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POSTNIKOV PIECES AND $B\mathbb{Z}/p$-HOMOTOPY THEORY

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ABSTRACT. We present a constructive method to compute the cellularization with respect to $B^m\mathbb{Z}/p$ for any integer $m \geq 1$ of a large class of H-spaces, namely all those which have a finite number of non-trivial $B^m\mathbb{Z}/p$-homotopy groups (the pointed mapping space $\mathrm{map}_*(B^m\mathbb{Z}/p, X)$ is a Postnikov piece). We prove in particular that the $B^m\mathbb{Z}/p$-cellularization of an H-space having a finite number of $B^m\mathbb{Z}/p$-homotopy groups is a p-torsion Postnikov piece. Along the way, we characterize the $B\mathbb{Z}/p^r$-cellular classifying spaces of nilpotent groups.

INTRODUCTION

The notion of A-homotopy theory was introduced by Dror Farjoun [9] for an arbitrary connected space A. Here A and its suspensions play the role of the spheres in classical homotopy theory, and so the A-homotopy groups of a space X are defined to be the homotopy classes of pointed maps $[\Sigma^i A, X]$. The analogue to weakly contractible spaces are those spaces for which all A-homotopy groups are trivial. This means that the pointed mapping space $\mathrm{map}_*(A, X)$ is contractible, i.e. X is an A-null space. On the other hand, the classical notion of CW-complex is replaced by the one of A-cellular space. Such spaces can be constructed from A by means of pointed homotopy colimits.

Thanks to work of Bousfield [2] and Dror Farjoun [9] there is a functorial way to study X through the eyes of A. The nullification $P_A X$ is the biggest quotient of X which is A-null, and $CW_A X$ is the best A-cellular approximation of the space X. Roughly speaking, $CW_A X$ contains all the transcendent information of the mapping space $\mathrm{map}_*(A, X)$, since the latter is equivalent to $\mathrm{map}_*(A, CW_A X)$. Hence, explicit computation of the cellularization would give access to information about $\mathrm{map}_*(A, X)$. The importance of mapping spaces (in the case $A = B\mathbb{Z}/p$) is well established thank to Miller's solution to the Sullivan conjecture [17] and later work.

While many computations of $P_A X$ are present in the literature, very few computations of $CW_A X$ are available. For instance, Chachólski describes a strategy to compute the cellularization $CW_A X$ in [7]. His method has been successfully

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applied in some cases (cellularization with respect to Moore spaces [21], $B\mathbb{Z}/p$-cellularization of classifying spaces of finite groups [10]), but it is in general difficult to apply.

An alternative way to compute $CW_A X$ is the following. The nullification map $l : X \to P_A X$ provides an equivalence $CW_A X \simeq CW_A \overline{P}_A X$, where, as usual, $\overline{P}_A X$ denotes the homotopy fiber of l. This equivalence gives a strategy when $\overline{P}_A X$ is known. Assume for example that X is A-null. Then $\overline{P}_A X$ is contractible, and thus, so is $CW_A X$. From the A-homotopy point of view, the next case in which the A-cellularization should be accessible is when X has only a finite number of A-homotopy groups, that is, some iterated loop space $\Omega^n X$ is A-null. Natural examples of spaces satisfying this condition are the n-connected covers of A-null spaces.

Let us specialize to H-spaces and $A = B^m\mathbb{Z}/p$. Bousfield has determined in [2] the fiber of the nullification map $X \to P_{B^m\mathbb{Z}/p} X$ when $\Omega^n X$ is $B^m\mathbb{Z}/p$-null. He shows that, for such an H-space, $\overline{P}_{B^m\mathbb{Z}/p} X$ is a p-torsion Postnikov piece F, whose homotopy groups are concentrated in dimensions from m to $m+n-1$. As F is also an H-space (because l is an H-map), we call it an *H-Postnikov piece*. The cellularization of X (which is again an H-space because CW_A preserves H-structures) therefore coincides with that of a Postnikov piece. In Section 3, we explain how to compute the cellularization of Postnikov pieces, and this enables us to obtain our main result.

Theorem 5.3. *Let X be a connected H-space such that $\Omega^n X$ is $B^m\mathbb{Z}/p$-null. Then*

$$CW_{B^m\mathbb{Z}/p} X \simeq F \times K(W, m),$$

where F is a p-torsion H-Postnikov piece with homotopy groups concentrated in dimensions from $m+1$ to $m+n-1$ and W is an elementary abelian p-group.

Thus, when X is an H-space with only a finite number of $B^m\mathbb{Z}/p$-homotopy groups, the cellularization $CW_{B^m\mathbb{Z}/p} X$ is a p-torsion H-Postnikov piece. This is not true in general if we do not assume X to be an H-space. For instance, the $B\mathbb{Z}/p$-cellularization of $B\Sigma_3$ is a space with infinitely many non-trivial homotopy groups [11]. Also, it is not true for an arbitrary space A that the A-cellularization of an H-space having a finite number of A-homotopy groups is always a Postnikov piece. This fails, for example, when A is the product of the $K(\mathbb{Z}/p, p)$'s, where p runs over the set of all primes, but it could be true for any n-supported p-torsion space A (in the terminology of [2]).

In our previous work [6], we analyzed a large class of H-spaces which fits into the present framework. Namely, if the mod p cohomology of an H-space X is finitely generated as an algebra over the Steenrod algebra, then there must exist an integer n such that $\Omega^n X$ is $B\mathbb{Z}/p$-null. Hence, we obtain the following.

Proposition 4.2. *Let X be a connected H-space such that $H^*(X; \mathbb{F}_p)$ is finitely generated as an algebra over the Steenrod algebra. Then*

$$CW_{B\mathbb{Z}/p} X \simeq F \times K(W, 1),$$

where F is a 1-connected p-torsion H-Postnikov piece and W is an elementary abelian p-group. Moreover, there exists an integer k such that $CW_{B^m\mathbb{Z}/p} X \simeq *$ for any $m \geq k$.

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Our results allow explicit computations which we exemplify by computing in Proposition 4.3 the $B\mathbb{Z}/p$-cellularization of the n-connected cover of any finite H-space, as well as the $B^m\mathbb{Z}/p$-cellularizations of the classifying spaces for real and complex vector bundles BU, BO, and their connected covers BSU, BSO, $BSpin$, and $BString$; see Proposition 5.6.

1. A double filtration of the category of spaces

As mentioned in the Introduction, the condition that $\Omega^n X$ be $B^m\mathbb{Z}/p$-null will enable us to compute the $B^m\mathbb{Z}/p$-cellularization of H-spaces. This section is devoted to giving a picture of how such spaces are related for different choices of m and n.

First of all, we present a lemma which collects various facts that are needed in the rest of the paper.

Lemma 1.1. *Let X be a connected space and $m > 0$. Then,*

(1) *If X is $B^m\mathbb{Z}/p$-null, then $\Omega^n X$ is $B^m\mathbb{Z}/p$-null for all $n \geq 1$.*
(2) *If X is $B^m\mathbb{Z}/p$-null, then it is $B^{m+s}\mathbb{Z}/p$-null for all $s \geq 0$.*
(3) *If ΩX is $B^m\mathbb{Z}/p$-null, then X is $B^{m+s}\mathbb{Z}/p$-null for all $s \geq 1$.*

Proof. For (1), simply apply $\mathrm{map}_*(B\mathbb{Z}/p, -)$ to the path fibration $\Omega X \to * \to X$.

Statement (2) is given by Dwyer's version of Zabrodsky's lemma [8, Prop. 3.4] applied to the universal fibration $B^m\mathbb{Z}/p \to * \to B^{m+1}\mathbb{Z}/p$.

Finally, (3) is proven like (2), using Zabrodsky's lemma in its connected version [8, Prop. 3.5] (see also Lemma 2.3). Recall that if ΩX is $B^m\mathbb{Z}/p$-null, then the component $\mathrm{map}(B^m\mathbb{Z}/p, X)_c$ of the constant map is weakly equivalent to X. $\square$

Of course, the converses of the previous results are not true. For the first statement, take the classifying space of a discrete group at $m = 1$. For the second and third, consider $X = BU$. It is a $B^2\mathbb{Z}/p$-null space (see Example 1.4), but neither BU nor ΩBU are $B\mathbb{Z}/p$-null. Observe that in fact $\Omega^n BU$ is never $B\mathbb{Z}/p$-null. The next result shows that this is the general situation. That is, if a connected space X is $B^{m+1}\mathbb{Z}/p$-null, then either ΩX is $B^m\mathbb{Z}/p$-null or none of the iterated loop spaces $\Omega^n X$ is $B^m\mathbb{Z}/p$-null for $n \geq 1$.

Theorem 1.2. *Let X be a $B^{m+1}\mathbb{Z}/p$-null space such that $\Omega^k X$ is $B^m\mathbb{Z}/p$-null for some $k > 0$. Then ΩX is $B^m\mathbb{Z}/p$-null.*

Proof. It is enough to prove the result for $k = 2$. Consider the fibration

$$K(Q, m+1) \to P_{\Sigma^2 B^m\mathbb{Z}/p} X \simeq X \to P_{\Sigma B^m\mathbb{Z}/p} X,$$

where the fiber is a p-torsion Eilenberg-Mac Lane space by Bousfield's description of the fiber of the $\Sigma B^m\mathbb{Z}/p$-nullification [2, Theorem 7.2]. The base space is $B^{m+1}\mathbb{Z}/p$-null by Lemma 1.1(3) and so is the total space, by assumption. Thus, the pointed mapping space $\mathrm{map}_*\big(B^{m+1}\mathbb{Z}/p, K(Q, m+1)\big)$ must be contractible as well, i.e. $Q = 0$. $\square$

The previous analysis leads to a double filtration of the category of spaces. Let $n \geq 0$ and $m \geq 1$. We introduce the notation

$$\mathcal{S}_m^n = \{X \mid \Omega^n X \text{ is } B^m\mathbb{Z}/p\text{-null}\}.$$

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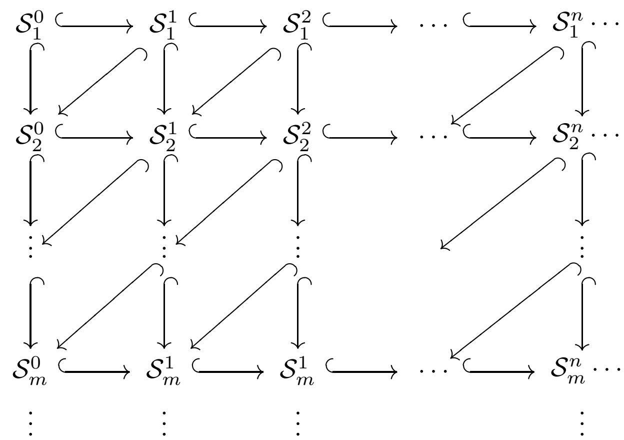
then Lemma 1.1 yields a diagram of inclusions:



Example 1.3. We give examples of spaces in every stage of the filtration.

(1) $\mathcal{S}_1^0$ are the spaces that are $B\mathbb{Z}/p$-null. This contains in particular any finite space (by Miller's theorem [17, Thm. A]), and, for a nilpotent space X (of finite type with finite fundamental group), to be $B\mathbb{Z}/p$-null is equivalent to its cohomology $H^*(X;\mathbb{F}_p)$ being locally finite by [22, Corollary 8.6.2].
(2) If $X\langle n\rangle$ denotes the n-connected cover of a space X, then the homotopy fiber of $\Omega^{n-1}X\langle n\rangle \to \Omega^{n-1}X$ is a discrete space. Hence, if $X \in \mathcal{S}_m^0$, then $X\langle n\rangle \in \mathcal{S}_m^{n-1}$.
(3) Observe that $\mathcal{S}_m^n \subset \mathcal{S}_{m+k}^{n-k}$ for all $0 \leq k \leq n$.
(4) The previous examples provide spaces in every stage of the double filtration. Consider a finite space. It is automatically $B\mathbb{Z}/p$-null and its n-connected cover $X\langle n\rangle$ lies in $\mathcal{S}_1^{n-1}$, hence also in $\mathcal{S}_{k+1}^{n-k-1}$ for all $0 \leq k \leq n$.

The next example provides a number of spaces living in $\mathcal{S}_m^0$ which do not come from the first row of the filtration. Of course their connected covers will be *new* examples of spaces living in $\mathcal{S}_m^n$.

Example 1.4. Let E_* be a homology theory. If $\tilde{E}^i\big(K(\mathbb{Z}/p,m)\big) = 0$ for all i, then the spaces E^i representing the corresponding homology theory are $B^m\mathbb{Z}/p$-null. If $\tilde{E}^j\big(K(\mathbb{Z}/p,m-1)\big) \neq 0$ for some j, then E^j is not $B^{m-1}\mathbb{Z}/p$-null. In particular, if E_* is periodic, it follows that the spaces E^i are $B^m\mathbb{Z}/p$-null for all i, but none of their iterated loops are $B^{m-1}\mathbb{Z}/p$-null.

A first example of such behavior is obtained from complex K-theory: BU is $B^2\mathbb{Z}/p$-null, but BU and U are not $B\mathbb{Z}/p$-null (see [18]). Note that real and quaternionic K-theory enjoy the same properties.

For every m, examples of homology theories following this pattern are given by p-torsion homology theories of type III-m as described in [1]. The mth Morava K-theory $K(m)_*$ for p odd is an example of such behavior with respect to Eilenberg-Mac Lane spaces. The spaces representing $K(m)_*$ are $B^{m+1}\mathbb{Z}/p$-null, but none of their iterated loops are $B^m\mathbb{Z}/p$-null.

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Our aim is to provide tools to compute the $B^m\mathbb{Z}/p$-cellularization of any H-space lying in the mth row of the above diagram. The key point is the following result of Bousfield [2], which determines the fiber of the nullification map.

Proposition 1.5. *Let $n \geq 0$ and let X be a connected H-space such that $\Omega^n X$ is $B^m\mathbb{Z}/p$-null. Then there is an H-fibration*

$$F \to X \to P_{B^m\mathbb{Z}/p}X,$$

where F is a p-torsion H-Postnikov piece whose homotopy groups are concentrated in dimensions from m to $m+n-1$. □

Therefore, since $F \to X$ is a $B^m\mathbb{Z}/p$-cellular equivalence, we only need to compute the cellularization of a Postnikov piece (which will end up being a Postnikov piece again; see Theorem 3.6). Actually, even more is true.

Proposition 1.6. *Let X be a connected space such that $CW_{B^m\mathbb{Z}/p}X$ is a Postnikov piece. Then there exists an integer n such that $\Omega^n X$ is $B^m\mathbb{Z}/p$-null.*

Proof. Let us loop once the Chachólski fibration $CW_{B^m\mathbb{Z}/p}X \to X \to P_{\Sigma B^m\mathbb{Z}/p}C$ (see [7, Theorem 20.5]). Since $\Omega P_{\Sigma B^m\mathbb{Z}/p}C$ is equivalent to $P_{B^m\mathbb{Z}/p}\Omega C$ by [9, Theorem 3.A.1], we get a fibration over a $B^m\mathbb{Z}/p$-null base space

$$\Omega CW_{B^m\mathbb{Z}/p}X \to \Omega X \to P_{B^m\mathbb{Z}/p}\Omega C.$$

Now there exists an integer n such that $\Omega^n CW_{B^m\mathbb{Z}/p}X$ is discrete, thus $B^m\mathbb{Z}/p$-null. Therefore, so is $\Omega^n X$. □

2. Cellularization of fibrations over BG

In general, it is very difficult to compute the cellularization of the total space of a fibration. In this section, we explain how to deal with this problem when the base space is the classifying space of a discrete group. The first step applies to any group. In the second step (see Proposition 2.4 below) we specialize to nilpotent groups.

Proposition 2.1. *Let $r \geq 1$ and let $F \to E \xrightarrow{\pi} BG$ be a fibration, where G is a discrete group. Let S be the (normal) subgroup generated by all elements $g \in G$ of order p^i for some $i \leq r$ such that the inclusion $B\langle g\rangle \to BG$ lifts to E. Then the pullback of the fibration along $BS \to BG$*

$$\begin{array}{ccccc} E' & \xrightarrow{f} & E & \xrightarrow{p} & B(G/S) \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \| \\ BS & \longrightarrow & BG & \xrightarrow{p'} & B(G/S) \end{array}$$

induces a $B\mathbb{Z}/p^r$-cellular equivalence $f : E' \to E$ on the total space level.

Proof. We have to show that f induces a homotopy equivalence on pointed mapping spaces $\operatorname{map}_*(B\mathbb{Z}/p^r, -)$. The top fibration in the diagram yields a fibration

$$\operatorname{map}_*(B\mathbb{Z}/p^r, E') \xrightarrow{f_*} \operatorname{map}_*(B\mathbb{Z}/p^r, E) \xrightarrow{p_*} \operatorname{map}_*\big(B\mathbb{Z}/p^r, B(G/S)\big).$$

Since the base is homotopically discrete, we only need to check that all components of the total space are sent by p_* to the component of the constant. Thus consider a map $h : B\mathbb{Z}/p^r \to E$. The composite $p \circ h$ is homotopy equivalent to a map

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induced by a group homomorphism $\alpha\colon \mathbb{Z}/p^r \to G$ whose image $\alpha(1) = g$ is in S by construction. Therefore $p \circ h = p' \circ \pi \circ h$ is null-homotopic. □

Remark 2.2. If the fibration in the above proposition is an H-fibration (in particular if G is abelian), the set of elements g for which there is a lift to the total space forms a subgroup of G. The central extension $Z(D_8) \hookrightarrow D_8 \to \mathbb{Z}/2 \times \mathbb{Z}/2$ of the dihedral group D_8 provides an example where the subgroup S is $\mathbb{Z}/2 \times \mathbb{Z}/2$, but the element in S represented by an element of order 4 in D_8 does not admit a lift.

The next lemma is a variation of Dwyer's version of Zabrodsky's Lemma in [8].

Lemma 2.3. *Let $F \to E \xrightarrow{f} B$ be a fibration over a connected base, and let A be a connected space such that ΩA is F-null. Then any map $g : E \to A$ which is homotopic to the constant map when restricted to the fiber F factors through a map $h : B \to A$ up to unpointed homotopy and, moreover, g is pointed null-homotopic if and only if h is so.*

Proof. Since ΩA is F-null, we see that the component $\mathrm{map}_*(F, A)_c$ of the constant map is contractible, and therefore, the evaluation at the base point $\mathrm{map}(F, A)_c \to A$ is an equivalence. By [8, Proposition 3.5], f induces a homotopy equivalence

$$\mathrm{map}(B, A) \simeq \mathrm{map}(E, A)_{[F]},$$

where $\mathrm{map}(E, A)_{[F]}$ denotes the space of maps $E \to A$ which are homotopic to the constant map when restricted to F.

Looking at the component of the constant map, we see that $\mathrm{map}(B, A)_c \simeq \mathrm{map}(E, A)_c$. Since any map homotopic to the constant map is also homotopic by a pointed homotopy, the result follows. □

Proposition 2.4. *Let $r \geq 1$ and let $F \xrightarrow{i} E \xrightarrow{\pi} BG$ be a fibration, where G is a nilpotent group generated by elements of order p^i with $i \leq r$. Assume that for each of these generators $x \in G$, the inclusion $B\langle x\rangle \to BG$ lifts to E. If F is $B\mathbb{Z}/p^r$-cellular, then so is E.*

Proof. In [7], Chachólski describes the cellularization $CW_{B\mathbb{Z}/p^r}E$ as the homotopy fiber of the composite

$$f : E \to C \to P_{\Sigma B\mathbb{Z}/p^r}C,$$

where C is the homotopy cofiber of the evaluation map $\bigvee_{[B\mathbb{Z}/p^r, E]} B\mathbb{Z}/p^r \to E$. This tells us that E is cellular if the map f is null-homotopic. Observe that if f is null-homotopic, then the fiber inclusion $CW_{B\mathbb{Z}/p^r}E \to E$ has a section and, therefore, E is cellular, since it is a retract of a cellular space ([9, 2.D.1.5]).

As the existence of an unpointed homotopy to the constant map implies the existence of a pointed one, we now work in the category of unpointed spaces. We remark that for any map $g : Z \to E$ from a $B\mathbb{Z}/p^r$-cellular space Z, the composite $f \circ g$ is null-homotopic, since g factors through the cellularization of E. In particular, the composite $f \circ i$ is null-homotopic. By Lemma 2.3, f factors through a map $\bar{f} : BG \to P_{\Sigma B\mathbb{Z}/p^r}C$ such that $\bar{f} \circ \pi \simeq f$ and, moreover, f is null-homotopic if and only if $\bar{f}$ is so.

We first assume that G is a finite group and show by induction on the order of G that $\bar{f}$ is null-homotopic. If $|G| = p$, the existence of a section $s : BG \to E$ implies that $f \circ s = \bar{f}$ is null-homotopic since $BG = B\mathbb{Z}/p$ is cellular.

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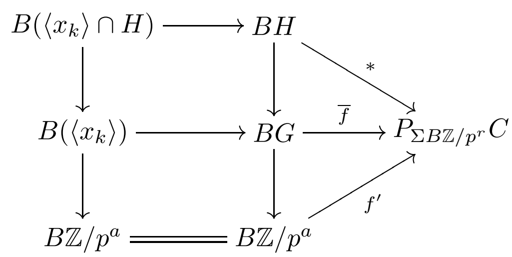
Let $\{x_1, \ldots, x_k\}$ be a minimal set of generators which admit a lift. Let $H \trianglelefteq G$ be the normal subgroup generated by $x_1, \ldots, x_{k-1}$ and their conjugates by powers of x_k. There is a short exact sequence

$$H \to G \to \mathbb{Z}/p^a,$$

where the quotient group is generated by the image of the generator x_k. Consider the fibration $F \to E' \to BH$ obtained by pulling back along $BH \to BG$, and denote by $h : E' \to E$ the induced map between the total spaces. Since H satisfies the assumptions of the proposition, the induction hypothesis tells us that E' is cellular and therefore, $f \circ h$ is null-homotopic. This implies that the restriction of $\bar{f}$ to BH is null-homotopic. Consider the following diagram:



By Lemma 2.3, it is enough to show that f' is null-homotopic. Again, applying Lemma 2.3 to the fibration on the left shows that f' is null-homotopic since $\bar{f}$ restricted to $\langle x_k \rangle$ is so. Therefore, $\bar{f}$ is null-homotopic.

Assume now that G is not finite. Any subgroup of G generated by a finite number of elements of order a power of p has a finite abelianization, and must therefore be itself finite by [20, Theorem 2.26]. Thus, G is locally finite, i.e. G is a filtered colimit of finite nilpotent groups generated by elements of order p^i for $i \leq r$. Likewise, BG is a filtered homotopy colimit of classifying spaces of finite groups (generated by finite subsets of the set of generators) which satisfy the hypotheses of the proposition. The total space E can be obtained as a pointed filtered colimit of the total spaces obtained by pulling back the fibration. By the case when G is finite, these total spaces are all cellular and therefore, so is E. □

Sometimes the existence of the "local" sections defined for every generator permits the construction of a global section of the fibration. By a result of Chachólski [7, Theorem 4.7], the total space of such a split fibration is cellular since F and BG are so. This is the case for an H-fibration, and E is then weakly equivalent to the product $F \times BG$.

A straightforward consequence of the above proposition (in the case when the fibration is the identity on BG) is the following characterization of the $B\mathbb{Z}/p^r$-cellular classifying spaces. For $r = 1$, we obtain R. Flores' result [10, Theorem 4.14].

Corollary 2.5. *Let $r \geq 1$ and let G be a nilpotent group generated by elements of order p^i with $i \leq r$. Then BG is $B\mathbb{Z}/p^r$-cellular.* □

Example 2.6. The quaternion group Q_8 of order 8 is generated by elements of order 4. Therefore, BQ_8 is $B\mathbb{Z}/4$-cellular. We do not know an explicit way to construct BQ_8 as a pointed homotopy colimit of a diagram whose values are copies of $B\mathbb{Z}/4$.

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We can now state the main result of this section. It provides a constructive description of the cellularization of the total space of certain fibrations over classifying spaces of nilpotent groups.

Theorem 2.7. *Let G be a nilpotent group and let $F \to E \to BG$ be a fibration with $B\mathbb{Z}/p^r$-cellular fiber F. Then the cellularization of E is the total space of a fibration $F \to CW_{B\mathbb{Z}/p^r}E \to BS$, where $S \lhd G$ is the (normal) subgroup generated by the p-torsion elements g of order p^i with $i \leq r$, such that the inclusion $B\langle g\rangle \to BG$ lifts to E.*

Proof. By Proposition 2.1, pulling back along $BS \to BG$ yields a cellular equivalence f in the following square:

$$\begin{array}{ccc} E_S & \xrightarrow{f} & E \\ \downarrow & & \downarrow \\ BS & \longrightarrow & BG. \end{array}$$

By Proposition 2.4, the total space E_S is cellular and therefore $E_S \simeq CW_{B\mathbb{Z}/p^r}E$. □

Corollary 2.8. *Let G be a nilpotent group and let $S \lhd G$ be the (normal) subgroup generated by the p-torsion elements g of order p^i with $i \leq r$. Then $CW_{B\mathbb{Z}/p^r}BG \simeq BS$. Moreover, when G is finitely generated, S is a finite p-group.*

Proof. We only need to show that S is a finite p-group. Note that the abelianization of S is p-torsion. Thus, S is also a torsion group (see [23, Cor. 3.13]). Moreover, since G is finitely generated, S is finite, by [23, 3.10]. □

In fact, Theorem 2.7 also holds when the base space is an Eilenberg-Mac Lane space $K(G,n)$.

Proposition 2.9. *Let n be an integer ≥ 2 and let G be a finitely generated abelian group of exponent dividing p^r. Consider a fibration $F \xrightarrow{i} E \xrightarrow{\pi} K(G,n)$ such that, for each generator $x \in G$, the inclusion $K(\langle x\rangle, n) \to K(G,n)$ lifts to E. If F is $B\mathbb{Z}/p^r$-cellular, then so is E.* □

3. Cellularization of nilpotent Postnikov pieces

In this section, we compute the cellularization with respect to $B\mathbb{Z}/p^r$ of nilpotent Postnikov pieces. The main difficulty lies in the fundamental group, so it will be no surprise that these results hold as well for cellularization with respect to $B^m\mathbb{Z}/p^r$ with $m \geq 2$. We will often use the following closure property [9, Theorem 2.D.11].

Proposition 3.1. *Let $F \to E \to B$ be a fibration where F and E are A-cellular. Then so is B.* □

Example 3.2 ([9, Corollary 3.C.10])**.** The Eilenberg-Mac Lane space $K(\mathbb{Z}/p^k, n)$ is $B\mathbb{Z}/p^r$-cellular for any integer k and any $n \geq 2$.

The construction of the cellularization is performed by looking first at the universal cover of the Postnikov piece. We start with the basic building blocks, the Eilenberg-Mac Lane spaces. For the structure results on infinite abelian groups, we refer the reader to Fuchs' book [12].

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Lemma 3.3. *An Eilenberg-Mac Lane space $K(A, m)$, with $m \geq 2$, is $B\mathbb{Z}/p^r$-cellular if and only if A is a p-torsion abelian group.*

Proof. It is clear that A must be p-torsion. Thus, assume that A is a p-torsion group. If A is bounded, it is isomorphic to a direct sum of cyclic groups. Since cellularization commutes with finite products, $K(A, m)$ is $B\mathbb{Z}/p^r$-cellular when A is a finite direct sum of cyclic groups. By taking a (possibly transfinite) telescope of $B\mathbb{Z}/p^r$-cellular spaces, we obtain that $K(A, m)$ is $B\mathbb{Z}/p^r$-cellular for any bounded group.

In general, A splits as a direct sum of a divisible group D and a reduced group T. A p-torsion divisible group is a direct sum of copies of $\mathbb{Z}/p^\infty$, which is a union of bounded groups. Thus, $K(D, m)$ is cellular. Now T has a basic subgroup $P < T$, which is a direct sum of cyclic groups, and the quotient T/P is divisible. So $K(T, m)$ is the total space of a fibration

$$K(P, m) \to K(T, m) \to K(D, m).$$

When $m \geq 3$, we are done because of the closure property Proposition 3.1. If $m = 2$, we have to refine the analysis of the fibration because $K(D, m-1)$ is not cellular. However, since D is a union of bounded groups $D[p^k]$, the space $K(T, 2)$ is the telescope of total spaces X_k of fibrations with cellular fiber $K(P, 2)$ and base $K(D[p^k], 2)$. We claim that these total spaces are cellular (and thus, so is $K(T, 2)$) and proceed by induction on the bound. Consider the subgroup $D[p^k] < D[p^{k+1}]$ whose quotient is a direct sum of cyclic groups $\mathbb{Z}/p$. Therefore, X_{k+1} is the base space in a fibration

$$K(\oplus\mathbb{Z}/p, 1) \to X_k \to X_{k+1},$$

where the fiber and total space are cellular. We are done. □

We are now ready to prove that any simply connected p-torsion Postnikov piece is a $B\mathbb{Z}/p^r$-cellular space.

Proposition 3.4. *A simply connected Postnikov piece is $B\mathbb{Z}/p^r$-cellular if and only if it is p-torsion.*

Proof. Let X be a simply connected p-torsion Postnikov piece. For some integer m, the m-connected cover $X\langle m\rangle$ is an Eilenberg-Mac Lane space, which is cellular by Lemma 3.3. Consider the principal fibration

$$K(\pi_m X, m-1) \to X\langle m\rangle \to X\langle m-1\rangle.$$

If $m \geq 3$, both $X\langle m\rangle$ and $K(\pi_m X, m-1)$ are cellular. It follows that $X\langle m-1\rangle$ is cellular by the closure property Proposition 3.1. An iteration of the same argument shows that $X\langle 2\rangle$ is cellular.

Thus, let us look at the fibration $X\langle 2\rangle \to X \to K(\pi_2 X, 2)$. The discussion in the proof of Lemma 3.3 also applies to the p-torsion group $\pi_2 X$. If this is a bounded group, say of exponent p^k, an induction on the bound shows that X is actually the base space of a fibration where the total space is cellular, because its second homotopy group is of exponent p^{k-1}, and the fiber is cellular because it is of the form $K(V, 1)$, with V a p-torsion abelian group of exponent $\leq p^r$. Then the closure property Proposition 3.1 ensures that X is cellular.

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If $\pi_2 X$ is divisible, X is a telescope of cellular spaces, hence cellular. If it is reduced, taking a basic subgroup $B < \pi_2 X$ yields a diagram of fibrations

$$\begin{array}{ccccc} X\langle 2\rangle & \longrightarrow & Y & \longrightarrow & K(B,2) \\ \| & & \downarrow & & \downarrow \\ X\langle 2\rangle & \longrightarrow & X & \longrightarrow & K(\pi_2 X, 2) \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & K(D,2) & = & K(D,2), \end{array}$$

which exhibits X as the total space of a fibration over $K(D,2)$ with D divisible and a $B\mathbb{Z}/p^r$-cellular fiber. Therefore, by writing D as a union of bounded groups as in the proof of Lemma 3.3, one obtains X as a telescope of cellular spaces. Thus, X is $B\mathbb{Z}/p^r$-cellular as well. □

Remark 3.5. The proof of the proposition holds in the more general setting, where X is a p-torsion space such that $X\langle m\rangle$ is $B\mathbb{Z}/p^r$-cellular for some $m \geq 2$. The proposition corresponds to the case when some m-connected cover $X\langle m\rangle$ is contractible.

Recall from [13, Corollary 2.12] that a connected space is nilpotent if and only if its Postnikov system admits a principal refinement

$$\cdots \to X_s \to X_{s-1} \to \cdots \to X_1 \to X_0.$$

This means that each map $X_{s+1} \to X_s$ in the tower is a principal fibration with fiber $K(A_s, i_s - 1)$ for some increasing sequence of integers $i_s \geq 2$. We are only interested in finite Postnikov pieces, i.e. nilpotent spaces that can be constructed in a finite number of steps by taking homotopy fibers of k-invariants $X_s \to K(A_s, i_s)$.

The key step in the study of the cellularization of a nilpotent finite Postnikov piece is the analysis of principal fibrations (given in our case by the k-invariants).

Theorem 3.6. *Let X be a p-torsion nilpotent Postnikov piece. Then there exists a fibration*

$$X\langle 1\rangle \to CW_{B\mathbb{Z}/p^r} X \to BS,$$

where S is the (normal) subgroup of $\pi_1 X$ generated by the elements g of order p^i with $i \leq r$, such that the inclusion $B\langle g\rangle \to B\pi_1 X$ admits a lift to X.

Proof. By Proposition 3.4, the universal cover $X\langle 1\rangle$ is cellular, and there is a fibration $X\langle 1\rangle \to X \to BG$, where $G = \pi_1 X$ is nilpotent. The result then follows from Theorem 2.7. □

4. Cellularization of H-spaces

In this section, we will use the computations of the cellularization of p-torsion nilpotent Postnikov systems to determine $CW_{B\mathbb{Z}/p} X$ when X is an H-space. We prove:

Theorem 4.1. *Let X be a connected H-space such that $\Omega^n X$ is $B\mathbb{Z}/p$-null. Then*

$$CW_{B\mathbb{Z}/p} X \simeq Y \times K(W,1),$$

where Y is a simply connected p-torsion H-Postnikov piece with homotopy groups concentrated in dimensions $\leq n$ and W is an elementary abelian p-group.

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Proof. The fibration in Bousfield's result Proposition 1.5 yields a cellular equivalence between a connected p-torsion H-Postnikov piece F and X. Theorem 3.6 thus applies. Moreover, since F is an H-space as well, the subgroup S is abelian and generated by elements of order p. Therefore, the H-fibration in Theorem 3.6 $F\langle 1\rangle \to CW_{B\mathbb{Z}/p}F \to K(W,1)$ admits a section (summing up the local sections), and the cellularization splits as a product. □

This result applies for H-spaces satisfying certain finiteness conditions.

Proposition 4.2. *Let X be a connected H-space such that $H^*(X;\mathbb{F}_p)$ is finitely generated as an algebra over the Steenrod algebra. Then*

$$CW_{B\mathbb{Z}/p}X \simeq F \times K(W,1),$$

where F is a 1-connected p-torsion H-Postnikov piece and W is an elementary abelian p-group. Moreover, there exists an integer k such that $CW_{B^m\mathbb{Z}/p}X \simeq *$ for any $m \geq k$.

Proof. In [6], we prove that if $H^*(X;\mathbb{F}_p)$ is finitely generated as an algebra over the Steenrod algebra, then $\Omega^n X$ is $B\mathbb{Z}/p$-null for some $n \geq 0$. Hence, Theorem 4.1 applies, and we obtain the desired result. In addition, Lemma 1.1 shows that X is $B^{n+s+1}\mathbb{Z}/p$-null for any $s \geq 0$, which implies the second part of the result. □

The technique we propose in this paper is not only a nice theoretical tool which provides a general statement about what the $B\mathbb{Z}/p$-cellularization of H-spaces looks like. Our next result shows that one can actually identify this new space precisely when dealing with connected covers of finite H-spaces. Recall that by Miller's theorem [17, Thm. A], any finite H-space X is $B\mathbb{Z}/p$-null and hence, $CW_{B\mathbb{Z}/p}X \simeq *$. The universal cover of X is still finite, and thus $CW_{B\mathbb{Z}/p}(X\langle 1\rangle)$ is contractible as well. We can therefore assume that X is 1-connected. The computation of the cellularization of the 3-connected cover is already implicit in [4].

Proposition 4.3. *Let X be a simply connected finite H-space and let k denote the rank of the free abelian group $\pi_3 X$. Then $CW_{B\mathbb{Z}/p}(X\langle 3\rangle) \simeq K(\bigoplus_k \mathbb{Z}/p, 1)$. For $n \geq 4$, up to p-completion, the universal cover of $CW_{B\mathbb{Z}/p}(X\langle n\rangle)$ is weakly equivalent to the 2-connected cover of $\Omega(X[n])$.*

Proof. By Browder's famous result [5, Theorem 6.11], X is even 2-connected and its third homotopy group $\pi_3 X$ is free abelian (of rank k) by Hubbuck and Kane's theorem [14]. This means we have a fibration

$$K(\bigoplus_k \mathbb{Z}_{p^\infty}, 1) \to X\langle 3\rangle \to P_{B\mathbb{Z}/p}X\langle 3\rangle,$$

which shows that $CW_{B\mathbb{Z}/p}X\langle 3\rangle \simeq K(\bigoplus_k \mathbb{Z}/p, 1)$.

We now deal with the higher connected covers. Consider the following commutative diagram of fibrations:

$$\begin{array}{ccccc}
F & = & F & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
\Omega X[n] & \longrightarrow & X\langle n\rangle & \longrightarrow & X \\
\downarrow & & \downarrow & & \| \\
P_{B\mathbb{Z}/p}(\Omega X[n]) & \longrightarrow & P_{B\mathbb{Z}/p}(x\langle n\rangle) & \longrightarrow & X
\end{array}$$

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where F is a p-torsion Postnikov piece by [2, Thm 7.2] and the fiber inclusions are all $B\mathbb{Z}/p$-cellular equivalences, because the base spaces are $B\mathbb{Z}/p$-null. Therefore,

$$CW_{B\mathbb{Z}/p}(X\langle n\rangle) \simeq CW_{B\mathbb{Z}/p}F \simeq F\langle 1\rangle \times K(W,1).$$

We wish to identify $F\langle 1\rangle$. Since the fibrations in the diagram are nilpotent, by [3, II.4.8] they remain fibrations after p-completion. By Neisendorfer's theorem [19], the map $P_{B\mathbb{Z}/p}(X\langle n\rangle) \to X$ is an equivalence up to p-completion, which means that $P_{B\mathbb{Z}/p}\big(\Omega(X[n])\big)^\wedge_p \simeq *$. Thus $F^\wedge_p \simeq \big(\Omega(X[n])\big)^\wedge_p$. Note that $\Omega(X[n])$ is simply connected and its second homotopy group is free by the above-mentioned theorem of Hubbuck and Kane (which corresponds up to p-completion to the direct sum of k copies of the Prüfer group $\mathbb{Z}/p^\infty$ in $\pi_1 F$). Hence, $F\langle 1\rangle$ coincides with $\big(\Omega(X[n])\big)\langle 2\rangle$ up to p-completion. □

To illustrate this result, we compute the $B\mathbb{Z}/2$-cellularization of the successive connected covers of S^3. The only delicate point is the identification of the fundamental group.

Example 4.4. Recall that S^3 is $B\mathbb{Z}/2$-null since it is a finite space. Thus, the cellularization $CW_{B\mathbb{Z}/2}S^3$ is contractible. Next, the fibration

$$K(\mathbb{Z}_{2^\infty},1) \to S^3\langle 3\rangle \to P_{B\mathbb{Z}/2}(S^3\langle 3\rangle)$$

shows that $CW_{B\mathbb{Z}/2}(S^3\langle 3\rangle) \simeq K(\mathbb{Z}/2,1)$. Finally, since $S^3[4]$ does not split as a product (the k-invariant is not trivial), we see that $CW_{B\mathbb{Z}/2}(S^3\langle 4\rangle) \simeq K(\mathbb{Z}/2,3)$. Likewise, for any integer $n \geq 4$, we have that $CW_{B\mathbb{Z}/2}(S^3\langle n\rangle)$ is weakly equivalent to the 2-completion of the 2-connected cover of $\Omega(S^3[n])$. The same phenomenon occurs at odd primes.

5. Cellularization with respect to $B^m\mathbb{Z}/p$

All the techniques developed for fibrations over BG apply to fibrations over $K(G,n)$ when $n > 1$, and we get the following results.

Lemma 5.1. *Let $m \geq 2$ and let X be a connected space. Then $CW_{B^m\mathbb{Z}/p^r}X$ is weakly equivalent to $CW_{B^m\mathbb{Z}/p^r}(X\langle n-1\rangle)$.*

Proof. Consider the fibrations $X\langle i\rangle \to X\langle i-1\rangle \to K(\pi_i X, i)$. For $i < m$, the base space is $B^m\mathbb{Z}/p^r$-null and so $CW_{B^m\mathbb{Z}/p^r}(X\langle i\rangle) \simeq CW_{B^m\mathbb{Z}/p^r}(X\langle i-1\rangle)$. □

Proposition 5.2. *Let $m \geq 2$ and let X be a p-torsion nilpotent Postnikov piece. Then there exists a fibration*

$$X\langle m\rangle \to CW_{B^m\mathbb{Z}/p^r}X \to K(W,m),$$

where W is a p-torsion subgroup of $\pi_m X$ of exponent dividing p^r. □

Theorem 5.3. *Let X be a connected H-space such that $\Omega^n X$ is $B^m\mathbb{Z}/p$-null. Then*

$$CW_{B^m\mathbb{Z}/p}X \simeq F \times K(W,m),$$

where F is a p-torsion H-Postnikov piece with homotopy groups concentrated in dimensions from $m+1$ to $m+n-1$, and W is an elementary abelian p-group. □

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Example 5.4. Let X denote "Milgram's space" (see [16]), the homotopy fiber of $Sq^2 : K(\mathbb{Z}/2, 2) \to K(\mathbb{Z}/2, 4)$. This is an infinite loop space. By Proposition 3.4, we know it is already $B\mathbb{Z}/2$-cellular. Since the k-invariant is not trivial, we see that

$$CW_{B^2\mathbb{Z}/2}X \simeq CW_{B^3\mathbb{Z}/2}X \simeq K(\mathbb{Z}/2, 3).$$

Finally, we compute the cellularization of the (infinite loop) space BU and its 2-connected cover BSU with respect to Eilenberg-Mac Lane spaces $B^m\mathbb{Z}/p$. By Bott periodicity, this actually tells us the answer for all connected covers of BU.

Example 5.5. First of all, recall from Example 1.4 that BU is $B^2\mathbb{Z}/p$-null since $\widetilde{K}^*(B^2\mathbb{Z}/p) = 0$ and its iterated loops are never $B\mathbb{Z}/p$-null. Therefore, the cellularization $CW_{B^m\mathbb{Z}/p}BU$ is contractible if $m \geq 2$. Since $BU \simeq BSU \times BS^1$, the same holds for BSU.

We now compute the $B^m\mathbb{Z}/p$-cellularization of BO and its connected covers BSO, $BSpin$, and $BString$.

Proposition 5.6. *Let $m \geq 2$. Then*

(1) $CW_{B^m\mathbb{Z}/p}BO \simeq CW_{B^m\mathbb{Z}/p}BSO \simeq CW_{B^m\mathbb{Z}/p}BSpin \simeq *$,
(2) $CW_{B^m\mathbb{Z}/p}BString \simeq *$ *if* $m > 2$,
(3) $CW_{B^2\mathbb{Z}/p}BString \simeq K(\mathbb{Z}/p, 2)$ *and* $\mathrm{map}_*(B^2\mathbb{Z}/p, BString) \simeq \mathbb{Z}/p$.

Proof. In [15], W. Meier proves that real and complex K-theory have the same acyclic spaces, hence BO is also $B^2\mathbb{Z}/p$-null. Therefore, $CW_{B^m\mathbb{Z}/p}BO$ is contractible for any $m \geq 2$. The 2-connected cover of BO is BSO, and there is a splitting $BO \simeq BSO \times B\mathbb{Z}/2$, so that $CW_{B^m\mathbb{Z}/p}BSO \simeq *$.

The 4-connected cover of BO is $BSpin$. From the fibration

$$BSpin \to BSO \xrightarrow{w_2} K(\mathbb{Z}/2, 2),$$

we infer that the homotopy fiber of $BSpin \to BSO$ is $B\mathbb{Z}/2$. Since BSO and $B\mathbb{Z}/2$ are $B^2\mathbb{Z}/p$-null, so is $BSpin$. Therefore, $CW_{B^m\mathbb{Z}/p}BSpin$ is contractible.

Finally, the 8-connected cover of BO is $BString$. It is the homotopy fiber of $BSpin \xrightarrow{p_1/4} K(\mathbb{Z}, 4)$, where p_1 denotes the first Pontrjagin class. Consider the fibration

$$K(\mathbb{Z}, 3) \to BString \to BSpin,$$

where the base space is $B^m\mathbb{Z}/p$-null for $m \geq 2$. Together with the exact sequence $\mathbb{Z} \to \mathbb{Z}[\frac{1}{p}] \to \mathbb{Z}/p^\infty$, this implies that

$$CW_{B^m\mathbb{Z}/p}BString \simeq CW_{B^m\mathbb{Z}/p}K(\mathbb{Z}, 3) \simeq CW_{B^m\mathbb{Z}/p}K(\mathbb{Z}/p^\infty, 2).$$

This is a contractible space unless $m = 2$, when we obtain $K(\mathbb{Z}/p, 2)$. The explicit description of the pointed mapping space $\mathrm{map}_*(B^2\mathbb{Z}/p, BString)$ follows. $\square$

Observe that the iterated loops of the m-connected covers of BO and BU are never $B\mathbb{Z}/p$-null. Hence, their cellularizations with respect to $B\mathbb{Z}/p$ must have infinitely many non-vanishing homotopy groups by Proposition 1.6.

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