ is the non-linear term, so we can choose as

$$Q^{(3)}=-3HU-\frac{9}{4}B_0^2a^4H^3C_b\,, \tag{3·16}$$

where C_b is a constant. In this case, the terms concerning the PN force become

$$\beta_{x,x}=-\frac{9}{4}B_0^2a^4H^3(S_{1,q}^2+C_b)\,, \tag{3·17}$$

$$X_{,x}=3B_0^2a^6H^4\left[\int\left(\frac{13}{4}S_{1,q}^2-\frac{9}{8}C_b\right)dq-3S_1S_{1,q}+C_x-B_0aS_{1,q}\left(\frac{3}{4}S_{1,q}^2+\frac{9}{8}C_b\right)\right], \tag{3·18}$$

and in total

$$F_x^{PN}=-3B_0^2a^6H^4\left[\int\left(\frac{13}{4}S_{1,q}^2-\frac{9}{8}C_b\right)dq-\frac{9}{4}S_1S_{1,q}+C_x-\frac{3}{8}B_0aS_{1,q}(S_{1,q}^2+C_b)\right], \tag{3·19}$$

where C_x is a constant. In this slice, no spurious gauge modes in X and F_x^{PN} ($\propto a^{-1}$) appear. This shows that the CPN slice is a good choice.

Then, let us consider the effects of the PN force on the evolution of the velocity field and the density fluctuation.*) First, we integrate the equation for u_x. To solve

*) Hereafter, the calculation is performed using the CPN slice.

it, we split u_x into the Newtonian and PN parts as

$$u_x = u_x{}^N + u_x{}^{PN}\,. \tag{3·20}$$

Since $u_x{}^N = a^2 v_N{}^x$, the equation for Newtonian order becomes

$$v^x_{N,t|q} + 2\frac{\dot{a}}{a} v_N{}^x = \frac{F_x{}^N}{a^2}\,. \tag{3·21}$$

Here, we know that the solution of $v_N{}^x$ is $B_0 \dot{a} S_{1,q}$. $u_x{}^{PN}$ should obey the equations of motion in the PN order, and the equation becomes

$$u^{PN}_{x,t|q} + u_x{}^{PN} v^x_{N,x} = F_x{}^{PN} + a^2\Big[\beta_x + \Big(\frac{(av_N{}^x)^2}{2} + 3U\Big) v_N{}^x\Big] v^x_{N,x}\,. \tag{3·22}$$

The left-hand side of Eq. (3·22) is rewritten as

$$\frac{1}{1+B_0 a S_{1,qq}} (u_x{}^{PN}(1+B_0 a S_{1,qq}))_{,t|q}\,. \tag{3·23}$$

Using this property, we can integrate Eq. (3·22) as follows:

$$\begin{aligned} a^{-2} u_x{}^{PN} = -2\frac{B_0{}^2}{1+B_0 a S_{1,qq}} H^3 a^4 \Big[&\int \Big(\frac{13}{4} S^2_{1,q} - \frac{9}{8} C_b\Big) dq + C_x - \frac{9}{4} S_1 S_{1,q} \\ &+ \frac{3}{5} B_0 a \Big\{ S_{1,qq} \Big(\int \Big(4 S^2_{1,q} - \frac{3}{4} C_b\Big) dq - \frac{15}{4} S_1 S_{1,q} + C_x \Big) - \frac{3}{8} S^3_{1,q} - \frac{3}{8} C_b S_{1,q} \Big\} \\ &+ \frac{3}{7} B_0{}^2 a^2 S_{1,q} S_{1,qq} \Big(-\frac{25}{24} S^2_{1,q} + \frac{3}{8} C_b \Big) \Big]\,, \end{aligned} \tag{3·24}$$

and v^x_{PN} becomes

$$\begin{aligned} v^x_{PN} = -\frac{B_0{}^2}{1+B_0 a S_{1,qq}} H^3 a^4 \Big[&\Big(2C_x + \frac{1}{4}\int (17 S^2_{1,q} - 9 C_b) dq\Big) \\ &+ \frac{1}{20} B_0 a \Big\{ S_{1,qq} \Big(\int (51 S^2_{1,q} - 63 C_b) dq + 24 C_x \Big) + 31 S^3_{1,q} - 54 C_b S_{1,q} \Big\} \\ &+ \frac{1}{28} B_0{}^2 a^2 S_{1,q} S_{1,qq} (31 S^2_{1,q} - 54 C_b) \Big]\,. \end{aligned} \tag{3·25}$$

Next, we calculate the time evolution of the density fluctuation, which is also straightforward. We write the conserved density D as

$$D = \rho_N a^3 (1+\delta_{PN})\,. \tag{3·26}$$

Then the evolution equation for δ_{PN} becomes

$$\delta_{PN,t|q} = -\Big(\frac{v^x_{PN}}{1+B_0 a S_{1,qq}}\Big)_{,q}\,. \tag{3·27}$$

Substituting Eq. (3·22) into Eq. (3·27), we obtain the evolution equation for density fluctuations generated by the PN force, but unfortunately, this cannot be calculated analytically. Although the calculation is easily done by numerical integration, we only perform an order estimate for the evolution of δ_{PN} using the linear approximation, i.e., $a \ll 1$. The order of magnitude of each mode in δ_{PN} becomes

$$\delta_{PN} \sim \frac{b_0{}^{n+1}}{(kt_0)^2} a^n \quad \text{for} \quad n \geq 1 . \tag{3·28}$$

Here, we note that there is no spurious mode in δ_{PN}. This is a natural consequence of the choice of $\rho_N = \rho(1-2U)$. The ratio of the Newtonian and PN modes of the density fluctuation can be also written as

$$r_\delta \sim \frac{b_0{}^n}{(kt_0)^2} a^{n-1} \quad \text{for} \quad n \geq 1 . \tag{3·29}$$

Hence, the PN effect is always a factor of $(kt_0)^{-2}$ smaller than the Newtonian effect. This means that for the small-scale density fluctuation ($kt_0 \gg 1$), the PN correction can be neglected. Also, the effect of the PN term is very small in the early stage of the evolution ($a \ll 1$). However, we cannot conclude that the PN terms are always negligible. In the last stage ($z \lesssim 1$) of the evolution of density fluctuations on a very large scale, the PN correction may become important. To explain this point, instead of r_δ, we use the ratio between the Newtonian velocity and the PN velocity, because we do not know the detailed behavior of r_δ in the last stage. In the late stage of the evolution, the order of the magnitude of r_δ for each mode is written as

$$r_v \sim \frac{b_0{}^n}{(kt_0)^2(1+B_0 a S_{1,qq})} a^{n-1} \sim \frac{b_0{}^n}{(kt_0)^2} \frac{\rho}{\rho_0} a^{n-1} \quad \text{for} \quad 1 \leq n \leq 3 . \tag{3·30}$$

Recall that b_0 determines the epoch when the first caustic ($\rho \to \infty$) is formed at a place $q = q_1$. If the epoch is a redshift of z_f, b_0 becomes $1+z_f$. In the realistic evolution of the density fluctuation, the density does not diverge, and instead, some structure will be formed around $q \sim q_1$. Also, in the other coordinate points, the density fluctuation will continue to grow. Let us consider such a point where $|\rho - \rho_0|/\rho_0$ becomes ≥ 1 at $z=0$. Assuming that the Zel'dovich solution can be a good approximation for $q \neq q_1$ and $z < z_f$, r_v for such points can be approximately written as

$$r_v \sim \frac{1}{(kt_0)^2} \frac{(1+z_f)^n}{(1+z)^{n-1}} \frac{\rho}{\rho_0} \quad \text{for} \quad 1 \leq n \leq 3 . \tag{3·31}$$

If $z_f \sim 5$ and $\rho/\rho_0 \sim 10$ with $n=3$, $r_v \sim \text{several} \times 10^3 (kt_0)^{-2}$ at $z=0$. Thus, in the late stage, the PN correction may contribute to the evolution of the very large-scale structure with the scale $kt_0 \lesssim 100$ ($2\pi k^{-1} \gtrsim 200 h^{-1}$Mpc).

§ 4. Summary

In this paper, we have investigated the PN equations of motion in the flat universe and their effects on the evolution of the large-scale structure of the universe. In the first part of this paper, we formulate the cosmological PN equations of motion making use of the (3+1)formalism in general relativity. To derive equations of motion which are useful for an actual analysis, there exist two important points, one being the choice of the density in the Newtonian order (ρ_N), and the other the choice of the gauge condition. As for ρ_N, we choose $\rho_N = \rho(1-2U)$ so that $\rho_N - \rho_0$ denotes a gauge invariant density in the early stage of the universe ($a \ll 1$). As for the gauge condition, we choose the pseudo TT gauge and CPN slice condition to eliminate the spurious gauge modes. In this formalism, the force term in the equations of motion is calculated from the six potentials which are derived by the six Poisson equations as is in the Newtonian case (although there is only one potential U in the Newtonian case). Therefore, these can be used for an actual numerical simulation if we use a standard numerical technique such as the particle-mesh method.[9]

In the second part of this paper, using the derived formula, we analyzed the PN correction of the Zel'dovich solution. In the analysis, we use not only the CPN slice, but also the constant mean curvature slice ($K=-3H$) to show the importance of the choice of the slice condition. It is found that in the case of the constant mean curvature slice, the spurious evolution mode for the metric appears, but for the CPN slice, it does not. This shows that we should carefully choose the slice condition to see the physical phenomena in the PN equations of motion correctly, and that for that purpose, the CPN slice is useful. We also perform an order estimate of the effect of the PN correction to the evolution of the density fluctuation. We find that 1) the PN correction is not important when $a \ll 1$ and/or when the scale of the density fluctuation is not too large ($\kappa \ll 1$), but 2) for the very large-scale structure of the universe (>100 h^{-1}Mpc), the PN correction may become important in the late stage of the evolution (i.e., in the highly non-linear regime of the density fluctuation). However, estimates we perform in this paper are crude, and to see the quantitative effect of the PN correction, a numerical simulation is required.

Although we do not have significant observational results as for the very large scale $\gtrsim 100$ h^{-1}Mpc,[3] there are observations which suggest that non-linear density fluctuations of very large scale do exist.[4),5)] Moreover, substantial progress is expected in the area of the observation of the large-scale structure because several projects such as the SDSS project[6] will be in operation, and the results will be brought out in this decade. Such observations may find very large-scale structure of the universe. Preparing for the time when we could confirm the existence of such very large-scale structure, we should investigate the PN correction quantitatively and clarify whether it is important or not. To answer these questions, we need to perform detailed numerical calculations of the evolution of the very large-scale structure including PN correction.

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Appendix A

In this appendix, we consider the linear approximation theory in the Einstein-de

Sitter universe (i.e., $\Lambda=0$) to determine the behavior of the metric quantities and the matter variables in the early stage of the universe. In the linear theory, the equations under the pseudo TT gauge condition become

$$\dot{\theta}+2H\theta+\dot{B}+2HB=a^{-2}\Delta A\,, \tag{A·1}$$

$$\dot{\delta}+6\dot{\varphi}+\theta=0\,, \tag{A·2}$$

$$6\dot{\varphi}+3HA+Q-B=0\,, \tag{A·3}$$

$$\Delta A=-4\pi\rho_0 a^2(\delta-3A)+a^2(\dot{Q}+2HQ)\,, \tag{A·4}$$

where

$$A=1-\alpha\,, \tag{A·5}$$

$$\delta=\frac{\rho}{\rho_0}-1\,, \tag{A·6}$$

$$\theta=v^i{}_{,i}\,, \tag{A·7}$$

$$B=\beta^i{}_{,i}\,, \tag{A·8}$$

$$\varphi=\psi-1\,. \tag{A·9}$$

From Eqs. (A·1)~(A·4), the equation for δ can be derived as

$$\ddot{\delta}+2H\dot{\delta}-\frac{3}{2}H^2\delta=-3H^2A+3H\dot{A}\,. \tag{A·10}$$

By using $\delta_m=\delta-3HC$, where $\dot{C}=A$ or $\Delta C=a^2(\theta+B)$, the equation is rewritten as

$$\ddot{\delta}_m+2H\dot{\delta}_m-\frac{3}{2}H^2\delta_m=0\,. \tag{A·11}$$

Here, δ_m denotes a gauge invariant quantity,*),12) and from this equation, we obtain the two evolution modes of the density fluctuation $\delta_m\propto a$ and $\propto a^{-3/2}$.

Now, let us suppose that we must solve the equations without the gauge invariant quantity. Then, we must solve the coupled equations (A·4) and (A·10). Since the following discussion does not depend on Q, for simplicity, we choose the CPN slice as

$$Q=-\frac{9}{2}H^2C\,. \tag{A·12}$$

In this case, Eq. (A·4) becomes a simple equation as

$$\Delta A=-4\pi\rho_0 a^2(\delta-3HC)(=-4\pi\rho_0 a^2\delta_m)\,. \tag{A·13}$$

If we use the gauge invariant quantity δ_m and consider only the growing mode $\propto a$, we immediately find that the solution for A becomes $\propto a^0$. Then, since $3HC\simeq 2A$

*) In the paper of Bardeen, this is expressed as ϵ_m. There also exists another gauge invariant density fluctuation ϵ_g, and we may use it or a linear combination of ϵ_m and ϵ_g instead of ϵ_m. In this paper, we use ϵ_m because its growing mode has a simpler form than that of ϵ_g. Thus, ϵ_m makes it easier for us to extract the growing mode, especially in numerical computations.

=constant, δ has the spurious mode $\sim 2A$. This means that if we use δ instead of δ_m, the undesirable mode will appear. Therefore, in order to see the physical density fluctuation, we must choose δ_m as a density fluctuation instead of δ (i.e., we had better choose $\rho_N=\rho(1-2U)$ as a density in the Newtonian order).

Finally, we note that the above argument is independent of the choice of Q; unless we adopt a gauge invariant quantity as the density fluctuation, the spurious mode necessarily appears irrespective of the choice of Q. Therefore, in order to delete the spurious gauge mode, the gauge invariant choice of the density fluctuation is only one method.

Appendix B

We discuss the energy momentum tensor for N-body systems in this appendix. In the case of the N-body system, the energy momentum tensor is not defined as Eq. (2·30), but as

$$T^{\mu\nu}=\sum_p \frac{m_p}{\alpha\psi^6 a^3 dt/ds}\frac{dz_p{}^\mu}{ds}\frac{dz_p{}^\nu}{ds}\delta^{(3)}(x^j-z_p{}^j(t))\equiv\sum_p\rho_p\frac{dz_p{}^\mu}{ds}\frac{dz_p{}^\nu}{ds}, \tag{B·1}$$

where m_p and $z_p{}^\mu(t)$ denote an inertial mass and a trajectory of a particle, respectively, and ρ_p denotes the mass density around a particle. Here, ρ and $\rho_d\equiv Da^{-3}$ become, respectively,

$$\rho=\sum_p m_p\psi^{-6}a^{-3}\left(a\frac{dt}{ds}\right)^{-1}\delta^{(3)}(x^j-z_p{}^j(t))\left(=\sum_p\rho_p\right),$$

$$\rho_d=\sum_p m_p a^{-3}\delta^{(3)}(x^j-z_p{}^j(t)). \tag{B·2}$$

In the PN approximation, ρ and ρ_N are expressed as

$$\rho=\sum_p m_p a^{-3}\left(1-\frac{a^2v^2}{\cdot 2}-3U\right)\delta^{(3)}(x^j-z_p{}^j(t)) \tag{B·3}$$

and

$$\rho_N=\sum_p m_p a^{-3}\left(1-\frac{a^2v^2}{2}-5U\right)\delta^{(3)}(x^j-z_p{}^j(t)). \tag{B·4}$$

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