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# POSTULATES FOR SUBADDITIVE PROCESSES[1,2]

BY J. M. HAMMERSLEY

*Trinity College, Oxford and*
*University of California, Berkeley*

The paper examines alternative postulates for subadditive processes, especially the ergodic theory thereof. It introduces superconvolutive sequences of distributions and proves limit laws for these, which generalize the weak law of large numbers, Chernoff's theorem, and Kesten's lemma. It discusses eigenshift and eigendistribution theory and concave recurrence relations in the convolutive semigroup, illustrating sundry conjectures with computer studies. It deals with applications of the theory to the first-death problem in branching processes, Bethe approximation of first-passage percolation, self-avoiding walks, maximal solutions of the generalized subconvolutive inequality, rates of convergence of a subadditive process, multidimensional subadditive processes in physics including the dimer problem and the overlapping-sphere model of liquid-vapor equilibrium, and Ulam's problem on the longest monotone subsequence of a random permutation.

**1. Introduction.** The postulates appropriate for any mathematical theory depend upon its purpose and uses. In his paper [10] on subadditive ergodic theory, Professor Kingman adopts the relevant ergodic postulates. This is proper and natural enough; but, of course, it does not guarantee that these same postulates will serve equally well for applications of subadditive processes to other situations. I want to illustrate this by looking at a few practical situations.

We shall need to deal with random variables which can take the value $+\infty$ with positive probability. If $X$ is such a random variable, and $F(x) = P(X \leqq x)$ is its cumulative distribution function, we adopt the usual convention that $F(\infty)$ denotes $\lim_{x\to\infty} F(x)$, so that $P(X = \infty) = 1 - F(\infty)$. A distribution is called *proper* if $F(\infty) = 1$, and *improper* if $F(\infty) < 1$. A sequence of improper distributions $F_n$ is *boundedly improper* if $F_n(\infty) \geqq \delta > 0$ for all $n$, and is *exponentially improper* if $[F_n(\infty)]^{1/n} \to \delta > 0$ as $n \to \infty$. We shall not need to deal with improprieties at $-\infty$; so we shall always assume $F(-\infty) = 0$.

**2. The first-death problem in an age-dependent branching process.** We define the branching process in the usual way:

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[1] Sections 1 through 10 of this paper were originally submitted as a contribution to the Discussion on Professor J. F. C. Kingman's paper [10]. Since it was too long for this purpose it is being published as a separate paper, together with the supplementary notes of Section 11.

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(i) The family tree originates from a single progenitor who dies at time $T = 0$.

(ii) Each person in the tree has a (birth to death) lifetime $U$ distributed with cumulative distribution function $G(u)$ and mean lifetime $\bar{u} = EU < \infty$. We suppose that $G(u) = 0$ for $u < 0$, and that $G(\infty) = 1$.

(iii) When anyone dies, he is replaced immediately by $j$ newly-born offspring with probability $p_j$ $(j = 0, 1, \cdots)$. We write $P(z) = \sum_{j=0}^{\infty} p_j z^j$; and we suppose $P(1) = 1$ and $p_0 < 1 < P'(1) = q$, say. Hence there is a positive chance $\rho > 0$ that the tree will propagate indefinitely, where $P(1 - \rho) = 1 - \rho$.

(iv) All members of the tree are independent, both in lifetimes and in the numbers of their offspring.

(v) The $r$th generation $(r = 1, 2, \cdots)$ consists of the offspring of the $(r - 1)$th generation, the zeroth generation being just the single progenitor.

The first-death problem for this branching process is to discuss at what moment of time a death first occurs to some member of the $r$th generation. (See Note 1 of Section 11). We can set up a subadditive process $x_{rs}$ for integers $0 \leqq r < s$ as follows:

(a) $x_{rs} = \infty$ if the subsequent recipe fails at any juncture through non-existence (for example, if the $r$th generation is empty).

(b) Let $T_1$ be the earliest time at which some person in the $r$th generation dies. Consider all members of the $s$th generation who are descendants of *this* particular person in the $r$th generation. Let $T_2$ be the earliest time at which one of *these* descendants dies. Define $x_{rs} = T_2 - T_1$.

In particular, $x_{0r}$ is the first-death time for the $r$th generation; and we write $F_r(x)$ for its cumulative disribution function. Since we are dealing with random variables which may be infinite, we shall have $\lim_{x\to\infty} F_r(x) = F_r(\infty) < 1$ when $p_0 > 0$.

Let us examine this subadditive process in relation to the postulates given by Kingman [10]. Clearly, postulate $S_1$ holds: (See Note 9 of Section 11).

$$x_{rt} \leqq x_{rs} + x_{st} \qquad (0 \leqq r < s < t)\,. \tag{2.1}$$

Postulate $S_3$ fails in the trivial sense that $Ex_{0r} = \infty$ when $p_0 > 0$; but we can easily replace it by a new postulate $S_3'$:

$$E(x_{0r} \mid x_{0r} < \infty) = g_r \leqq r\bar{u} < \infty\,. \tag{2.2}$$

The corresponding conditional form of postulate $S_2'$ is clearly satisfied; but the conditional version of postulate $S_2$ is *not*. To see this, we write $I_j$ for the individual in the $j$th generation $(j = 0, 1, \cdots, 4)$ who figures the defining process (b) above for one of the three random variables $x_{02}, x_{13}, x_{24}$. Write $I_j \to I_k$ for the event that $I_k$ is the offspring of $I_j$. Now $x_{02}$ and $x_{13}$ are independent except when $I_1 \to I_2 \to I_3$; and $x_{13}$ and $x_{24}$ are independent except when $I_1 \to I_2 \to I_3 \to I_4$. In both these exceptional cases, the dependence between $x_{02}$ and $x_{13}$ is the same as the dependence between $x_{13}$ and $x_{24}$. Since the event $I_1 \to I_2 \to I_3 \to I_4$ is less probable than $I_1 \to I_2 \to I_3$, the positive correlation between $x_{02}$ and $x_{13}$ exceeds the positive correlation between $x_{13}$ and $x_{24}$. Hence the joint distribution of $(x_{02}, x_{13})$ is not equal to the joint distribution of the shifted pair $(x_{13}, x_{24})$. So (the conditional version of) postulate $S_2$ is false. So Kingman's Theorem 1 will not apply, and we cannot conclude (conditionally on $x_{0r} < \infty$) that

$$x_{0r}/r \to \gamma = \inf_r g_r/r \tag{2.3}$$

with probability 1 as $r \to \infty$. This is a pity, because (2.3) doubtless holds in this particular instance: so we need some more pure mathematical work to establish the analogue of Theorem 1 when $S_2'$ (with perhaps some additional conditions?) replaces $S_2$. (See Note 2 of Section 11.)

A tempting additional condition is the following postulate, given in [8]: a subadditive process is called an *independent* subadditive process if the variables $x_{rs}, x_{tu}, \cdots, x_{vw}$ are mutually independent whenever the *open* intervals $(r, s)$, $(t, u), \cdots, (v, w)$ are mutually disjoint. I call this a tempting condition because at first sight it seems to be obviously true. However, on reflection, one sees that it is not satisfied in the unconditional case (where $x_{0r} = \infty$ is allowed). For, if $0 < r < s$ and if we are given that $x_{rs} < \infty$, then $x_{0r}$ *must* be finite. I am not at all sure whether the process is an independent subadditive process in the conditional case of finite random variables, or whether it can be made so by some adjustment of the definitions. If it is, or can be made so, then the conditional process is *self-smothering* (see [8] for a definition of this term) and (2.3) would follow in the sense of convergence in probability (as opposed to convergence with probability 1). However, by means of different tactics (superconvolutive sequences as defined later) I shall prove that (2.3) is in fact true in conditional probability (again, as opposed to conditional probability 1). This is no great loss from the practical point of view, because convergence in probability is the more important *practical* conclusion.

We next obtain a recurrence relation for the distributions $F_r(x)$. Consider the conditional situation given that the progenitor has exactly $j$ offspring. The time from the death of a given one of these offspring to the first death amongst *his* descendants in the $r$th generation has distribution function $F_{r-1}$; and his lifetime has distribution $G$. Hence $x_{0r}$ is the least of $j$ independent observations, each with distribution $F_{r-1} * G$; and so has the distribution function $1 - (1 - F_{r-1} * G)^j$. This holds for $j = 0$ as well, in the sense that the distribution function is identically zero. Hence unconditionally

$$\begin{aligned} F_r(x) &= \textstyle\sum_{j=0}^{\infty} p_j[1 - \{1 - (F_{r-1} * G)(x)\}^j] \\ &= 1 - P[1 - (F_{r-1} * G)(x)]\,, \qquad (r = 1, 2, \cdots)\,. \end{aligned} \tag{2.4}$$

This recurrence relation stars from $F_0(x) = 1$ or $0$ according as $x \geqq 0$ or $x < 0$. We shall presently obtain an explicit expression of the time constant $\gamma$ from (2.4); but first we examine a more general situation.

**3. Concave recurrence relations and superconvolutive sequences on the convolution semigroup.** Let **D** denote the convolution semigroup: the elements of **D** are (proper and improper) distribution functions, that is to say non-decreasing functions $L(x)$ with $L(-\infty) = 0$ and $L(+\infty) \leqq 1$; and the binary operation of **D** is convolution

$$(L * M)(x) = \int_{-\infty}^{\infty} L(x-y)\, dM(y) . \tag{3.1}$$

We shall obtain three theorems for **D**, and the reader who prefers to skip the lengthy proofs should proceed at once to their statement at the end of this section.

We shall write

$$L \leqq M \qquad (x \leqq a) \tag{3.2}$$

as shorthand for $L(x) \leqq M(x)$ for all $x \leqq a$. Usually $a = +\infty$, in which case we simply write $L \leqq M$ for our partial ordering on **D**. From (3.1) it is clear that (3.2) implies

$$L * N \leqq M * N \qquad (x \leqq a) ; \tag{3.3}$$

provided, when $a < \infty$, that $N(y) = 0$ for $y < 0$.

Let $Q$ be a concave non-decreasing function defined on $0 \leqq z \leqq 1$, with $Q(0) = 0$ and $Q(1) \leqq 1$. Suppose also that

$$1 < Q'(0) = q \leqq \infty . \tag{3.4}$$

Clearly $Q: L(x) \to Q[L(x)]$ is an isotonic mapping of **D** into itself in the sense that (3.2) implies

$$Q(L) \leqq Q(M) \qquad (x \leqq a) . \tag{3.5}$$

Also by (3.1), concavity of $Q$ implies that

$$Q(L * M) \geqq Q(L) * M . \tag{3.6}$$

Let $H_a(x)$ denote the step function

$$H_a(x) = 1 \quad x \geqq a; \qquad = 0 \quad x < a . \tag{3.7}$$

We are going to study the recurrence relation

$$F_r = Q(F_{r-1} * G) \qquad (r = 1, 2, \cdots), \qquad F_0 = H_0 , \tag{3.8}$$

where $G \in \mathbf{D}$ is given. It is easy to see that (2.4) is the particular case of (3.8) when $Q(z) = 1 - P(1-z)$ and that (3.4) then holds. We shall write $G_r$ for the $r$th-fold convolution of $G$:

$$G_r = G_{r-1} * G \quad (r = 1, 2, \cdots), \qquad G_0 = H_0 . \tag{3.9}$$

We claim that (3.8) implies

$$F_{r+s} \geqq F_r * F_s . \tag{3.10}$$

For, if (3.10) is true, then

$$F_{r+s+1} = Q(F_{r+s} * G) \geqq Q(F_r * F_s * G) \geqq Q(F_r * G) * F_s = F_{r+1} * F_s \tag{3.11}$$

by (3.8), (3.10), (3.3), (3.5) and (3.6). Hence (3.10) follows by induction on $r$, since it is plainly true for $r = 0$.

A sequence of distributions satisfying (3.10) for all $r$, $s$ will be called *superconvolutive*. It follows immediately from (2.1) that any independent subadditive process possesses a superconvolutive sequence of distribution functions. The converse is however false. For consider the counter-example in which $F_1$, $F_2$, $F_3$ are defined by the discrete distributions:

$$\begin{aligned} &F_1: \quad P(x_{01} = 0) = \tfrac{1}{2}, \qquad P(x_{01} = 1) = \tfrac{1}{2} \\ &F_2: \quad P(x_{02} = 0) = \tfrac{1}{4}, \qquad P(x_{02} = 1) = \tfrac{3}{4} \\ &F_3: \quad P(x_{03} = 0) = \tfrac{1}{8}, \qquad P(x_{03} = 1) = \tfrac{1}{2}, \qquad P(x_{03} = 2) = \tfrac{3}{8}. \end{aligned} \tag{3.12}$$

These distributions satisfy $F_2 > F_1 * F_1$ and $F_3 = F_1 * F_2$. But, if $x_{rs}$ is an independent subadditive process, then $x_{03} < x_{01} + x_{13}$ and $x_{03} < x_{02} + x_{23}$; and $x_{03} = 2$ can only occur if $x_{01} = x_{23} = 1$; and we have the contradiction

$$\tfrac{3}{8} = P(x_{03} = 2) \leqq P(x_{01} = x_{23} = 1) = P(x_{01} = 1)P(x_{23} = 1) = \tfrac{1}{4}. \tag{3.13}$$

Superconvolutive sequences have several interesting properties; and the following one will be important in the sequel: there exists a concave non-positive non-decreasing function $\psi(x)$, which may take the value $-\infty$, such that

$$F_r(rx) \leqq e^{r\psi(x)} \tag{3.14}$$

and

$$r^{-1} \log F_r(rx) \to \psi(x) \qquad \text{as } r \to \infty. \tag{3.15}$$

To prove this we note that (3.1) and (3.10) imply

$$F_{r+s}(x + y) \geqq F_r(x)F_s(y). \tag{3.16}$$

First, write $rx$ for $x$ and $sx$ for $y$ in (3.16). This shows that $F_r(rx)$ is a supermultiplicative function of $r$ for fixed $x$; which implies (3.14) and (3.15) for some function $\psi(x)$, and this function must be non-positive and non-decreasing because $F_r(rx) \leqq F_r(ry) \leqq 1$ for $x \leqq y$. Second, write $rx$ for $x$ and $sy$ for $y$ in (3.16), take logarithms, divide by $r + s$, and let $r, s \to \infty$ such that $r/(r + s) \to \alpha$ and $s/(r+s) \to \beta$ where $\alpha$, $\beta$ are arbitrary non-negative numbers such that $\alpha + \beta = 1$. This shows that $\psi(\alpha x + \beta y) \geqq \alpha\psi(x) + \beta\psi(y)$, and hence $\psi$ is concave.

A second property of superconvolutive sequences is that the corresponding cumulant generating functions

$$K_r(\theta) = \log \int_{-\infty}^{\infty} e^{-\theta x}\, dF_r(x) \tag{3.17}$$

are superadditive functions of $r$ for each fixed $\theta \geqq 0$:

$$K_{r+s}(\theta) \geqq K_r(\theta) + K_s(\theta) \qquad (\theta \geqq 0). \tag{3.18}$$

To prove this, let $X$ and $Y$ be independent random variables with distributions

$F_r$ and $F_s$ respectively. Then (3.10) implies that

$$Z = F_{r+s}^{-1}[(F_r * F_s)(X+Y)] \leqq X + Y . \tag{3.19}$$

The relation (3.19) is to be interpreted in an obvious limiting sense if $F_r * F_s$ has discontinuities: we replace such a discontinuity by a sufficiently steep linear slope and then allow the slope to tend to infinity. This limiting process is used too at $+\infty$ if $F_r * F_s$ is improper. With this interpretation $Z$ will be a random variable with distribution $F_{r+s}$. Hence

$$Ee^{-\theta Z} \geqq Ee^{-\theta X} Ee^{-\theta Y} , \tag{3.20}$$

from which (3.18) follows at once. (Of course, we allow the terms in (3.18) to be infinite if the expectations in (3.20) are infinite or zero.) From (3.18) we deduce the existence of a (possibly infinite) function $K(\theta)$ such that

$$K_r(\theta) \leqq rK(\theta) , \qquad r^{-1}K_r(\theta) \rightarrow K(\theta) \qquad \text{as } r \rightarrow \infty . \tag{3.21}$$

Next we shall establish a relation between $\psi(x)$ and $K(\theta)$, namely

$$\psi(x) = \inf_{\theta \geqq 0} [K(\theta) + \theta x] . \tag{3.22}$$

In the particular case when equality holds in (3.10), $K(\theta) = K_1(\theta)$, and (3.22) reduces to a familiar formula on the extreme tails of a convolution proved by Chernoff [1]. Thus (3.22) is the superconvolutive generalization of Chernoff's convolutive formula.

Since $\psi(x)$ is concave and non-decreasing, there exists a number $\omega$ satisfying $0 \leqq \omega \leqq \infty$ such that

$$\psi(x) \sim \omega x \qquad \text{as } x \rightarrow -\infty . \tag{3.23}$$

We shall show that $K(\theta) = +\infty$ whenever $\theta > \omega$. We have

$$\begin{aligned} K_r(\theta) &\geqq \textstyle\int_{-\infty}^{-ra} e^{-\theta x}\, dF_r(x) \geqq e^{\theta ra} F_r(-ra) \\ &\geqq \exp[r[\theta a + \psi(-a) + o(1)]] \qquad \text{as } r \rightarrow \infty . \end{aligned} \tag{3.24}$$

If $\theta > \omega$, we can choose $a$ to make $\theta a + \psi(-a) > 0$, and $K(\theta) = +\infty$ by (3.21) and (3.24). (See Note 10 of Section 11).

Next we dispose of the special case $\omega = 0$. In this case (3.23) implies that $\psi(x)$ is a constant, since $\psi$ is concave and non-decreasing. (See Note 3 of Section 11.) Also $K(\theta) = \infty$ for $\theta > 0$, so (3.22) becomes $\psi(\infty) = \psi(x) = K(0) = \lim_{r\rightarrow\infty} r^{-1} \log F_r(\infty)$ which is true because of (3.15) and (3.17). Thus (3.22) holds for $\omega = 0$, and we may hereafter suppose that $\omega > 0$. From (3.17), we see that $\exp[K_r(\theta)]$ is a convex function of $\theta$, and hence $e^{K(\theta)}$ is also convex. Hence, if $e^{K(\theta)}$ is not continuous on the left at $\theta = \omega$, then $e^{K(\omega)} = \infty$. Similarly, $e^{K(\theta)}$ must be continuous on the right at $\theta = 0$, because $e^{K(0)} < \infty$. Since further $K(\theta) = \infty$ for $\theta > \omega$, we see that it suffices to prove (3.22) in the equivalent form

$$\psi(x) = \inf_{0<\theta<\omega} [K(\theta) + \theta x] , \tag{3.25}$$

and hereafter we suppose that $0 < \theta < \omega$.

By (3.14),

$$0 \leqq e^{-\theta x}F_r(x) \leqq \exp\left\{-x\left[\theta - \frac{\psi(x/r)}{x/r}\right]\right\} \tag{3.26}$$

and

$$\begin{aligned} \theta - \frac{\psi(x/r)}{x/r} &\to \theta > 0 && \text{as } x \to \infty \\ &\to \theta - \omega < 0 && \text{as } x \to -\infty . \end{aligned} \tag{3.27}$$

Hence we see that $K_r(\theta)$ is finite for all $r$ when $0 < \theta < \omega$; and we may integrate by parts to obtain

$$\begin{aligned} \exp[K_r(\theta)] &= \theta \int_{-\infty}^{\infty} e^{-\theta x}F_r(x)\,dx \\ &\leqq \theta \int_{-\infty}^{\infty} \exp\left\{-x\left[\theta - \frac{\psi(x/r)}{x/r}\right]\right\} dx \\ &\leqq 2\theta r \int_a^b \exp\{r[\psi(y) - \theta y]\}\,dy \end{aligned} \tag{3.28}$$

for suitably chosen finite $a$, $b$ (independent of $r$) after the substitution $y = x/r$. Hence

$$\exp[K_r(\theta)] \leqq 2\theta r(b-a)\exp\{r \sup_y [\psi(y) - \theta y]\} . \tag{3.29}$$

On taking logarithms, dividing by $r$, and letting $r \to \infty$, we deduce

$$K(\theta) \leqq \sup_y [\psi(y) - \theta y] . \tag{3.30}$$

Now the slope of the concave function $\psi(y)$ decreases from $\omega$ at $y = -\infty$ to 0 at $y = +\infty$. Hence for $0 < \theta < \omega$ we may find a finite value $y_\theta$ such that $\theta$ lies between the right-hand and left-hand derivatives of $\psi$ at $y_\theta$:

$$\psi'(y_\theta + 0) \leqq \theta \leqq \psi'(y_\theta - 0) . \tag{3.31}$$

It is readily verified that $\psi(y) - \theta y$ attains its supremum at $y = y_\theta$. Hence

$$K(\theta) \leqq \psi(y_\theta) - \theta y_\theta . \tag{3.32}$$

The concavity of $\psi$ also guarantees that $y_\theta$ is a non-increasing function of $\theta$.

On the other hand, if the random variable $X$ has the distribution $F_r$, the Chebyshev inequality yields

$$\begin{aligned} F_r(rx) = P(X \leqq rx) &= P(e^{-\theta X} \geqq e^{-\theta r x}) \\ &\leqq E(e^{-\theta X})/e^{-\theta r x} = \exp[K_r(\theta) + \theta r x] . \end{aligned} \tag{3.33}$$

If we take logarithms, divide by $r$, and let $r \to \infty$, we get

$$\psi(x) \leqq K(\theta) + \theta x . \tag{3.34}$$

Putting $x = y_\theta$ in (3.34), and comparing the result with the opposite inequality (3.32), we have

$$\psi(y_\theta) = K(\theta) + \theta y_\theta . \tag{3.35}$$

If $x$ is such that either $0 < \psi'(x - 0)$ or $\psi'(x + 0) < \omega$, we can find a solution of (3.31) such that $y_\theta = x$ for a suitable $\theta$ satisfying $0 < \theta < \omega$; and (3.34) and (3.35) will establish (3.25) for this $x$. If $\psi'(x+0) = \omega$, we proceed instead in the following way. We note that $y_\theta > x$ for $\theta < \omega$ and that

$$\psi(x) \geqq \psi(y_\theta) - (y_\theta - x)\psi'(x+0) \tag{3.36}$$

by the concavity of $\psi$. Inserting (3.35) into (3.36) we get

$$\psi(x) \geqq K(\theta) + \theta x + (y_\theta - x)[\theta - \psi'(x+0)]\,. \tag{3.37}$$

If we let $\theta \to \omega - 0$, and note that $y_\theta$ is non-increasing, we again obtain (3.25) from (3.34) and (3.37). A similar argument applies to the case $\psi'(x-0) = 0$, and completes the proof of (3.25). The reciprocal equation

$$K(\theta) = \sup_x [\psi(x) - \theta x] \qquad (\theta \geqq 0) \tag{3.38}$$

can also be established by analogous arguments.

It is a familiar result that convolutive sequences obey the conditional versions of weak law of large numbers if we assume the existence of the conditional means

$$E(x_r \mid x_r < \infty) = g_r\,. \tag{3.39}$$

(The counter-example of the Cauchy distribution shows that some such assumption as (3.39) is needed.) Can this result be generalized to superconvolutive sequences? I do not know the answer to this question; but the answer is affirmative if we make the additional assumption that all the distributions are proper. Thus we now suppose that $x_r$ has a distribution $F_r$ with $F_r(\infty) = 1$ and that

$$g_r = E(x_r) < \infty\,. \tag{3.40}$$

Taking expectations of (3.19), we find

$$g_{r+s} \leqq g_r + g_s\,; \tag{3.41}$$

so $g_r$ is a subadditive function of $r$ and hence there exists $\gamma$ such that

$$\gamma \leqq g_r/r \to \gamma \qquad \text{as } r \to \infty\,. \tag{3.42}$$

To sketch in the remainder of the proof without going into all details of rigor, we fix an $\varepsilon > 0$ and choose $r$ so that $g_r/r < \gamma + \varepsilon$. For large $n$ we write $n = rs + t$ where $0 < t \leqq r$; and we write $X_1, X_2, \cdots, X_s$ for independent random variables each with distribution $F_r$, and $Y$ for a further independent random variable with distribution $F_t$. By an obvious analogue of (3.19) we have

$$Z/n \leqq (X_1 + X_2 + \cdots + X_s + Y)/n\,, \tag{3.43}$$

where $Z$ is a random variable defined in terms of $X_1, X_2, \cdots, X_s, Y$ and where $Z$ has distribution $F_n$. As $n \to \infty$, the right-hand side of (3.43) will converge in probability to $g_r/r$; and hence $P(x_n/n > \gamma + 2\varepsilon) \to 0$ as $n \to \infty$. Similarly, if $L_n$ is the distribution function for $x_n/n$, we can make $\int_{\gamma+2\varepsilon}^{\infty} x\, dL_n$ arbitrarily small by bounding it with the corresponding integral for the right-hand side of (3.43).

Hence

$$(3.44) \qquad \gamma = \lim_{n\to\infty} g_n/n = \lim_{n\to\infty} \int_{-\infty}^{\infty} x\,dL_n = \lim_{n\to\infty} \int_{-\infty}^{\gamma+2\varepsilon} x\,dL_n .$$

For fixed $\xi < \gamma$, we have

$$(3.45) \qquad \int_{-\infty}^{\gamma+2\varepsilon} x\,dL_n \leqq \xi L_n(\xi) + (\gamma + 2\varepsilon)[1 - L_n(\xi)] ;$$

and (3.44) and (3.45) give

$$(3.46) \qquad 0 \leqq 2\varepsilon - (\gamma + 2\varepsilon - \xi) \limsup_{n\to\infty} L_n(\xi) .$$

Thus

$$(3.47) \qquad 0 \leqq \limsup_{n\to\infty} L_n(\xi) \leqq 2\varepsilon/(\gamma + 2\varepsilon - \xi) .$$

On letting $\varepsilon \to 0$, we conclude that $L_n(\xi) \to 0$ as $n \to \infty$ for each fixed $\xi < \gamma$; and consequently $x_n/n \to \gamma$ in probability as $n \to \infty$ as required for a superconvolutive weak law of large numbers.

There seem to be difficulties in extending this argument to cover improper distributions in general. However, by means of an *ad hoc* procedure, we can prove convergence in probability for the special case of the improper distributions arising from the recurrence relation (3.8). We write

$$(3.48) \qquad f_r = F_r(\infty) .$$

Putting $x = \infty$ in (3.8) we deduce

$$(3.49) \qquad f_0 = 1 , \qquad f_r = Q(f_{r-1}) \qquad (r = 1, 2, \cdots)$$

because $G(\infty) = 1$. Hence the sequence $f_r$ is non-increasing by the assumed properties of the function $Q$ and

$$(3.50) \qquad f_r \to \rho > 0 \qquad \text{where } Q(\rho) = \rho .$$

Thus we are dealing with a boundedly improper superconvolutive sequence $F_r$. Now define

$$(3.51) \qquad C_r(x) = F_r(x)/f_r .$$

Thus $C_r$ is the conditional distribution of $x_r = x_{0r}$ given $x_r < \infty$. Since $G(u) = 0$ for $u < 0$ by hypothesis, we see that $x_r$ is a nonnegative random variable; and hence the cumulant-generating functions $K_r(\theta)$ exist for all $\theta \geqq 0$. On dividing (3.18) by $\theta$ and letting $\theta \to 0$ we deduce

$$(3.52) \qquad f_{r+s} g_{r+s} \leqq f_r g_r + f_s g_s ,$$

where

$$(3.53) \qquad g_r = E(x_r \,|\, x_r < \infty) .$$

Note that sign of the equality in (3.52) is the reserves of that in (3.18) because $f_r g_r$ is the coefficient of $-\theta$ in the expansion of $K_r(\theta)$ by virtue of (3.17). Now (3.52) shows that $f_r g_r$ is a subadditive function of $r$; so $f_r g_r/r$ tends to a limit as $r \to \infty$. We write $\gamma\rho$ for this limit, thus defining $\gamma$; and now (3.50) establishes the existence of a conditional time-constant

$$\gamma = \lim_{r\to\infty} g_r/r\,. \tag{3.54}$$

By substituting (3.51) into (3.8), we find that the recurrence relation governing the distributions $C_r$ is

$$C_0 = H_0\,, \qquad C_{r+1} = Q_r(C_r * G) \qquad (r = 0, 1, 2, \cdots) \tag{3.55}$$

where

$$Q_r(z) = f_{r+1}^{-1} Q(f_r z)\,. \tag{3.56}$$

Each of the functions $Q_r$ is concave and non-decreasing and satisfies $Q_r(0) = 0$, $Q_r(1) = 1$. Therefore the same properties are enjoyed by the function

$$R(z) = \inf_r Q_r(z)\,. \tag{3.57}$$

Consequently if we define a sequence of distributions $D_r$ by

$$D_0 = H_0\,, \qquad D_{r+1} = R(D_r * G) \qquad (r = 0, 1, 2, \cdots) \tag{3.58}$$

we shall have

$$D_r \leqq C_r\,. \tag{3.59}$$

The distributions $D_r$ are all proper because $G$ is proper and $R(1) = 1$. Hence, if $y_r$ is a sequence of random variables with respective distributions $D_r$, we know that $y_r/r$ converges in probability to some time constant $\beta$ as $r \to \infty$. In other words, as $r \to \infty$

$$\begin{aligned} D_r(rx) &\to 1 & (x > \beta) \\ &\to 0 & (x < \beta)\,. \end{aligned} \tag{3.60}$$

Suppose that $\psi(x)$ is the $\psi$-function associated with the sequence $D_r$:

$$\psi(x) = \lim_{r\to\infty} r^{-1} \log D_r(rx)\,, \tag{3.61}$$

and define $\alpha$ by the equation

$$\alpha = \sup\{x : \psi(x) < 0\}\,. \tag{3.62}$$

It is clear from (3.60) and (3.62) that

$$\alpha \leqq \beta\,, \tag{3.63}$$

and we shall eventually prove that $\alpha = \beta$.

For $0 < \delta < 1$ define

$$q_\delta = Q(\delta)/\delta\,. \tag{3.64}$$

By the concavity of $Q$ we have

$$q_\delta z \leqq Q(z) \leqq qz \qquad (0 \leqq z \leqq \delta) \tag{3.65}$$

where

$$1 < q = Q'(0) = \lim_{\delta\to 0} q_\delta\,. \tag{3.66}$$

Hence from (3.57) and (3.65)

$$\frac{qf_r}{f_{r+1}} \geqq \frac{Q_r(z)}{z} \geqq \frac{R(z)}{z} \geqq \inf_r \frac{q_\delta f_r}{f_{r+1}} = q_\delta \qquad (0 < z \leqq \delta) \tag{3.67}$$

because $f_r$ is a decreasing sequence which tends to $\rho > 0$ as $r \to \infty$. Letting $r \to \infty$ in (3.67), and then letting $\delta \to 0$ we deduce

$$R'(0) = q > 1 . \tag{3.68}$$

Next consider arbitrary numbers $a$ and $b$ such that $a < b < \beta$. By (3.60), there exists $n = n(b, \delta)$ such that

$$D_r(rb) < \delta , \qquad (r \geqq n) . \tag{3.69}$$

Let $r$, $s$, $t$ be positive integers such that

$$0 < t \leqq s , \qquad n \leqq r - s , \qquad ra \leqq (r-s)b \leqq (r-t)b . \tag{3.70}$$

Then for $x \leqq ra$

$$D_{r-t}(x) \leqq D_{r-t}(ra) \leqq D_{r-t}[(r-t)b] < \delta . \tag{3.71}$$

Hence, by (3.3), since $G(u) = 0$ for $u < 0$

$$D_{r-t} * G < \delta \qquad (x \leqq ra) ; \tag{3.72}$$

whereupon (3.58) and (3.67) yield

$$D_{r-t+1} = R(D_{r-t} * G) \geqq q_\delta D_{r-t} * G \qquad (x \leqq ra) . \tag{3.73}$$

Hence by (3.3) and (3.9)

$$q_\delta^{t-1} D_{r-t+1} * G_{t-1} \geqq q_\delta^{t} D_{r-t} * G_t \qquad (x \leqq ra) . \tag{3.74}$$

Since (3.74) holds for $t = 1, 2, \cdots, s$, we deduce

$$\begin{aligned} e^{r\psi(a)} &\geqq D_r(ra) \geqq q_\delta^s (D_{r-s} * G_s)(ra) \\ &\geqq q_\delta^s D_{r-s}(ra - sa) G_s(sa) \\ &= q_\delta^s G_s(sa) \exp[(r-s)\psi(a) + o(r-s)] . \end{aligned} \tag{3.75}$$

By (3.22) applied to the convolutive sequence $G_s$, we have

$$\lim_{s\to\infty} s^{-1} \log G_s(sa) = \log J(a) , \tag{3.76}$$

where

$$J(a) = \inf_{\theta \geqq 0} \int_{-\infty}^{\infty} e^{\theta(a-u)} \, dG(u) . \tag{3.77}$$

If we take logarithms of (3.75), divide by $r$ and let $r, s \to \infty$ such that $(r-s)/r \to a/b$, which is consistent with (3.70), we get

$$\psi(a) \geqq \frac{a}{b} \psi(a) + \left(1 - \frac{a}{b}\right) \log [q_\delta J(a)] . \tag{3.78}$$

Letting $\delta \to 0$ in (3.78), we deduce

$$e^{\psi(a)} \geqq qJ(a) \qquad (a < \beta) . \tag{3.79}$$

On the other hand (3.56) and (3.67) give

$$C_{r+1} \leqq (f_r/f_{r+1})qC_r * G\,; \tag{3.80}$$

whence (3.59) provides

$$D_r \leqq C_r \leqq (\textstyle\prod_{s=1}^{r} f_{s-1}/f_s)q^r C_0 * G_r = q^r G_r/f_r \leqq q^r G_r/\rho\,. \tag{3.81}$$

Therefore (3.61) and (3.76) yield

$$\psi(x) \leqq \limsup_{r\to\infty} r^{-1} \log C_r(rx) \leqq \log\,[qJ(x)]\,. \tag{3.82}$$

Combining (3.79) and (3.82) and noting that $\psi(x)$ and $J(x)$ are continuous, we deduce (See Note 4 of Section 11.) that

$$e^{\psi(x)} = qJ(x) \qquad (x \leqq \beta)\,. \tag{3.83}$$

In view of (3.62) and (3.63) we now deduce

$$qJ(\alpha) = 1\,. \tag{3.84}$$

Since $q > 1$, we have $J(\alpha) < 1$; and therefore $J(x)$ is a strictly increasing function at $x = \alpha$. So if $\alpha < \beta$ were true we should have $\psi(\beta) > 0$ from (3.83) in contradiction of $\psi \leqq 0$. Hence

$$\alpha = \beta\,. \tag{3.85}$$

Now (3.59), (3.60), (3.82), and (3.84) provide

$$\begin{aligned} C_r(rx) &\to 1 \quad (x > \alpha) \\ &\to 0 \quad (x < \alpha) \end{aligned} \qquad \text{as } r \to \infty\,. \tag{3.86}$$

So the random variables $x_r/r$ converge to $\alpha$ in probability conditionally on $x_r < \infty$. Finally the contribution to the conditional mean

$$g_r = \int_{-\infty}^{\infty} x\,dC_r(x) \tag{3.87}$$

from the top part of this integral $\int_{r(\alpha+\varepsilon)}^{\infty} x\,dC_r(x)$ is bounded by $\int_{r(\alpha+\varepsilon)}^{\infty} x\,dD_r(x)$, and this latter integral tends to zero as in the proof of the superconvolutive version of the weak law. The contribution to (3.87) from the lower part $\int_{-\infty}^{r(\alpha-\varepsilon)} x\,dC_r(x)$ also tends to zero because of (3.82) and the fact that $x_r \geqq 0$. Hence by (3.42)

$$\alpha = \gamma\,, \tag{3.88}$$

whereupon (3.77) and (3.84) yield and explicit equation for the conditional time-constant $\gamma$. Finally, we have $\gamma < \bar{u}$ because $J(\bar{u}) = 1$. We can sum up our results in this section by stating them as theorems.

THEOREM 1. *If a sequence of distributions $F_r$ in the convolutive semigroup* **D** *is superconvolutive, namely if*

$$F_{r+s}(x) \geqq (F_r * F_s)(x) \qquad \textit{for all } x\,, \tag{3.89}$$

*then the function*

$$K_r(\theta) = \log \int_{-\infty}^{\infty} e^{-\theta x}\, dF_r(x) \tag{3.90}$$

*is a superadditive function of r for each fixed* $\theta \geqq 0$, *and* $\log F_r(rx)$ *is a superadditive function of r for each fixed x. Also the limits*

$$\psi(x) = \lim_{r\to\infty} r^{-1} \log F_r(rx)\,, \qquad K(\theta) = \lim_{r\to\infty} r^{-1} K_r(\theta) \qquad (\theta \geqq 0) \tag{3.91}$$

*satisfy the reciprocal relations*

$$\psi(x) = \inf_{\theta\geqq 0} [K(\theta) + \theta x]\,, \qquad K(\theta) = \sup_x [\psi(x) - \theta x] \qquad (\theta \geqq 0)\,. \tag{3.92}$$

*If, in addition, the distributions* $F_r$ *are all proper, there exists a constant* $\gamma$ *such that*

$$\begin{aligned} F_r(rx) &\to 1 \quad (x > \gamma) \\ &\to 0 \quad (x < \gamma) \qquad \text{as } r \to \infty\,. \end{aligned} \tag{3.93}$$

THEOREM 2. *If* $Q(z)$ *is a concave non-decreasing function defined on* $0 \leqq z \leqq 1$ *such that for some* $\rho$ *satisfying* $0 < \rho \leqq 1$

$$Q(\rho) = \rho \qquad \text{and} \qquad q = Q'(0) > 1\,, \tag{3.94}$$

*and if the sequence of distributions* $F_r$ *is given by*

$$F_{r+1}(x) = Q[(F_r * G)(x)] \qquad (r = 0, 1, 2, \cdots) \tag{3.95}$$

*where* $F_0(x) = 0$ *or* $1$ *according as* $x < 0$ *or* $x > 0$, *and where G is a proper distribution of a nonnegative random variable with a mean* $\bar{u}$, *then the sequence* $F_r$ *is superconvolutive and*

$$\begin{aligned} F_r(rx) &\to \rho \quad (x > \gamma) \\ &\to 0 \quad (x < \gamma) \qquad \text{as } r \to \infty\,, \end{aligned} \tag{3.96}$$

*where* $\gamma$ *is the unique* (See Note 5 of Section 11.) *root of*

$$\inf_{\theta>0} \int_{0-}^{\infty} e^{\theta(\gamma-u)}\, dG(u) = 1/q\,, \qquad \gamma < \bar{u}\,. \tag{3.97}$$

*Moreover, if* $x_r$ *is a random variable with distribution* $F_r$, *then conditionally on* $x_r < \infty$ *the random variable* $x_r/r$ *converges in probability to* $\gamma$ *as* $r \to \infty$, *and*

$$\gamma = \lim_{r\to\infty} E(x_r/r \mid x_r < \infty)\,. \tag{3.98}$$

THEOREM 3. *An independent subadditive process is characterised by a superconvolutive sequence of distributions, but the converse is false in general.*

Theorem 1 is the superconvolutive generalization of two familiar convolutive theorems, namely the weak law of large numbers and Chernoff's theorem [1] on the extreme tails of distribution. Theorem 2 yields an explicit equation (3.97) for the time constant $\gamma$ of the subadditive process for the first-death problem in an age-dependent branching process, when we take $Q(z) = 1 - P(1 - z)$. The discussion in this section exhibits some of the shortcomings of the postulates for subadditive processes in [8] and [10]; and in particular, it suggests that the postulates need widening to include improper distributions. Further, Theorem

1 ought to be extended, if possible, to remove the final clause under which the distributions $F_r$ are all assumed proper. The fact that we are able to prove (3.88) in the course of proving Theorem 2 seems to be a fortunate consequence of being able to work with an explicit recurrence relation (3.95): an analogous (3.88) is also likely to be true in other situations though its proof in general promises to be difficult—it appears, for example, as one of the long-standing conjectures in the theory of self-avoiding walks (see below).

The result (3.97) for the particular case of a branching process is an old unpublished result which I obtained in 1959 in the early days of work on subadditive processes, but the idea of setting it in the more general framework of superconvolutive sequences is new. (See Note 6 of Section 11.)

**4. Eigenshift problems in the convolution semigroup.** We have seen in the previous section that the concave recurrence relation (3.94) yields conclusions which amount to convergence in probability; and it seems likely that corresponding results involving convergence with probability 1 could be established for some suitably imposed underlying probability space. However, computer studies (See Note 8 of Section 11.) of particular cases of (3.94) suggest the truth of a very much stronger conclusion than the almost sure convergence of a sequence like $x_r/r$: they suggest that $x_r - g_r$ converges in distribution, or perhaps the even bolder conjecture that $x_r - \gamma r$ converges in distribution. We shall now look at the consequences of this second conjecture.

The distribution of the random variable $x_r - \gamma r$ is $F_r * H_{-\gamma r}$, where $H$ denotes the step function defined in (3.7). Shift operators of the form $H_\alpha *$ commute with any function like $Q(\cdot)$; so we have

$$(4.1)\qquad \begin{aligned} H_\gamma * (F_{r+1} * H_{-\gamma(r+1)}) &= H_{-\gamma r} * Q(F_r * G) \\ &= Q(F_r * H_{-\gamma r} * G)\,. \end{aligned}$$

Hence, if it is true that

$$(4.2)\qquad F_r * H_{-\gamma r} \to F \qquad \text{as } r \to \infty\,,$$

then this limiting distribution $F$ must satisfy

$$(4.3)\qquad Q(F * G) = H_\gamma * F\,.$$

Now we look at (4.3) in a more general framework. Suppose we are given a transformation $T$ which maps the convolution semigroup $\mathbf{D}$ into itself. We can ask for what values $\gamma$ (if any) does the equation

$$(4.4)\qquad T(F) = H_\gamma * F$$

have a solution $F \in \mathbf{D}$. We call any such solution $F$ an *eigendistribution* corresponding to the *eigenshift* $\gamma$. In the particular case when $T(\cdot) = Q(\cdot * G)$, where $Q$ and $G$ have the properties specified in Theorem 2, we know that $\gamma$ cannot be an eigenshift unless it satisfies (3.96) (or rather, to be more precise, it is the only eigenshift resulting from iterating the recurrence relation with $H_0$ as starting point—conceivably there might be other values of $\gamma$ if we started the iteration from some other original $F_0$). But, if $\gamma$ does satisfy (3.96), is it an eigenshift; and if so, how do we then solve (4.3) for $F$ and to what extent will $F$ be unique; and, finally, is (4.2) true for some eigendistribution $F$ of (4.3)? I do not know the answers, let alone to the corresponding questions for (4.4).

However, we can create special cases in which (4.3) does possess an eigendistribution by the simple device of choosing $F$ and $G$ arbitrarily, and then solving (4.3) for the function $Q(\cdot)$. From several possible examples, I choose the following, which actually began life as a putative counterexample to Theorem 2! Suppose

$$F(x) = 1 - e^{-x} \qquad (x \geqq 0)\,, \tag{4.5}$$

and

$$\begin{aligned} G(x) &= 0 && (x < 0) \\ &= 1 - \tfrac{1}{2}e^{-x} && (x \geqq 0)\,. \end{aligned} \tag{4.6}$$

Then

$$(F * G)(x) = 1 - (1 + \tfrac{1}{2}x)e^{-x} \qquad (x \geqq 0)\,; \tag{4.7}$$

and so, for $\gamma = 0$, we may define $Q$ implicitly by

$$Q[1 - (1 + \tfrac{1}{2}x)e^{-x}] = 1 - e^{-x} \qquad (x \geqq 0)\,. \tag{4.8}$$

It is easy to verify that $Q$ satisfies the conditions of Theorem 2, and that $Q'(0) = 2$. Also we find

$$\int_{0-}^{\infty} e^{\theta(\gamma-u)}\, dG(u) = \frac{e^{\theta\gamma}(1 + \frac{1}{2}\theta)}{1 + \theta} = \varphi(\gamma, \theta)\,, \quad \text{say}\,; \tag{4.9}$$

and, according to Theorem 2,

$$\inf_{\theta \geqq 0} \varphi(\gamma, \theta) = \tfrac{1}{2}\,, \qquad \gamma < \tfrac{1}{2} \tag{4.10}$$

*ought* to have the unique solution $\gamma = 0$. Since $\varphi(\gamma, 0) = 1 > \frac{1}{2}$, we see that $\theta$ must satisfy $\partial\varphi(\theta, \gamma)/\partial\theta = 0$, which leads to

$$\gamma = 1/(1 + \theta)(2 + \theta)\,, \tag{4.11}$$

and (at first sight) it looks as though $\gamma = 0$ is incompatible with (4.11). However, solving (4.11) for $\theta > 0$ and substituting the result into (4.10), we find

$$\log 2 - \gamma\left[\tfrac{3}{2} - \frac{1}{2}\left(1 + \frac{4}{\gamma}\right)^{\frac{1}{2}}\right] + \log\left\{\frac{1 + (1 + 4/\gamma)^{\frac{1}{2}}}{-2 + 2(1 + 4/\gamma)^{\frac{1}{2}}}\right\} = 0 \tag{4.12}$$

and this *does* have the solution $\gamma = 0$, and moreover this can be proved to be the only solution satisfying $\gamma < \frac{1}{2}$. So, after all, this is a verification of Theorem 2, and not a counterexample: in fact (4.11) is satisfied with $\theta = \infty$.

With appropriate choices of $G$, one can convert the problem of solving (4.3) into an eigenshift problem for differential-difference equations. For example, with $G(x) = 1 - e^{-x}$ $(x \geqq 0)$ and $y(x) = Q^{-1}[F(x)]$, and $\rho = Q(\rho)$, we obtain

$$y'(x) + y(x) = Q[y(x+\gamma)]\,, \qquad y(-\infty) = 0,\, y(+\infty) = \rho\,; \tag{4.13}$$

and Theorem 2 suggests that (4.13) cannot have a non-decreasing solution unless $Q'(0)\gamma e^{1-\gamma} = 1$. Whether it actually has such a solution under this condition requires further investigation.

**5. Bethe approximations to first-passage percolation.** In studies of cooperative phenomena in physics, it is a great mathematical simplification (though admittedly somewhat of a distortion of the physical reality) to replace the interatomic bound structure (of, say, an atomic lattice in solid-state physics, or of a coagulating gel in polymer chemistry) by a tree structure. The independence of the branches of the tree removes difficult and embarassing correlation terms. Physicists call such a replacement a *Bethe approximation*.

We can use the same device to find bounds in first-passage percolation by stripping out the tree of self-avoiding walks on a linear graph. In this situation, we start with a linear graph $\Gamma$, possibly directed, on which we have a distinguished node $N$ together with a set of nodes $B_t$ depending on an integer $t$. Each edge of $\Gamma$ carries a nonnegative random variable $U$ with common distribution $G$, and all the $U$'s are independent. Each walk along $\Gamma$ from $N$ to $B_t$ has a passage time equal to the sum of all $U$ traversed on the walk; and the first-passage time form $N$ to $B_t$ is the infimum of the passage-times taken over all such walks. Clearly we need only consider self-avoiding walks, namely walks which never traverse any edge more than once. We now construct a tree $\Delta$, having $N$ for its root. Each walk $\omega_\Delta$ on $\Delta$ from $N$ to a terminal node of $\Delta$ corresponds to a self-avoiding walk $\omega_\Gamma$ from $N$ to $B_t$ on $\Gamma$ and vice versa. The nodes of $\Delta$ which are visited by some $\omega_\Delta$ correspond to nodes on $\Gamma$ visited (in the same order) by the corresponding $\omega_\Gamma$; and the structure of $\Delta$ is determined by the following rule: if $\omega_\Delta'$ and $\omega_\Delta''$ coincide from $N$ to some node $N_\Delta'$ and never thereafter ($\Delta$ being a tree), then the corresponding $\omega_\Gamma'$ and $\omega_\Gamma''$ coincide from $N$ to the corresponding $N_\Gamma'$ whereat they immediately separate (though they may coalesce once again on $\Gamma$ a few steps later). Now suppose $B_t$ consists of all nodes on $\Gamma$ which can be reached from $N$ by some self-avoiding walk $\omega_\Gamma$ of exactly $r$ steps. The corresponding tree $\Delta$ will have $r$ generations with $N$ as progenitor. It is fairly simple to prove that the expected time to reach the $r$th generation on $\Delta$ does not exceed the expected time to reach $B_r$ on $\Gamma$. Consequently the time-constant $\gamma_\Delta$ for the first-death problem of the age-dependent branching process on $\Delta$ is a lower bound for the time-constant $\gamma_\Gamma$ of the percolation process on $\Gamma$ (see [4] and [15] for an example of this procedure).

At first sight, this argument would seem to require that the tree-structure $\Delta$ should be one which can be generated by a branching process with some generating function $P(z)$; and it is by no means certain (or even likely) that $\Delta$ will be of this special form. At any rate, far too little is known about the asymptotic properties of the number of self-avoiding walks to afford a reasonable hope of being able to prove such a fact. On the other hand, the final conclusion (that $\gamma_\Gamma \geqq \gamma_\Delta$) can be proved rigorously [4]; and this is because we only need the easy half of the argument in Section 3 to prove $\gamma_\Gamma \geqq \gamma_\Delta$: the more difficult half of the argument in Section 3 would not be needed unless we wanted to prove that a Bethe approximation was incapable of providing a sharper lower bound than this. But, in turn, these considerations raise two questions.

First, is it possible to get a sharper lower bound to $\gamma_\Gamma$ by some sort of truncated Bethe approximation? Speaking very roughly, if two self-avoiding walks on $\Gamma$ coincide from $N$ to $A$ and then separate at $A$ but meet once again at a subsequent node $A'$, should it not be possible to delete from *one* of the corresponding walks on $\Delta$ all steps after $A'$? The answer would clearly be "Yes", *provided* we knew which of the two walks we ought to select for truncation. But what happens if, not knowing which to select, we select one at random?

Second, if we can manage to get a sharp lower bound to $\gamma_\Gamma$ in this sort of fashion, could we then prove that it is sharp? Effectively, we should need to prove Theorems 1 and 2 under much less stringent postulates; and this draws attention to how unsatisfactory these postulates are at present. The theorems for subadditive processes have very weak conclusions: they merely say that some sequence tends to a limit in some sense (in probability, or with probability 1, etc.). For such feeble conclusions should it really be necessary to impose such draconian postulates as stationarity, or subadditivity, or superconvolutivity? It ought to be enough to postulate that the sequence approaches stationarity (or subadditivity, etc.) as we approach infinity. This sort of thing works for ordinary subadditive functions. For example, we can relax the ordinary subadditive inequality

$$x_{r+s} \leqq x_r + x_s \tag{5.1}$$

to the weaker generalized form

$$x_{r+s} \leqq x_r + x_s + y_{r+s}\,, \tag{5.2}$$

and yet still reach the same conclusion

$$x_r/r \to \gamma \tag{5.3}$$

provided [2] that $y_r$ is non-decreasing and satisfies

$$\sum y_r/r^2 < \infty\,. \tag{5.4}$$

What is the stochastic analogue to (5.2) and (5.4)? For example, consider a weakened superconvolutive inequality of the form

$$F_{r+s} \geqq F_r * F_s * H_{h(r+s)}\,, \tag{5.5}$$

where the non-decreasing function $h(r)$ satisfies

$$\sum h(r)/r^2 < \infty\,. \tag{5.6}$$

Then (because we are merely shfting the origin by amounts which satisfy a relation like (5.2)) an easy combination of Theorem 1 and [2] shows that the essential conclusions of Theorem 1 remain true. How far can we generalize this result with some more elaborate distribution in place of $H_{h(r+s)}$ in (5.5)? (See Note 7 of Section 11.)

**6. Self-avoiding walks.** Analysis of self-avoiding walks has hitherto depended upon combinatorial arguments because we have lacked a probabilistic framework for them. However, it now seems that the ideas in Section 3 can provide a satisfactory framework. In the first place, although Section 3 envisages scalar random variables $x$, we can easily extend it to random vectors $\mathbf{x}$, with the usual convention that $\mathbf{x} \leqq \mathbf{y}$ if and only if the coordinates of $\mathbf{x}$ do not exceed the corresponding coordinates of $\mathbf{y}$. For simplicity, let us consider self-avoiding walks on the square lattice (the nodes of $\Gamma$ are the points with integer coordinates in the Euclidean plane, and the edges of $\Gamma$ are undirected and join pairs of nodes unit distance apart). Consider all $n$-stepped walks starting from the origin, and assign probability $4^{-n}$ to each of them. Let $\mathbf{x}$ be a vector associated with each walk as follows: if the walk is self-avoiding, then $\mathbf{x}$ is the position vector of the node reached at the end of the walk; whereas, if the walk is not self-avoiding, then $\mathbf{x} = \infty = (+\infty, +\infty)$. Let $F_n(\mathbf{x})$ denote the cumulative distribution function of this improper distribution. It is easy to see that the sequence $F_n$ is subconvolutive

$$F_{r+s} \leqq F_r * F_s \,. \tag{6.1}$$

So there is a $\psi$-function attached to the sequence:

$$\psi(\mathbf{x}) = \lim_{r\to\infty} r^{-1} \log F_r(r\mathbf{x}) \,, \tag{6.2}$$

and

$$\psi(\mathbf{x}) = \kappa - \log 4 \qquad \text{for } \mathbf{x} \geqq \mathbf{0} \,, \tag{6.3}$$

where $\kappa$ is the connective constant of the lattice [3]. A long-standing unproved conjecture about self-avoiding walks is that the distance between the two ends of an $r$-stepped self-avoiding walk is $o(r)$ as $r \to \infty$ for "almost all" such walks. More precisely, this conjecture may be formulated as

$$\psi(\mathbf{x}) < \kappa - \log 4 \qquad \text{for } \mathbf{x} < \mathbf{0} \,. \tag{6.4}$$

This conjecture may be compared with the analogous result (3.88) proved for concave recurrence relations.

Self-avoiding walks give rise to an exponentially improper sequence of subconvolutive distributions, whereas in Section 3 we have considered superconvolutive sequences. When the sequence is proper or at most boundedly improper, we can infer results about subconvolutive sequences from the corresponding superconvolutive results by reversing the signs of all the random variables. To do this for an exponentially improper sequence would however throw away the baby with the bath water; so a separate treatment is needed for expontially improper subconvolutive sequences. I imagine this will go through without difficulty, but I have not examined the issue in detail.

**7. Maximal solutions of the generalized subconvolutive inequality.** Suppose we are given a sequence of distributions $G_r$ (not necessarily successive convolutions of a distribution $G$), and we consider a sequence $F_r$ satisfying

$$F_{r+s} \leqq F_r * F_s * G_{r+s} . \tag{7.1}$$

The maximal solution of (7.1) is clearly given by the recurrence

$$F_n(x) = \min_{1<r\leqq\frac{1}{2}n} (F_r * F_{n-r} * G_n)(x) \qquad (n = 2, 3, \cdots) , \tag{7.2}$$

when $F_1$ is prescribed. Suppose that $r = \rho(n, x)$ is the (or a) value of $r$ which achieves the minimum on the right of (7.2). What can we say about the function $\rho$? How does it depend upon $F_1$ and the sequence $G_r$, and what are the properties of the resulting sequence (7.2)? These questions are the obvious analogues of the corresponding questions for ordinary subadditive sequences [6], and are clearly germane to any study of the inequality (7.1).

**8. Rates of convergence.** Kingman [10] asks how fast a subadditive process converges. This question can be split into two parts, and a complete answer given to one part. First, we can ask how fast does the sequence of means $g_r/r$ converge to $\gamma$; and second, we can ask what is the order of magnitude of the deviations $x_{0r} - g_r$: for example, under what conditions is this deviation $O(r^{\frac{1}{2}} \log\log r)$?

As to first question, $g_r$ can be any subadditive sequence; and hence $g_r/r$ may tend to $\gamma$ arbitrarily slowly. For, in particular, $g_r$ is subadditive when $g_r/r$ is an arbitrary decreasing sequence, because

$$g_{r+s} = r\frac{g_{r+s}}{r+s} + s\frac{g_{r+s}}{r+s} \leqq r\frac{g_r}{r} + s\frac{g_s}{s} = g_r + g_s . \tag{8.1}$$

In the absence of additional information in the problem, this gives a complete (and completely disappointing) answer to the first question. In most applications, however, there is additional information; and the usual technique is to show that a subadditive function $g_r$ can be made superadditive by modifying it with some small correction. Another way of looking at this technique is to say that $g_r$ satisfies both the subadditive relation (8.1) and also a suitable generalized superadditive relation

$$g_{r+s} \geqq g_r + g_s + h_{r+s} . \tag{8.2}$$

This highlights the importance of studying inequalities like (7.1). For an example of how these techniques work, see [3] and [9].

The second question has not yet received any satisfactory answer. We note, however, that certain subadditive processes appear to have deviations which are of much smaller order than the usual $O(r^{\frac{1}{2}} \log\log r)$: for example, the concave recurrence relations discussed in Section 3, and Ulam's problem [5].

The considerations in this section show that the existing postulates for subadditive processes are too general, in the sense that we need additional restrictions if we are to get good descriptions of the phenomena which arise in various practical problems.

**9. Questions of structure.** Kingman [10] asks whether a subadditive process can always be represented as the supremum of additive processes. The answer is no for ordinary subadditive sequences, and *a fortiori* no for subadditive processes. The subadditive sequence

$$x_r = 1 \quad r \text{ odd} \qquad \qquad (9.1)$$
$$= 0 \quad r \text{ even}$$

is not the supremum of a family of additive sequences.

**10. Multidimensional processes.** The existing postulates of subadditive processes contemplate random variables indexed by a pair of integers, each of which is of course scalar. On the other hand, many of the most interesting physical applications require the random variable to be indexed by a pair of vectors. Suppose, then, that $\mathbf{r}$, $\mathbf{s}$, $\mathbf{t}$ are vectors (all of the same finite dimensionality) with integer coordinates, and that $x_{\mathbf{rs}}$ is a random variable indexed by $\mathbf{r}$ and $\mathbf{s}$. We can postulate a subadditive inequality of the form

$$x_{\mathbf{rt}} \leqq x_{\mathbf{rs}} + x_{\mathbf{st}} \qquad (\mathbf{r} \leqq \mathbf{s} \leqq \mathbf{t}) \,. \qquad (10.1)$$

The other postulates can be straightforward analogues of $S_2$ (or $S_2'$) and $S_3$. We write $\mathbf{r} \to \infty$ to denote that the coordinates of $\mathbf{r}$ tend to infinity independently; and we define $\prod \mathbf{r}$ to be the product of the coordinates of $\mathbf{r}$. It is true that

$$x_{0\mathbf{r}}/\prod \mathbf{r} \to \gamma \qquad \text{as } \mathbf{r} \to \infty \qquad (10.2)$$

in some sense (ergodically, in probability, etc.)?

To fix the ideas, let us consider a practical example from the statistical theory of liquid-vapor equilibrium [11], [12], [13], [14]. Suppose that $\mathbf{r}$, $\mathbf{s}$ are 3-dimensional vectors; and let $V_{\mathbf{rs}}$ denote the rectangular parallelopiped, with edges parallel to the coordinate axes of 3-dimensional Euclidean space, and a pair of opposite vertices at the points with coordinates $\mathbf{r}$ and $\mathbf{s}$. Consider a Poisson process of density $\lambda$ on the Euclidean space, let $\alpha$ be a fixed positive number, and suppose there is a sphere of radius $\alpha$ centered at each point of the Poisson process. Let $S_{\mathbf{rs}}$ denote the set of such spheres which lie wholly within $V_{\mathbf{rs}}$; and let $x_{\mathbf{rs}}$ denote the total volume covered by $S_{\mathbf{rs}}$ (counting each element of volume once only, irrespective of the number of spheres covering it). Clearly (10.1) is satisfied and $x_{\mathbf{rs}}$ is a multidimensional subadditive process. By *ad hoc* arguments [12] one can show that (10.2) holds for this example, and one can calculate the first five cumulants of the distribution of $x_{0\mathbf{r}}$; and all five are asymptotically proportional to $\prod \mathbf{r}$.

How does one extend subadditive theory to cover situations of this type?

More generally, suppose that $x(E)$ is a random variable defined for each subset $E$ of a vector space; and let

$$x(E_1 \cup E_2) \leqq x(E_1) + x(E_2) \tag{10.3}$$

for all suitably restricted $E_1$, $E_2$ (e.g., Borel sets). Under what conditions does there exist a measure $\mu(E)$ such that

$$x(E)/\mu(E) \to \gamma \qquad \text{as } \mu(E) \to \infty \tag{10.4}$$

with some suitable definition of convergence? The answers to questions like this are of great importance in studying cooperative phenomena in physics and chemistry; and they depend greatly upon what one means by $\mu(E) \to \infty$. Roughly speaking, the desired results require that $E$ should become infinite in all directions (and should not be a long needle of large volume but fixed cross-section) and that the "boundary" of $E$ should be "reasonably smooth." Even a single tiny irregularity on the boundary can catastrophically change the results. For example consider the dimer problem [7] in solid-state chemistry. In the two-dimensional version of this one has a region consisting of a whole number of squares of an infinite chessboard together with a set of dominoes, each of which will cover a pair of neighboring squares. The problem is to determine (for large regions) the number of ways of covering the region with dominoes such that each square is covered by one and only one domino. If the region is a rectangle of area $A$ with an even number of squares, then the number of ways turns out to be $\exp[\sigma A + o(A)]$ as $A \to \infty$, where

$$\sigma = \frac{1}{2\pi} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^2} \,. \tag{10.5}$$

But if the region has an irregularity on its boundary, which prevents the number of black squares equalling the number of white squares in the region, then obviously the number of ways is zero because each domino must cover one white and one black square. In the dimer problem, the number of ways of covering a region satisfies a supermultiplicative relation analogous to (10.3). Similarly, in the treatment of Ulam's problem via Poisson processes [5], one encounters (10.3). There are many other examples [16]; but enough has been said to indicate the desirability of extending subadditive processes and their postulates to the multidimensional case.

**11. Notes.** The following notes have been added to the original discussion given above to bring things up to date: Note 1, 2, 3, and 6 to deal with points raised by the referee; Note 7 to cover work arising from Kesten's contribution to the discussion on Kingman's paper [10]; Note 8 to cover some recent computations at Berkeley performed by Dr. Robert Davies, of D.S.I.R., Wellington, New Zealand and by Mr. Bob Traxler, Statistics Department, Berkeley; and Notes 4 and 5 to remedy an oversight in Theorem 2 revealed by the computer in the course of these calculations. Note 8 is written jointly with Dr. Davies and Mr. Traxler.

NOTE 1. I do not know of any previously published material on the first-death problem. In our original work on percolation (in 1954 on the design of gas-masks for coal miners) Broadbent and I noticed that branching processes were the simplest special case of percolation processes, and critical branching was the inspiration for the concept of the critical percolation probability. Hence in 1958, when extending ordinary percolation to first-passage percolation, I naturally considered the simplest case—namely the first-death problem—and the original computations on it were done on Illiac I in 1959. Nor do I know how much this theory has been used for applications, though I have used it myself as a rough model for detonation fronts in initiators of some explosive devices. Maybe the theory might apply to the onset of cancer in multistage carcinogenesis. There is of course a corresponding last-death problem, and the time-lapse between first and last deaths gives an epoch for the generation, which might be useful in palaeological dating of fossil records.

NOTE 2. The extra work is now supplied by Kesten's lemma in the discussion of Kingman's paper and its extension in Note 7 below.

NOTE 3. If $\phi'$ is the right-hand derivative of $\phi$ and $x \leqq y$ and $x < 0$, then

$$\frac{\phi(x)}{x} \geqq \frac{\phi(y)}{x} + \left(1 - \frac{y}{x}\right)\phi'(y) .$$

Letting $x \to -\infty$, we get $0 = \omega \geqq \phi'(y) \geqq 0$, and hence

$$\begin{aligned} \phi(x) &= \phi(0) + \textstyle\int_0^x \phi'(y)\,dy \\ &= \phi(0) . \end{aligned}$$

NOTE 4. We need a gloss here to cover one possible exception. The function $\phi(x)$ is concave, non-decreasing, and bounded above by zero. Hence $e^{\phi(x)}$ is continuous everywhere except perhaps at a single point $\xi$; and if there is a discontinuity at $\xi$ then $e^{\phi(x)} = 0$ for $x < \xi$. The function $J(x)$ arises from the convolutive sequence $G_r$ in the same way as $e^{\phi(x)}$ arises from the superconvolutive sequence $F_r$; and so $J(x)$ can have at most one discontinuity, at $\eta$ say; and if this discontinuity exists, then $J(x) = 0$ for $x < \eta$. Now (3.82) shows that $e^{\phi}(x) \leqq qJ(x)$ for all $x$; while (3.62) shows that $e^{\phi(x)} = 1$ for $x > \alpha$. Hence neither of the functions $e^{\phi(x)}$ and $J(x)$ can be discontinuous in the open interval $x > \alpha$. Also $e^{\phi(x)} = qJ(x)$ for $x < \beta$ by (3.79) and (3.82). So if $\alpha < \beta$, we get (3.83) as before, and (3.84) holds in the modified form $qJ(\alpha + 0) = 1$ and $J(x)$ is a strictly increasing function in some right-hand neighborhood of $\alpha$ and this leads to (3.85) as before. If either side of $e^{\phi(x)} = qJ(x)$ is continuous in the open interval $x < \beta$, then both sides are discontinuous at the same point $\xi = \eta < \beta$; and, by working in the half-open interval $\xi < x \leqq \beta$ we deduce that $qJ(\beta) = 1$. The only remaining case to consider is when either $e^{\phi(x)}$ or $J(x)$ is discontinuous at $x = \beta$ itself. In this case we find $e^{\phi(x)} = qJ(x) = 0$ for $x < \beta$, while $qJ(x) \geqq e^{\phi(x)} = 1$ for $x > \beta$; so both functions must be discontinuous at $x = \beta$, and we then have the pair of relations

$$qJ(\beta - 0) = 0\,, \qquad qJ(\beta + 0) \geqq 1$$

in place of the previous relation $qJ(\beta) = 1$.

NOTE 5. Equation (3.97) may fail to have a solution in one exceptional case. In that case, if $J(\gamma)$ denotes the left-hand side of (3.97), then $\gamma$ is the unique solution of $0 = J(\gamma - 0) < 1/q \leqq J(\gamma + 0)$. See Note 4 above. If $q = \infty$, we choose the "largest possible" solution of $J(\gamma) = 0$.

NOTE 6. In the branching process case, $\gamma$ has the following rough interpretation. If the $r$th generation is not void, we should expect it to have roughly $q^r$ members. Any one of these members will die at a time (after the tree starts) equal to the sum of $r$ independent observations from the distribution $G$. The times for different members of the $r$th generation will be dependent. However, if we ignore this dependence, we still get the same asymptotic value of $\gamma$. In other words, $\gamma r + o(r)$ is the expected value of the least of $q^r$ independent sums, each sum being the sum of $r$ independent identically distributed variables from $G$. The dependence which actually occurs, strongly near the top of the tree and more weakly as we descend the tree, is asymptotically "forgotten."

NOTE 7. Some answers to these questions are given in the following extension of Kesten's contribution [10], which I read after writing the main text of this paper.

LEMMA. *Let* $X_s$ $(s = 1, 2, \cdots)$ *be a given sequence of real random variables with distribution functions* $F_s$ *and finite second moments* $(EX_s^2 < \infty)$. *Suppose that there exists an absolute constant* $\beta$ *(independent of* $s$*) such that*

$$E[(X_s + s\beta)^-]^2 \leqq A_s^2\,, \tag{11.1}$$

*where* $\{A_s\}$ *is a real sequence satisfying*

$$0 \leqq A_1 \leqq A_2 \leqq \cdots \qquad \text{and} \qquad \textstyle\sum_{s=1}^{\infty} A_s/s^2 < \infty\,. \tag{11.2}$$

*Suppose that we can also find (for each pair of positive integers* $s$, $t$*) a pair of random variables* $X'_{s,t}$ *and* $Y_{s,t}$ *with the properties*:

(i) $EY_{s,t}^2 \leqq A_{s+t}^2$;

(ii) $X'_{s,t}$ *has the distribution function* $F_t$;

(iii) *the correlation coefficient* $\rho_{s,t}$ *between* $X_s$ *and* $X'_{s,t}$ *satisfies* $\rho_{s,t} \leqq \delta < 1$ *for some absolute constant* $\delta$ *(independent of* $s$ *and* $t$*); and*

(iv) $F_{s+t} \geqq G_{s,t}$ *where* $G_{s,t}$ *is the distribution function of* $X_s + X'_{s,t} + Y_{s,t}$.

*Let* $s(n) = 2^n m$, *where* $m$ *and* $n$ *are positive integers. Then there exists an absolute constant* $\gamma$ *(independent of* $m$*) such that* $X_{s(n)}/s(n) \to \gamma$ *almost surely as* $n \to \infty$ *for each fixed* $m$. *Finally* $X_s/s \to \gamma$ *in quadratic mean (and hence probability) as* $s \to \infty$.

REMARK 1. The form of condition (11.1) allows some flexibility in applications because we can choose $\beta$ suitably. However, if we write $X_s - s\beta$ and $X'_{s,t} - t\beta$

in place of $X_s$ and $X'_{s,t}$, we see that there is no loss of generality in proving the lemma under the assumption $\beta = 0$, in which case (11.1) reduces to

$$E(X_s^-)^2 \leqq A_s^2 . \tag{11.1'}$$

REMARK 2. The additional assumption that $\{X_s\}_{s=1,2,\dots}$ be almost surely a monotone sequence will entail in the usual way the stronger conclusion that $X_s/s \to \gamma$ almost surely as $s \to \infty$.

REMARK 3. Our lemma generalizes Kesten's lemma [10]: he considers the case when $A_s$ is a constant $C$ and when $X'_{s,t}$ is independent of $s$ and of $\{X_s\}_{s=1,2,\dots}$. It is also a stochastic generalization of the following real variable theorem [2], to which it reduces when $X_s = B_s$ almost surely. Moreover, this theorem shows that condition (11.2) in the lemma cannot be weakened.

THEOREM 4. *Let $\{A_s\}_{s=1,2,\dots}$ and $\{B_s\}_{s=1,2,\dots}$ be real sequences such that $\{A_s\}$ is non-decreasing and such that*

$$B_{s+t} \leqq B_s + B_t + A_{s+t} . \tag{11.3}$$

*Then the convergence of $\sum_{s=1}^{\infty} A_s/s^2$ is a sufficient condition that $B_s/s$ shall tend to a limit (which may be $-\infty$) as $s \to \infty$. It is also a necessary condition in the sense that, if $\sum A_s/s^2$ is any given divergent sequence, we can always find a corresponding solution $\{B_s\}$ of (11.3) such that $B_s/s$ fails to converge to any (finite or infinite) limit as $s \to \infty$.*

PROOF OF THE LEMMA. Define

$$B_s = EX_s , \qquad C_s = \surd EX_s^2 , \qquad D_s = C_s^2 - B_s^2 , \tag{11.4}$$

where (here and later) $\surd$ denotes the positive square root. We have

$$(EY_{s,t})^2 \leqq EY_{s,t}^2 E1 \leqq A_{s+t}^2 \tag{11.5}$$

by the Schwarz inequality and condition (i). By conditions (iv) and (ii)

$$B_{s+t} = EX_{s+t} \leqq E(X_s + X'_{s,t} + Y_{s,t}) \leqq B_s + B_t + A_{s+t} . \tag{11.6}$$

So the Theorem above exhibits a constant $\gamma$ such that

$$\lim_{s\to\infty} B_s/s = \gamma \qquad (-\infty \leqq \gamma < \infty) . \tag{11.7}$$

By (11.1′) and condition (iv), together with Schwarz inequalities like $2EX_sX'_{s,t} \leqq 2[(EX_s^2)^{\frac12}][(EX'^2_{s,t})^{\frac12}]$, we have

$$\begin{aligned} C_{s+t}^2 &= E(X_{s+t}^+)^2 + E(X_{s+t}^-)^2 \leqq E(X_{s+t}^+)^2 + A_{s+t}^2 \\ &\leqq E(X_s + X'_{s,t} + Y_{s,t})^2 + A_{s+t}^2 \\ &\leqq ((EX_s^2)^{\frac12} + (EX'^2_{s,t})^{\frac12} + (EY^2_{s,t})^{\frac12})^2 + A_{s+t}^2 \\ &\leqq (C_s + C_t + A_{s+t})^2 + A_{s+t}^2 \leqq (C_s + C_t + 2A_{s+t})^2 . \end{aligned} \tag{11.8}$$

Hence

$$0 \leqq C_{s+t} \leqq C_s + C_t + 2A_{s+t} , \tag{11.9}$$

and the Theorem exhibits a constant $\theta$ such that

$$\lim_{s\to\infty} C_s/s = \theta \qquad (0 \leqq \theta < \infty) . \tag{11.10}$$

However $B_s^2/s^2 \leqq C_s^2/s^2$, because $D_s$ is nonnegative (variance of $X_s$). So (11.7) and (11.10) show that $\gamma^2 \leqq \theta^2 < \infty$. Hence $\gamma > -\infty$. By (11.4), (11.8), and condition (iii)

$$\begin{aligned} D_{s+t} + B_{s+t}^2 &= C_{s+t}^2 \leqq E(X_s + X'_{s,t} + Y_{s,t})^2 + A_{s+t}^2 \\ &\leqq E(X_s + X'_{s,t})^2 + 2EY_{s,t}(X_s + X'_{s,t}) + 2A_{s+t}^2 \\ &\leqq E(X_s + X'_{s,t})^2 + 2(EY_{s,t}^2)^{\frac{1}{2}}[E(X_s + X'_{s,t})^2]^{\frac{1}{2}} + 2A_{s+t}^2 \\ &\leqq E(X_s + X'_{s,t})^2 + 2A_{s+t}(C_s^2 + 2C_sC_t + C_t^2)^{\frac{1}{2}} + 2A_{s+t}^2 \\ &= D_s + 2\rho_{s,t}(D_sD_t)^{\frac{1}{2}} + D_t + (B_s + B_t)^2 \\ &\qquad + 2A_{s+t}(C_s + C_t + A_{s+t}) \\ &\leqq D_s + 2\delta(D_sD_t)^{\frac{1}{2}} + D_t + (B_s + B_t)^2 \\ &\qquad + 2A_{s+t}(C_s + C_t + A_{s+t}) . \end{aligned} \tag{11.11}$$

However (2) shows that $A_s = o(s)$ as $s \to \infty$; so (11.10) guarantees the existence of a constant $M$ such that

$$C_s + C_t + A_{s+t} \leqq M(s + t) . \tag{11.12}$$

Putting $s = t = s(n-1) = 2^{n-1}m$ in (11.11) and writing

$$\gamma_n = B_{s(n)}/s(n) , \tag{11.13}$$

we find from (11.11) and (11.12)

$$\frac{D_{s(n)}}{[s(n)]^2} \leqq \frac{(1+\delta)}{2} \frac{D_{s(n-1)}}{[s(n-1)]^2} + \gamma_{n-1}^2 - \gamma_n^2 + 2M\frac{A_{s(n)}}{s(n)} . \tag{11.14}$$

Since $A_s$ is non-decreasing in $s$, we have for any integer $N \geqq 2$

$$\begin{aligned} \sum_{n=2}^N \frac{A_{s(n)}}{s(n)} &\leqq \sum_{n=2}^\infty \frac{A_{s(n)}}{s(n)} = 2\sum_{n=2}^\infty A_{s(n)} \sum_{s=s(n)}^{s(n+1)-1} \frac{1}{s(s+1)} \\ &\leqq 2\sum_{n=2}^\infty \sum_{s=s(n)}^{s(n+1)-1} \frac{A_s}{s(s+1)} \leqq 2\sum_{s=s(2)}^\infty \frac{A_s}{s(s+1)} \\ &\leqq 2\sum_{s=1}^\infty \frac{A_s}{s^2} < \infty . \end{aligned} \tag{11.15}$$

Also

$$\frac{D_{s(1)}}{[s(1)]^2} = \frac{D_{2m}}{(2m)^2} \leqq \left(\frac{C_{2m}}{2m}\right)^2 \leqq \left(\frac{C_{2m}}{2m}\right)^2 + \frac{1+\delta}{2}\frac{D_{s(N)}}{[s(N)]^2} . \tag{11.16}$$

Summing (11.14) from $n = 2$ to $n = N$, and adding (11.15) and (11.16) to the result, we find

$$\sum_{n=1}^N \frac{D_{s(n)}}{[s(n)]^2} \leqq \frac{1+\delta}{2}\sum_{n=1}^N \frac{D_{s(n)}}{[s(n)]^2} + \left(\frac{C_{2m}}{2m}\right)^2 + \gamma_0^2 - \gamma_N^2 + 4M\sum_{s=1}^\infty \frac{A_s}{s^2} \tag{11.17}$$

and hence

$$\sum_{n=1}^{N} \frac{D_{s(n)}}{[s(n)]^2} \leqq \frac{2}{1-\delta}\left[\left(\frac{C_{2m}}{2m}\right)^2 + \left(\frac{B_m}{m}\right)^2 + 4M \sum_{s=1}^{\infty} \frac{A_s}{s^2}\right]. \tag{11.18}$$

The right-hand side of (11.18) is independent of $N$, and the left-hand side consists of nonnegative terms. Hence

$$\sum_{n=1}^{\infty} D_{s(n)}/[s(n)]^2 < \infty ; \tag{11.19}$$

and the Chebyshev inequality now implies that for every $\delta > 0$

$$\sum_{n=1}^{\infty} P\left[\left|\frac{X_{s(n)}}{s(n)} - \frac{B_{s(n)}}{s(n)}\right| > \varepsilon\right] < \infty . \tag{11.20}$$

So the Borel–Cantelli lemma ensures that

$$\frac{X_{s(n)}}{s(n)} - \frac{B_{s(n)}}{s(n)} \to 0 \tag{11.21}$$

almost surely as $n \to \infty$. Combination of (11.7) and (11.21) implies that $X_{s(n)}/s(n) \to \gamma$ almost surely as $n \to \infty$.

Finally we have

$$\theta^2 - \gamma^2 = \lim_{s\to\infty} D_s/s^2 = 0 . \tag{11.22}$$

Indeed, the existence of the limit in (11.22) is guaranteed by (11.4), (11.7), and (11.10); and, if its value were not zero, the series in (11.19) would not converge. Hence $X_s/s \to \gamma$ in quadratic mean as $s \to \infty$.

NOTE 8. We have done some computations of the iteration $F_{n+1} = Q(F_n * G)$, with various $Q$ and $G$. Writing $\mu_n$ and $\sigma_n{}^2$ for the mean and variance of $F_n$, we may ask

(i) how does $\mu_n - n\gamma = o(n)$ behave?
(ii) does $\mu_{n+1} - \mu_n \to \gamma$?
(iii) is $\sigma_n$ bounded?

For simplicity, we confined the calculations to discrete distributions on the nonnegative integers. If $\gamma$ is not an integer, the centered version of $F_n$ cannot converge in distribution, but one would expect it to reflect the behavior of the continuous case. Table 1 shows the results for

$$\begin{aligned} G(u) &= 0 && (u < 0) \\ &= p && (0 \leqq u < 1) \\ &= 1 && (u \geqq 1) \end{aligned}$$

and $Q(x) = 1 - (1 - x)^2$, corresponding to a branching process with exactly 2 offspring for each parent.

TABLE 1

| $p$ | $\gamma$ | $n$ | $\mu_n$ | $\mu_n - n\gamma$ | $\mu_n/n$ | $\mu_n - \mu_{n-1}$ | $\sigma_n^2$ |
|---|---|---|---|---|---|---|---|
| 0.25 | 0.1893 | 1 | .56 | .37 | .563 | .563 | .25 |
| | | 2 | .97 | .59 | .486 | .409 | .38 |
| | | 5 | 1.95 | 1.01 | .341 | .303 | .57 |
| | | 10 | 3.28 | 1.39 | .328 | .249 | .70 |
| | | 20 | 5.58 | 1.79 | .279 | .219 | .79 |
| | | 50 | 11.77 | 2.31 | .235 | .200 | .85 |
| | | 100 | 21.62 | 2.69 | .216 | .195 | .87 |
| 0.50 | 0.0 | 1 | .25 | .25 | .250 | .250 | .19 |
| | | 2 | .41 | .41 | .203 | .156 | .27 |
| | | 5 | .57 | .57 | .140 | .078 | .41 |
| | | 10 | .96 | .96 | .096 | .041 | .49 |
| | | 20 | 1.22 | 1.22 | .061 | .019 | .55 |
| | | 50 | 1.55 | 1.55 | .031 | .007 | .59 |
| | | 100 | 1.77 | 1.77 | .018 | .003 | .60 |
| 0.75 | 0.0 | 1 | .06 | .063 | .063 | .063 | .06 |
| | | 2 | .09 | .088 | .044 | .026 | .08 |
| | | 5 | .11 | .109 | .022 | .003 | .10 |
| | | 10 | .11 | .112 | .011 | .000 | .10 |
| | | 20 | .11 | .112 | .006 | .000 | .10 |
| | | 50 | .11 | .112 | .002 | .000 | .10 |
| | | 100 | .11 | .112 | .001 | .000 | .10 |

For $p = 0.75$, both $\mu_n - n\gamma$ and $\sigma_n$ tend to constants. But, for $p = 0.50$ and $p = 0.25$, the quantity $\mu_n - n\gamma$ seems to be increasing to infinity rather slowly (perhaps logarithmically) and it is not very clear whether $\sigma_n$ converges to a finite limit (probably it does, we believe). The critical distinction appears to be whether $pQ'(0) > 1$ or $pQ'(0) \leqq 1$. We have also looked at a number of other cases, which seem to confirm the above findings. These cases include

$$Q(z) = 1 - (1 - z)^k \qquad \text{for } k = \tfrac{3}{2}, 2, 3$$

$$Q(z) = 3z - 2z^{\frac{3}{2}}$$

$$Q(z) = 1 - \{(1 - z) + (1 - z)^2 + (1 - z)^3\}/3$$

taken (in various combinations) with $G$ equal to binomial distribution, a geometric distribution, a mildly truncated Poisson distribution, and a $J$-shaped distribution (low on the left and high on the right). Results are rather different when $Q'(0) = \infty$, for example, when $Q(z) = z^{\frac{1}{2}}$, and in this case, although $\sigma_n$ converges for the discrete distributions we have used, there are indications that $\sigma_n$ would diverge to $+\infty$ for continuous distribution $G$.

NOTE 9. This note, and the next note, constitute corrigenda added at the page proof stage. Dr. V. M. Joshi, of Bombay, kindly pointed out to me (in a letter dated 10 April, 1974) that the inequality (2.1) is false in the case of the first death problem: indeed, here is a case of one of those "clearly true" false statements!

Dr. Joshi writes "In the $r$th generation let $A$ be the person whose death occurs first; in the $s$th, among the descendants of $A$, let $B$ be the person whose death occurs first; in the $t$th, amongst persons descended from $A$, let $C$ be the person whose death occurs first. In the whole $s$th generation, let $B'$ be the person whose death occurs first; and in the $t$th, amongst the descendants of $B'$, let $C'$ be the person whose death occurs first. Let the times of death of $A$, $B$, $C$, $B'$, $C'$ be respectively $T_1$, $T_2$, $T_3$, $T_2'$, $T_3'$. Then according to the definition of the process $x_{rs} = T_2 - T_1$, $x_{rt} = T_3 - T_1$, $x_{st} = T_3' - T_2'$. If $B$ and $B'$ are different persons, of which there is positive probability, $T_3 - T_2$ and $T_3' - T_2'$ are independently and identically distributed random variables. Hence, with positive probability, $T_3' - T_2' < T_3 - T_2$ so that $x_{rt} > x_{rs} + x_{st}$." Thus the first-death problem for an age-dependent branching process is not a subadditive process, at least if formulated as here. It does, however, still provide an example of a superconvolutive sequence, since Theorem 2 remains valid. It also illustrates the false converse in Theorem 3. Dr. Joshi's letter continues "I also point out here a minor discrepancy in your joint paper with D. J. A. Welsh [8]. The Theorem 8.3.1 (page 108 of that paper) states that $\tau(0, m + n) \geqq \phi(0, m) + \phi(0, n)$. But this is incorrect. The route $r$ of $t_{0,m+n}(\omega)$ meets the line $X = n$ at the point $P$ and $r_1$, $r_2$ are the portions of $r$ which run from (0, 0) to $P$ and $(m + n, 0)$ to $P$ respectively. In the proof of the theorem it is assumed that $t(r_2, \omega) \geqq s_{m+n,n}(\omega)$ which is not correct because the path of $s_{m+n,n}(\omega)$ is subject to the restriction that it has no arc along the line $X = m + n$, a restriction which does not apply to $r_2$. What can be asserted therefore is only $t(r_2) \geqq b_{m+n,n}(\omega)$ leading to $\tau(0, m + n) \geqq \phi(0, n) + \beta(0, m)$." I am grateful to Dr. Joshi for giving me the present opportunity of correcting these errors.

NOTE 10. The argument after (3.24) needs further clarification, From (3.24) we conclude (on letting $r \to \infty$) that $K(\theta) \geqq \theta a + \phi(-a)$ for any $a$. However $\phi(-a) \sim -\omega a$ as $a \to \infty$. Hence $K(\theta) \geqq (\theta - \omega)a + o(a)$ as $a \to \infty$. Letting $a \to \infty$, we deduce $K(\theta) = +\infty$ for $\theta > \omega$, as required. There is also an incorrect statement in Section 3 four lines before (3.25): the convexity of $\exp\{K_r(\theta)\}$ need *not* imply the convexity of $e^{K(\theta)}$. Accordingly the three lines before (3.25) should be deleted. Nevertheless, (3.22) *is* equivalent to (3.25). To see this, we first carry through the proof of (3.25) under the restriction $0 < \theta < \omega$ as in the text. Next notice that the argument in (3.33) remains valid under the weaker condition $0 < \theta$. Thus (3.34) holds for $\theta \geqq 0$, since it is trivially true for $\theta = 0$. Consequently $\phi(x) \leqq \inf_{\theta \geqq 0} \{K(\theta) + \theta x\} \leqq \inf_{0<\theta<\omega} [K(\theta) + \theta x] = \phi(x)$, by (3.34) and (3.25). This establishes (3.22), as required.

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UNIVERSITY OF OXFORD
INSTITUTE OF ECONOMICS AND STATISTICS
ST. CROSS BUILDING
MANOR ROAD
OXFORD OX13UL
ENGLAND