Pot and ladle: a formula for estimating the distribution of seats under the Jefferson–D'Hondt method

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Abstract

We propose a simple yet new formula for estimating national seat shares and quantifying seat biases in elections employing the Jefferson–D'Hondt (JDH) method for seat allocation. It is based solely on the national vote shares and fixed parameters of the given electoral system. The proposed formula clarifies the relationship between seat bias on the one hand, and the number of parties and the number of districts on the other. We demonstrate that the formula provides a good estimate of seat allocations in real-life elections even in the case of minor violations of the underlying assumptions. With that aim in mind, we have tested it for all nine EU countries that employ the JDH method in parliamentary elections. Moreover, we discuss the applications of the formula for modeling the effects of vote swings, coalition formation and breakup, spoiler effects, electoral engineering, artificial thresholds and political gerrymandering. By not requiring district-level vote shares, our formula simplifies electoral simulations using the JDH method.

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1 Introduction

The *Jefferson–D'Hondt method* is one of the most popular ways for allocating parliamentary seats to party lists in proportional representation electoral systems (Colomer 2004; Bormann and Golder 2013; Carey 2017).[1] Originally devised in 1792 by Thomas Jefferson to apportion seats in the US House of Representatives among the states (Jefferson 1792),[2] it was later reinvented by a Belgian mathematician and lawyer D'Hondt (1882, 1885) for use in parliamentary elections, though it is unclear whether D'Hondt actually knew of Jefferson's work on the subject.[3] It is used to allocate all or nearly all parliamentary seats in Turkey, Spain, Argentina, Poland, Peru, Chile, the Netherlands, Belgium, the Czech Republic, Israel, Switzerland, Paraguay, Serbia, Finland, Croatia, Albania, Macedonia, East Timor, Fiji, Montenegro, Luxembourg, Suriname, Cape Verde, São Tome and Príncipe, Aruba and Greenland.[4] It also is employed as a part of a mixed system or as a tier within the multi-tiered seat allocation procedures in Japan, the Dominican Republic, Austria, Denmark, Iceland and the Faroe Islands. In addition, most EU member states use the method to allocate seats in the European Parliament elections (Poptcheva 2016).

Jefferson–D'Hondt method's bias in favor of larger parties is well known (see, e.g., Humphreys 1911; Huntington 1921, 1928, 1931; Morse et al. 1948; Rae 1967; Taagepera and Laakso 1980; Lijphart 1990; Benoit 2000; Balinski and Young 2001; Marshall et al. 2002; Pukelsheim 2014, 2017). Its magnitude has been estimated by, among others, Sainte-Laguë (1910), Pólya (1918a, b, 1919a, b), Schuster et al. (2003), Drton and Schwingenschlögl (2005), Pukelsheim (2014), and Janson (2014). However, though earlier research focused on a single-district scenario, the majority of countries employing the Jefferson–D'Hondt method allocate seats within each of their multiple electoral districts separately. In those countries, the political effects of the advantage provided by the Jefferson–D'Hondt method to larger parties can be assessed only on a national scale.[5]

The Jefferson–D'Hondt method requires the vote shares of all parties in all districts to be known in order to obtain nationwide seat allocation results, and therefore cannot be applied if district-level results are unavailable, as in the case of national polls, or nonexistent, as in the case of counterfactuals. In this article, we propose a new formula for estimating those

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[1] Authors focusing on apportionment methods sometimes refer to the Jefferson–D'Hondt method as the *method of greatest divisors* (Huntington 1921, 1928, 1931), the *method of highest averages* (Carstairs 1980, pp. 17–19), or the *method of rejected fractions* (Chafee 1929). In Switzerland it is known as the *Hagenbach-Bischoff method* (Szpiro 2010, p. 204), after the Swiss physicist Hagenbach-Bischoff (1888, 1905), who developed and popularized an alternative but equivalent formulation for the method, and in Israel as the *Bader-Ofer system* after two members of the Knesset who proposed it in 1975: Yohanan Bader, an eminent alumnus of the authors' Alma Mater, and Avraham Ofer.

[2] The equivalency of the Jefferson and D'Hondt proposals has been noted in passing by James (1897, p. 36), but to the best of our knowledge, Balinski and Young (1978a) are the first modern authors to recognize that fact.

[3] Other authors rediscovered the Jefferson method independently prior to D'Hondt (see Mora 2013, p. 6 for details), but did not obtain wider recognition. The origins of D'Hondt's rediscovery and the spread of his idea that led to its practical application are discussed in greater detail by Dančišin (2013).

[4] Countries are sorted degressively according to population.

[5] Pukelsheim (2017, p. 133) noted that seat biases depend on the number of districts, but did not provide an explicit formula. He primarily has been concerned with an expected seat bias for the k-th largest party, rather than making it a function of a party vote share. Unless one assumes that the distribution of party vote shares on the probability simplex is so concentrated that the order of parties is the same in every district (an assumption that is empirically unjustified), a simple summation of the expected biases over districts will not produce a nationwide seat bias.

Party	$p_i s$	$-c/2$	$p_i(cn/2)$	s_i	Actual seats
PiS	207.29	− 20.5	46.29	233.08	235
PO	132.88	− 20.5	29.67	142.06	138
K'15	48.60	− 20.5	10.85	38.95	42
N	41.93	− 20.5	9.36	30.79	28
PSL	28.30	− 20.5	6.32	14.12	16

Table 1 The "pot and ladle" formula as applied to the Polish general election of 2015

Notation p_i is the normalized vote share of the i-th party, s_i is the estimated seat count of the i-th party, s is the number of seats to be allocated, c is the number of districts, and n is the number of relevant parties

results solely from the nationwide electoral results and two fixed parameters: assembly size and the number of electoral districts. The estimated number of seats for the i-th party (s_i) is given by the *"pot and ladle" formula*:

$$s_i = p_i \cdot s + p_i \cdot \frac{cn}{2} - \frac{c}{2}, \tag{1}$$

where p_i is the share of votes cast for that party (normalized after removing non-relevant parties), s is the total number of seats, c is the number of districts, and n is the number of "relevant" parties (see Sect. 2.2; note that it is a function of the vector of national vote shares and not an independent parameter). The formula is exact if the three underlying assumptions discussed also in Sect. 2.2 are fully satisfied, and it provides an approximation if they are satisfied only approximately, as is usually the case in real-life elections.

We can think of the formula in terms of a *potluck* metaphor. First, all relevant parties receive their proportional shares, $p_i s$. Each party also provides an identical contribution to the *common bounty pot* ($c/2$, or half a seat per district). Then the pot is divided among the parties with the size of the bounty being proportional to each party's *ladle*, namely its renormalized vote share (p_i).[6] Thus, small parties are disadvantaged, since they contribute more than they get back from the pot, while large parties receive a bonus. The formula makes clear that the size of the bonus depends not only on the size of the ladle, but also on the size of the bounty pot ($cn/2$). That is, in fact, the basic mechanism the formula reveals: the bonus created by the Jefferson–D'Hondt system is a function of both the number of districts and of the number of parties.

We illustrate the working of the formula with the results of the most recent Polish general election (Table 1), showing how the bonus helped the winning party in attaining the majority.

Scholars traditionally have thought of electoral systems in terms of *seat shares* $\left(q_i := s_i/s\right)$, which are more suitable for international comparisons. The *seat share formula*, obtained by transforming the "pot and ladle" formula above, is more conveniently expressed in terms of the mean district magnitude ($m := s/c$):[7]

$$q_i = \left(1 + \frac{n}{2m}\right) p_i - \frac{1}{2m}. \tag{2}$$

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[6] See Janson (2014, Remark 3.9) for a similar heuristic when $c=1$.

[7] For another alternative form of (2), see (5).

The difference between the seat share and the vote share is the *bias* for the *i*-th party ($\Delta_i := q_i - p_i$), given by the following *seat bias formula*:

$$\Delta_i = \frac{n}{2m}\Big(p_i - \frac{1}{n}\Big). \tag{3}$$

Since the sign of the bias is determined by the difference between the party's vote share and the mean vote share ($1/n$), it is evident that a party gains or loses from the Jefferson–D'Hondt method depending on its vote share being above or below the mean. The system is neutral (i.e., $\Delta_i = 0$) only towards those parties for which $p_i = 1/n$.

We discuss the mathematical background of the formula in a separate article (Flis et al. 2019). In the present article, we focus on the formula's empirical accuracy and on its applications. The first application is primarily predictive. The Jefferson–D'Hondt method requires that district-level results be known, and as all divisor methods can be sensitive to small variations in vote shares, those results have to be known exactly. By contrast, our formula provides a surprisingly good prediction of the nationwide seat allocation (with accuracy within 1.5% of the national seat total for more than 94.2% of parties in the nine countries for which data are analyzed), while requiring that only aggregate party vote shares be known. Hence, it can be used to model seat allocation accurately on the basis of opinion polls, exit polls and preliminary election results, when aggregate vote shares usually are all that is known.[8]

Second, because the "pot and ladle" formula does not depend on the precise vote shares of other parties, it can be used to model the effects of vote swings, political strategies (e.g., party consolidation or fragmentation) and electoral engineering (such as changing the number of districts or introducing statutory thresholds) on particular or hypothetical parties without involving complicated models of voter preference distributions.

Third, by providing a functional form of the relation between seats and votes, the formula explains the magnitude of the seat bias in terms of two explanatory variables: the number of relevant parties and the district magnitude. Such dependence cannot easily be seen when the Jefferson–D'Hondt apportionment algorithm is stated in its original form. While an empirical connection between the seat bias and the district magnitude has been recognized before, this article provides a firm theoretical basis for its existence.

Fourth, the model can also be applied to detect gerrymandering in electoral systems employing the Jefferson–D'Hondt method. In fact, it generalizes and extends the McGhee-Stephanopoulos efficiency gap test (McGhee 2014; Stephanopoulos and McGhee 2015), currently one of the most prominent methods for detecting gerrymandering in two-party first-past-the-post (FPTP) systems, in three aspects: it allows for relaxation of a restrictive assumption that voter turnout is equal across all districts and it permits the test to be extended to multiparty systems, as well as to systems with multimember districts.

In Sect. 2, we discuss the basic features and assumptions of the proposed formula and demonstrate how it fits into the earlier relevant literature. In Sect. 3, we analyze actual election data from nine European countries to demonstrate that the formula provides reasonably accurate estimates of actual seat allocations and, moreover, is quite robust to minor violations of the assumptions. In Sect. 4 we discuss how the formula can be applied to

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[8] For earlier attempts to estimate seat allocations on the basis of nationwide polls, see, e.g., Pavia and García-Cárceles (2016) and Udina and Delicado (2005). Those prior works attempt to fit a statistical model to data (a task heavily reliant on overt and latent distributional assumptions), while we derive a theoretical model directly from the Jefferson–D'Hondt method and use the empirical data only for test purposes.

analyze electoral coalitions, spoiler effects, and electoral engineering. In Sect. 5 we briefly discuss future research directions, such as the formula's application in detecting gerrymandering, and potential further corrective adjustments.

2 Basic features and assumptions

2.1 An overview of the Jefferson–D'Hondt method and of the literature on seat bias

Two formulations of the Jefferson–D'Hondt seat allocation method are commonly used. The one first proposed by D'Hondt (1882) closely tracked an earlier proposal by Jefferson (1792) for apportioning seats among the states in the US House of Representatives. It called for finding a *divisor D* such that if each party (or state) were to be allocated as many seats as its number of votes (or population) divided by D, rounded down to the nearest integer, then no seats would remain unallocated. The fraction of a seat that is discarded when rounding down is called a *rounding residual*. It is easy to demonstrate that in almost all cases[9] infinitely many such divisors exist and all yield the same distribution of seats. When dealing with vote shares rather than counts, it is convenient to replace the divisor D by a *multiplier* $L := v/D$, where v is the number of all the votes cast.

An alternative formulation of the Jefferson–D'Hondt method was first introduced by D'Hondt himself (1885) and is far more popular among legislators and political scientists.[10] Let s be the number of seats to be allocated within a given district and v_i be the number of votes cast for the i-th party ($i = 1, \dots, n$) in that district. We define a k-th quotient for the i-th party as v_i/k for $k = 1, 2, \dots$. Let q_s be the s-th highest quotient overall, i.e., across all parties. The number of seats s_i allocated to the i-th party is then defined as the number of quotients for the i-th party larger than or equal to q_s (if $q_{s+1} = q_s$, an electoral tie occurs). It is well known that both formulations are equivalent (see Equer 1911, for an early proof).

Analytic formulae for the relationship between single-district seat bias and a party's vote share have been developed independently by Bochsler (2010), Janson (2014) and Pukelsheim (2014, 2017). While appearing identical to each other and matching our seat bias formula (3) for $c = 1$, they address different problems and employ different assumptions. Pukelsheim's and Janson's results are asymptotic, guaranteed to be correct only as the district magnitude approaches infinity. Bochsler instead assumes that the rounding residuals are distributed uniformly on $(0, 1)$, which in general also is true only asymptotically (Pukelsheim 2017, Sect. 6.10). Such assumptions impose significant limitations, as real-life district magnitudes are not only finite, but quite small (typically between three and 15 seats). In contrast, our formula relies also on averaging across districts, and the errors decrease with $(m\sqrt{c})^{-1}$ (Flis et al. 2019).

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[9] The set of exceptional cases, which amount to an electoral tie (as some extrinsic rule has to be applied to allocate the last seat or seats), is of Lebesgue measure zero.

[10] For instance, all EU countries employing the Jefferson–D'Hondt method for legislative elections except for Luxembourg include this formulation in their electoral legislation.

Table 2 Identification of relevant parties in the Portuguese general election of 2015

Party	Threshold	Normalization con-	PàF	Socialist	BE	PCP-PEV	PAN
Votes	(t_n) (%)	stant $p_1 + \cdots + p_n$	2,074,345	1,747,730	550,945	445,901	75,170
Step (n)			p_1 (%)	p_2 (%)	p_3 (%)	p_4 (%)	p_5 (%)
1	4.56	2,074,345	100.00				
2	4.37	3,822,075	54.27	45.73			
3	4.18	4,373,020	47.44	39.97	12.60		
4	4.01	4,818,921	43.05	36.27	11.43	9.25	
5	3.86	4,894,091	42.38	35.71	11.26	9.11	1.54

Notation p_i is the normalized vote share of the i-th relevant party

2.2 Assumptions

The "pot and ladle" formula can be proven to produce exact results under three assumptions discussed informally in this subsection. If those assumptions are not satisfied, errors occur. However, the formula remains robust against minor violations in the sense of providing approximately correct results.

It is clear that the "pot and ladle" formula cannot be applied to every party, because for sufficiently small parties the resulting number of seats turns out to be negative. That result reveals that a threshold exists below which a party always obtains zero seats, a conclusion well in accord with prior works by D'Hondt (1883), Rokkan (1968), Rae et al. (1971), Lijphart and Gibberd (1977) and Palomares and Ramírez (2003) on the existence of a natural threshold. Only those parties with vote shares above the maximum of the natural threshold and any applicable statutory threshold are here considered *relevant*. By transforming formula (2), we obtain our estimate of the *natural threshold*:

$$t := t_n = \frac{1}{2m + n}. \tag{4}$$

Our estimate of the natural threshold depends on the number of relevant parties, but the condition of relevance also depends on the threshold. Such circular dependency can be resolved by using an iterative algorithm for determining the number of relevant parties. First, we sort all parties in descending order according to their original (non-renormalized) vote share. Then we start with the largest party ($n = 1$) and continue to add others, according to the sort order, as long as the condition $p_n > t_n$ is satisfied for the n-th party.[11] Note that in each step, p_n is defined as the share of votes for the n-th party among votes cast for the n largest parties.[12]

For example, let us consider the Portuguese general election of 2015. There were 230 seats to be allocated and 22 districts (giving the mean district magnitude of about $m = 10.45$). In the first step of the algorithm, the set of relevant parties consists only of the largest party, PàF (Portugal Ahead). Then we continue to add parties as detailed in Table 2.

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[11] For a formal proof that the above algorithm identifies all relevant parties correctly, an interested reader is referred to Flis et al. (2019, Sect. 7).

[12] Thus, the first party always qualifies (as p_1 equals 1 in the first step).

Table 3 Seat allocation in the Portuguese general election of 2015

Party	PàF	Socialist	BE	PCP-PEV	PAN
Vote share (p_i)	43.05%	36.27%	11.43%	9.25%	0%
Seat share (s_i)	106.95	88.37	20.33	14.35	0
Actual seats	107	86	19	17	1
Error	−0.05	2.37	1.33	−2.65	−1

It can be seen that for $n = 1, \ldots, 4$, the normalized vote share p_n is larger than the natural threshold t_n, but the inequality no longer holds for $n = 5$, as $p_5 \approx 1.54\% < 3.86\% \approx t_5$. Hence, only the first four parties can be considered relevant in the sense defined above. Next, applying the "pot and ladle" formula we obtain the results presented in Table 3.

With the concept of party relevance explained, we can proceed to state our three assumptions:

A1 There exists such a selection of a multiplier for each district that for every relevant party (a) the rounding residuals average to 1/2 over all districts, and (b) the multipliers are not correlated with normalized party vote shares;[13]

A2 Normalized party vote shares average to national vote shares over all districts;[14]

A3 Non-relevant parties get no seats.

Assumption A1 is fundamental to the working of the formula, but at the same time, highly technical in nature. As we discuss in Flis et al. (2019, Sect. 4), it can be justified partially under the probabilistic model of elections based on certain assumptions concerning the distribution of party vote shares.

Assumption A2 is fairly intuitive and we would expect it to be satisfied approximately in most real-life elections. A major violation of A2 likely would be indicative of some form of gerrymandering. Minor violations may occur in countries, such as Spain, where the largest electoral districts are in urban areas and a significant urban–rural divide exists in politics, or when one party's vote share is correlated with turnout at the level of the electoral district or with the aggregate support for non-relevant parties.

Assumption A3 can be violated in two instances:

(1) If the electoral support of some parties is tightly concentrated in a small number of districts, while their nationwide vote shares are insufficient for those parties to be included in the set of relevant parties. In the most extreme case, some parties are regional (e.g., represent a national or ethnic minority) and register party lists only in a single region, where they are relevant. That problem may be solved by employing a *regional correction*, described in detail in "Appendix 1".

(2) If the variances in district magnitude cause large variances in natural thresholds such that those parties too small to qualify for seat allocation in an average-sized district

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[13] It can be noted that if A1 is satisfied, the multiplier averaged over districts equals the average unbiased multiplier $(m + n/2)$ considered by Gfeller (1890), Joachim (1917) and Happacher and Pukelsheim (1996, 2000).

[14] This is equivalent to the assumption that normalized party vote shares are not correlated with district size measured by the sum of votes cast for relevant parties.

nevertheless gain seats in the larger ones. For instance, in Portugal the average district magnitude is about 10.45 seats, but (as of the 2015 election) 47 seats have been allocated in the Lisbon district and 39 seats in the Porto district. Accordingly, outside the two the natural threshold with four relevant parties equals approximately 10.87%, while in Lisbon it is more than five times smaller—just roughly 2.04%. For a recent discussion of this effect, see Barceló and Muraoka (2018).

2.3 Determinants of the seat bias

As can be seen from the seat share formula (2), the seat share of each relevant party depends on only three variables: its vote share, the mean district magnitude, and the number of relevant parties. It should be noted that the relationship is affine, but not linear, suggesting that the Jefferson–D'Hondt method should not be regarded as strictly proportional. However, do note that (2) can be expressed in an equivalent form:

$$q_i = \frac{1}{2mt}\left(p_i - t\right),\tag{5}$$

which demonstrates that the seat shares are proportional but to the over-the-threshold vote shares.

Of the three variables named above, the mean district magnitude is, strictly speaking, the only parameter of the formula, being fixed in advance by electoral rules. In seven of the nine countries discussed in Sect. 3, the mean district magnitude varies between 10 and 16. The exceptions are the Netherlands (with a single 150-seat district) and Spain (with $m = 6.73$, one of the lowest in Europe, see Baldini and Pappalardo 2009, pp. 67–69).

When the formula is applied to estimate seat allocations, the number of relevant parties is not a parameter, but instead is obtained from nationwide electoral results using the procedure described in Sect. 2.2. However, when the formula is used to analyze counterfactuals, and only one party's vote share is fixed, the number of relevant parties can be treated as a parameter, albeit a constrained one: fixing a number of relevant parties limits the feasible values of the vote share and, vice versa, fixing a vote share limits the feasible numbers of relevant parties.[15] As expected, for the countries under consideration the number of relevant parties correlates with the mean district magnitude.[16]

Some of the relationships revealed by the seat bias formula (3) are well known to students of electoral systems. For instance, there is nothing new about the finding of the negative effects of small districts on small parties, as the same follows from the well-known *micromega rule* ("the large prefer the small and the small prefer the large"; see Colomer 2004) and already has been well established by Rae (1967), Taagepera and Laakso (1980), Taagepera (1986) and Taagepera and Shugart (1989). Nevertheless,

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[15] As each relevant party must cross the natural threshold defined in (4), and their vote shares must sum up to 1, when we fix m and n, p_i can vary only between t and $(2m + 1)t$, where t is the natural threshold defined in (4). Conversely, with m and p_i fixed, n must lie between $1/p_i - 2m$ and $(2m + 1)/p_i - 2m$.

[16] The number of relevant parties for use in counterfactual models can be based on earlier elections, polls, or statistical models connecting it with exogenous parameters (such as the seat-product model proposed by Shugart and Taagepera 2017a, b; see also Taagepera 2007; Li and Shugart 2016; Shugart and Taagepera 2017a, b). In the latter case, care should be taken to use a model providing an expected number of parties under the condition that a party of a given size exists, rather than just the absolute (unconditional) expectation.
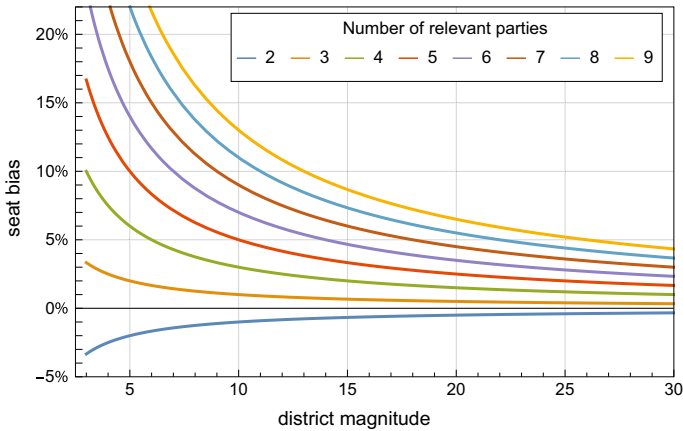
Fig. 1 Seat bias of a party with a normalized vote share of 40% as a function of mean district magnitude
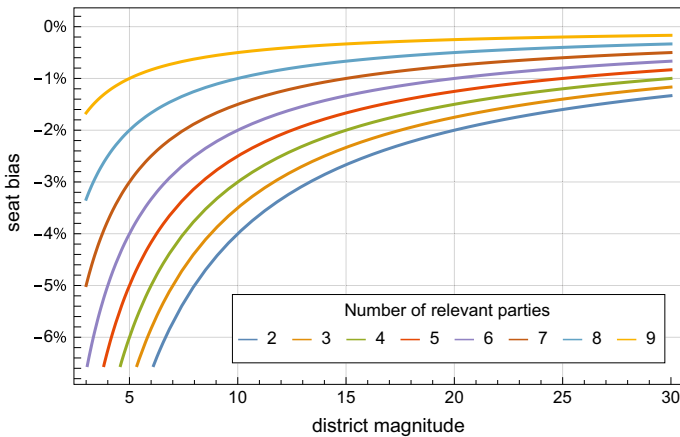

Fig. 2 Seat bias of a party with a normalized vote share of 10% as a function of district magnitude

note that under the Jefferson–D'Hondt method this effect is magnified compared to other apportionment rules, as the negative seat bias arises in addition to the exclusionary effect of the small district magnitude documented by Lijphart and Gibberd (1977).

The effect of the number of parties on the magnitude of the seat bias appears to have escaped the attention of many electoral scholars. It demonstrates, however, an important self-correcting aspect of electoral systems based on the Jefferson–D'Hondt method: as the number of parties increases, so does the bias of the largest party, at least partially alleviating difficulties in government formation caused by legislative fragmentation.

To illustrate the relationships revealed by the formula, on Figs. 1 and 2 we plot the seat biases for two hypothetical parties with a normalized vote share of, respectively, 40% and 10%—as they vary depending on the number of parties (from two to nine) and the district magnitude; on Fig. 3 we plot the bias as a function of the vote share when the number of parties is either four or eight and the mean district magnitude is either three or sixteen.
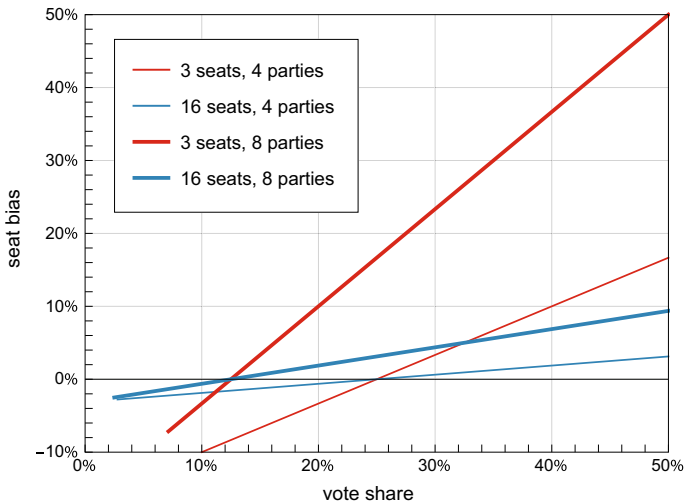
Fig. 3 Seat bias (Δ) as a function of vote share (p) for specified electoral parameters (m, n)

3 Empirical test

As noted above, the Jefferson–D'Hondt method is commonly used worldwide. Owing to data availability constraints, we restrict our empirical test to those cases that meet the following criteria:

(1) Post-1945 national lower house elections in EU member states…
(2) … with a single-tiered electoral system and mostly multi-member districts…
(3) … where the Jefferson–D'Hondt method continues to apply as of March 2019.

Eight countries satisfy those criteria: Belgium, the Czech Republic, Finland, Luxembourg, the Netherlands, Poland, Portugal and Spain. We also include Croatia, although it does not fully satisfy criterion (3): it uses FPTP to allocate seats in special districts set aside for ethnic minorities. Since the number of those minority seats is relatively small (six out of about 150) and elections for those seats are held at different dates, consequently we shall omit them from our calculations. We do not include pre-2014 Belgian elections, as the overlap of the Walloon and Flemish regions in the Brussels–Halle–Vilvoorde district prevents the use of the regional correction, which is otherwise necessary (see "Appendix 1"). For Finland, we omit elections prior to 2003 because of the lack of data on apparentments, i.e., formal electoral alliances, at the district level. For Poland, we omit the 1993 and 1997 elections owing to the use of the second electoral tier (the national list).

Table 4 sets forth the general parameters of the electoral systems of our countries of interest.

In seven of those nine countries at least some regional parties have won seats, such as the Convergence and Union (CiU), the Republican Left of Catalonia (ERC), the Basque Nationalist Party (EAJ/PNV), and many others in Spain; the Swedish People's Party (SFP/RKP) in Finland; the Croatian Democratic Alliance of Slavonia and Baranja (HDSSB) and the Istrian Democratic Party (IDS) in Croatia; the German Minority (MN) in Poland; the Independent Democratic Association of Macau in Portugal in 1975; or the Party of Independents of the

Table 4 General parameters of electoral systems of the test country dataset

Country	Earliest election included	Number of elections	Number of seats (s)	Number of districts (c)	Number of relevant parties (n)	
					National	Regional
Belgium	2014	1	150	11	0	14
Croatia	2000	6	143–146	11	4–7	0–2
Czech Republic	2002	5	200	14	4–9	0
Finland	2003	4	200	13–15	6–7	2
Luxembourg	1945	17	26–64	2–4	3–7	0–1
Netherlands	1948	21	100–150	1	7–14	0
Poland	2005	4	460	41	4–6	1
Portugal	1975	15	230–263	22–25	3–5	0–2
Spain	1977	13	350	52	2–5	4–9

Fig. 4 Kernel density estimate of the distribution of party errors. Bandwidth fitted by least squares cross-validation

N = 661   Bandwidth = 0.1658
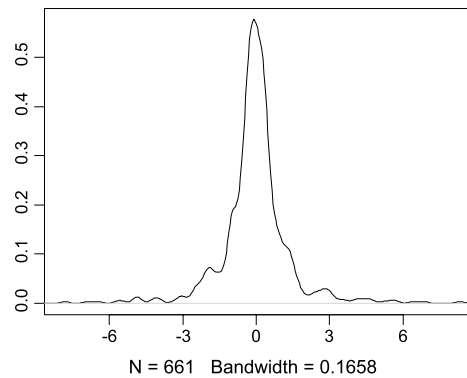
East in Luxembourg in 1945. In Belgium, all relevant parties have been regional. Accordingly, in those cases we employ the regional correction.

The results of the empirical test confirm that our formula does indeed work as expected and is robust against minor violations of its assumptions. As seen in Fig. 4, showing the kernel density estimate of the distribution of party errors (i.e., the differences between actual and estimated seat shares for a given party), in more than 94.2% of the cases the error is within the $(-1.5\%, 1.5\%)$ interval. That result is explained by the fact that under typical conditions encountered in real-life elections, the errors introduced at different stages of approximation tend largely to cancel each other out, thereby making the overall error quite small.

To measure the aggregate error per election, we use the total variation distance between actual and estimated seat share vectors (which is equivalent, up to a constant, to the ℓ_1 (taxi-cab) metric; see Deza and Deza 2014, p. 260):

$$\varepsilon := \frac{1}{2s} \sum_{i=1}^{n} \left| s_i^{act} - s_i \right|, \tag{6}$$

where s_i^{act} is the number of seats awarded to the i-th party under the actual allocation. The intuition behind (6) is that it corresponds to the share of misallocated seats. Again, the

results are promising, as only in five out of 86 elections were more than 4% of all seats misallocated.[17] Compared to the leading alternative formula, the modified cube law for proportional elections (Taagepera 1986), our formula is more accurate in 82 of 86 cases. The detailed error values for each election can be found in "Appendix 2".

4 Political applications

4.1 Advantages of the formula for modeling political counterfactuals

Until now, we primarily have investigated the simplest case for our formula: allocating seats on the basis of the nationwide vote totals, when the results for all parties are known. But as was noted in Sect. 2.3, one of the main features of the formula consists in the fact that we do not need that much information. Hence, we can employ the formula to analyze various counterfactuals, such as vote swings, party splits and mergers, changes in the number of districts, or the introduction of statutory thresholds.

Of course, it is theoretically possible to investigate such counterfactuals even without the "pot and ladle" formula. That is, however, a taxing undertaking that can be accomplished only at the cost of many arbitrary or oversimplifying assumptions. First, one needs to translate aggregate results into district-level results, a task requiring one to model the distribution of district-level election results. As attested by the voluminous literature on partisan bias, that is a difficult problem in and of itself, especially if the voting patterns are not stable over time (see, e.g., Gudgin and Taylor 1979; Katz and King 1999; Linzer 2012; Calvo and Rodden 2015). It also is mathematically challenging, as one has to generate a random matrix with constraints on both rows and columns. If the counterfactual situation involves changes in party vote shares, one then needs to translate those changes into district-level vote swings. Again, even in the simplest case of a two-party system, the swing patterns are quite complex (Blau 2001). While none of those difficulties are insurmountable, solving them is complicated and time-consuming. Moreover, it is difficult to distinguish the true effects of the counterfactual of interest from artifacts arising from the technical assumptions introduced along the way. The proposed formula avoids those technical differences. At the cost of a relatively small approximation error, it enables one to model the generic effects of counterfactuals in a simpler and more transparent manner.

4.2 Changes in the structure of the party system: mergers and splits

It is well known that the Jefferson–D'Hondt system encourages coalitions (cf. Balinski and Young 1978b; Bochsler 2010). The seat bias formula (3) facilitates an easy assessment of the theoretical *merger bonus*, i.e., the difference between the estimated bias $\Delta_{i,j}$ of the coalition of parties i and j and the sum of the estimated biases of the individual parties $\Delta_i + \Delta_j$ under the assumption that the vote shares of all other parties remain constant:

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[17] Three of those five cases are the Spanish elections of 1977, 1979 and 2015, when significant correlations are observed between the vote shares of the left-wing parties (PSOE and PCE in 1977 and 1979, Podemos in 2015) and district size. The fourth case is the Luxembourg election of 1948, which was held in only two districts. The last one is the Luxembourg election of 2018, when deviations from assumption A1 happened, resulting from a coincidence of low probability events (very narrow margins of victory) in two out of four districts (i.e., 50%), which would be highly improbable in countries with more districts.

$$\Delta_{i,j} - \left(\Delta_i + \Delta_j\right) = \frac{1 - \left(p_i + p_j\right)}{2m}. \tag{7}$$

Janson 2014, Theorem 8.1, provides an analogous asymptotic formula for $c = 1$ and $s \to \infty$. Note that the merger bonus does not depend on the number of third parties, but only on their total vote share, and that it is negatively related to the sum of the coalition parties' vote shares. Also note that while the coalition, taken together, always benefits and never loses from a merger, it does not necessarily follow that each member thereof will have adequate incentives to join: that will depend on how the seats are allocated within the coalition (see Kaminski 2001; Leutgäb and Pukelsheim 2009; Janson 2014; Karpov 2015).

Of course, the seat bias formula is by itself not sufficient for assessing the exact effects of coalition formation, since a coalition may alienate each party's fringe electorate or attract additional voters owing to the bandwagon effect. Kaminski (2001) demonstrates that additivity of voter support is infrequent. However, our formula provides an initial estimate of the merger bonus and its derivative with respect to vote share changes.

Formula (7) can easily be transformed to model the reverse case: a breakup of a party or a coalition. Again, as long as all successor parties are relevant, the sum of the successor parties' seat shares remains invariant with respect to the distribution of their vote shares. It also can be noted that the gains from the breakup of a party or coalition accrue to every competing party in proportion to its normalized vote share.

Another change in the structure of the party system that can be modeled is the appearance of spoiler parties. With the "pot and ladle" formula, the counterfactual estimates of election results without the spoiler and under various assumptions about the transfer of the spoiler's votes can be conducted very easily, while performing district-level simulations would be very complex. For such an application of our formula to the study of spoiler effects in the Polish parliamentary elections of 1993 and 2015, see Kaminski (2018).

Of course, while all parties seek to maximize their vote shares, most elections—and especially those in parliamentary systems—are still primarily about winning the legislative majority. The seat share formula (2) can be transformed to yield, for any combination of parameters m (the mean district magnitude) and n (the number of relevant parties), the minimum vote share p_{Maj} that translates to at least half of the total number of seats:

$$p_{Maj} = \frac{1 + 1/m}{2 + n/m} = (m + 1)t, \tag{8}$$

where t is the natural threshold defined in (4). This is illustrated in Fig. 5 by a contour plot.

4.3 Electoral engineering and reform

Political strategies involve not only changes in party identities but also electoral engineering. In proportional systems, such engineering usually takes the form of changes in the seat allocation method, statutory thresholds, or the number (and, consequently, the magnitude) of electoral districts (Kaminski 2002). In the former case, the "pot and ladle" formula can be used to model a change in the seat allocation method in countries that already use another form of proportional representation, as well as to provide a first approximation of the consequences of the introduction of the Jefferson–D'Hondt method in countries that employ an altogether different electoral system. Such an approximation especially could be useful if the electoral system change were a subject of major public discussion akin to

Fig. 5 The minimum share of votes necessary for a party to obtain at least half of the total number of seats as a function of the number of relevant parties and the mean district magnitude. Midpoints of country code locations correspond to the value of the two parameters in the most recent election
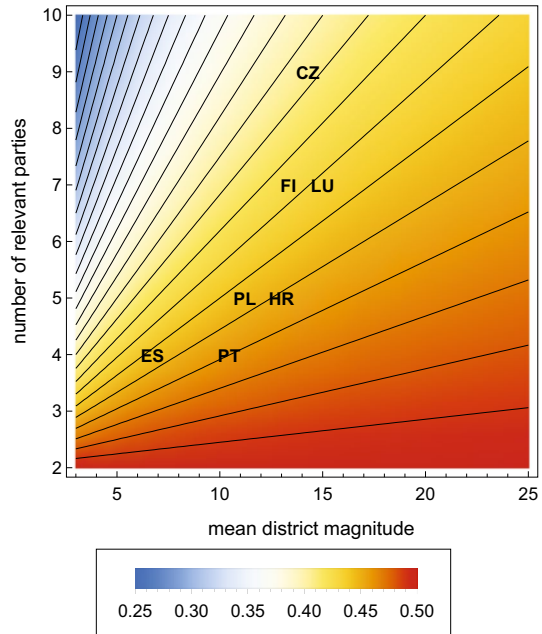
the ones held in New Zealand and the United Kingdom before the electoral reform referendums in those countries.

In the case of the smaller adjustments, and assuming that the Jefferson–D'Hondt method is used for seat allocation, the "pot and ladle" formula can be applied to estimate the effects of the change in electoral rules, again subject to a possible correction for secondary effects (such as induced coalition formation among the opposition or changes in the distribution of vote shares). For instance, from the "pot and ladle" formula we can calculate the number of districts that need to be added for the i-th party ($p_i > 1/n, i = 1, \ldots, n$) to gain a single seat:

$$\delta_i^1 := \left\lceil \frac{2}{p_i n - 1} \right\rceil. \tag{9}$$

Somewhat counterintuitively, δ_i^1 does not depend on the initial number of districts c. Moreover, it follows from (9) that δ_i^1 has a singularity at $1/n$, which is not surprising, since no bias exists for mean-sized parties, so they will gain no seats no matter how many districts are added (as long as the number of relevant parties remains unchanged). Of course, in practice the interval wherein no change in seats is possible is wider, as the number of districts that can be added is bounded by the total number of seats. Note that for small parties, $p_i < 1/n$, δ_i^1 is negative, as they lose rather than gain seats when districts are added. The dependence of δ_i^1 on p_i is plotted on Fig. 6.

Statutory (artificial) vote thresholds are another device of electoral engineering. Unlike the natural thresholds, see Sect. 2.2, they (if greater than the latter) give rise to discontinuities in the seats-votes curves of all parties.[18] Such discontinuities easily can be quantified.

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[18] Four out of the nine countries discussed in Sect. 3 employ national statutory thresholds. In the Netherlands, it equals the Hare quota (1/150), while in Croatia, the Czech Republic and Poland—5%. In the latter two countries coalition thresholds are also in place (8% in Poland and 5% times the number of member parties in the Czech Republic).
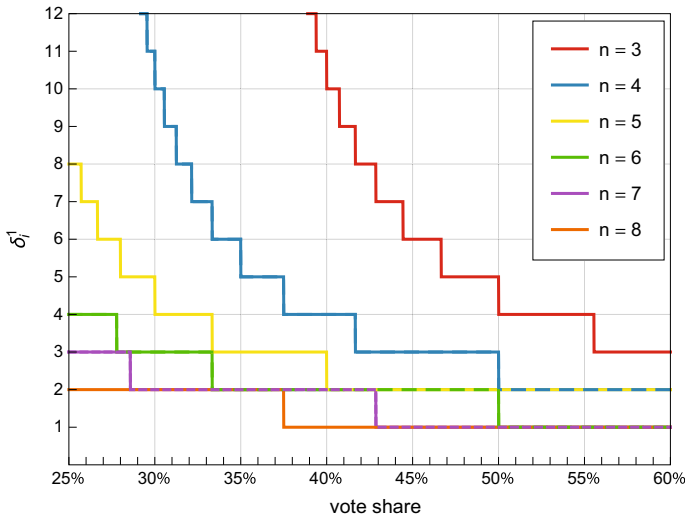
Fig. 6 The number of districts δ_i^1 that need to be added (to some initial number of districts c) for the i-th party to gain a single seat, depending on its vote share p_i and the number of relevant parties n

Let $\tau > t$ be the statutory threshold, $n - 1$ parties be certain to clear it and only the n-th party be uncertain. Note that it is not necessary for the n-th party to be the smallest one—because of different thresholds for different types of electoral competitors, it is possible for one competitor to fail to clear its threshold, while other smaller competitors manage to clear their own.[19] Then, the i-th party's ($i = 1, \ldots, n - 1$) seat share gain arising from the n-th party's failure to clear the threshold is:

$$T_i = \frac{p_i}{2m}\left(\frac{\tau/t - 1}{1 - \tau}\right),\tag{10}$$

where t is given by (4). Note that for $\tau = t$ (i.e., the artificial threshold equaling the natural threshold), $T_i = 0$, and the discontinuity disappears. Further corollaries to this result are explored in "Appendix 3", and Figs. 7 and 8 illustrate the dependence of T_i on τ.

With that result, we can answer another question of interest to political strategists. Let us consider two blocs of parties: A and B, with, respectively, n_A and n_B parties and aggregate vote shares p_A and p_B; a single swing party C hovers over the statutory threshold and there are no other relevant parties. Who would benefit from C's failure to clear the threshold? If the benefit was to be measured by the number of seats acquired, the answer would be simple: it follows from (10) (which is additive) that the block with the larger aggregate vote share would capture a greater number of seats vacated by C. However, if the benefit is to be measured by capturing the legislative majority, the problem becomes a little less trivial. If C were to fail to clear the statutory threshold, bloc A would be able to govern if and only if its benefit in seat shares exceeds the additional seat share needed to form the majority, i.e., if, by (10) and (2),

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[19] For instance, in Poland in 2015 the United Left coalition failed to clear the coalition threshold of 8% with a vote share of 7.55%, while the Polish People's Party qualified for seats with a smaller vote share of 5.13%.
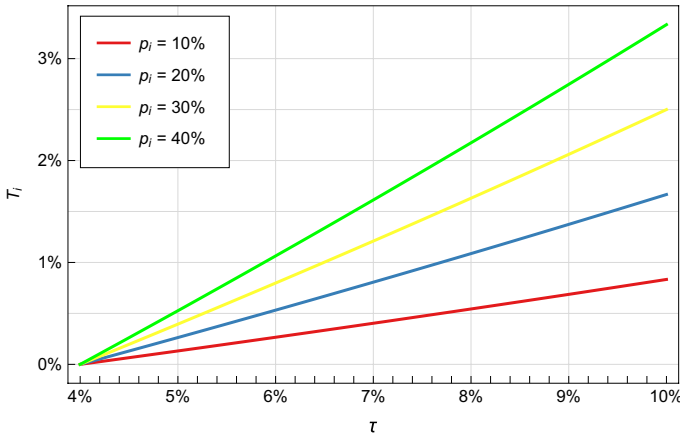
Fig. 7 Absolute seat share gains from a single competitor's failure to clear statutory threshold (estimated for $m = 10$ and $n = 5$, where $t = 1/25$)
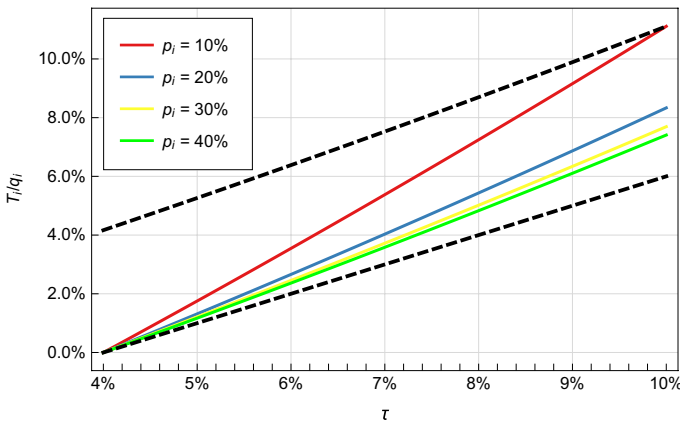

Fig. 8 Relative seat share gains from a single competitor's failure to clear the statutory threshold (estimated for $m = 10$ and $n = 5$, where $t = 1/25$). The dotted lines represent the lower and the upper bound given by, respectively, (20)—a straight line and (21)—a fragment of hyperbola

$$\frac{p_A}{2m}\left(\frac{\tau/t - 1}{1 - \tau}\right) > \frac{1}{2} - p_A\left(1 + \frac{n}{2m}\right) + \frac{n_A}{2m}. \tag{11}$$

As shown in "Appendix 3", the requirement is equivalent to

$$\frac{p_A}{m + n_A} > \frac{p_B}{m + n_B}. \tag{12}$$

Accordingly, a beneficiary of a swing party's failure to clear the artificial threshold depends not only on the size of the two contending blocks, but also on their degree of consolidation.

5 Future research directions

Potential future research directions for which the "pot and ladle" formula likely is to be useful have not escaped our notice. Some of those involve adjustments of the formula to deal with modifications of the Jefferson–D'Hondt method, such as multi-tiered seat allocation, or with major deviations from our assumptions. Others involve applications of the formula to the theoretical study of seats-votes relationships. In that field, we note that the formula can be used to generalize the McGhee-Stephanopoulos efficiency gap test for gerrymandering. Also, in a two-party case the analysis of the relationship between the ratio of the seat shares and the ratio of the vote shares yields a functional relation that approaches Taagepera's (1986) generalized cube law for proportional systems, but with Euler's number $e \approx 2.71$ instead of number 3 as the base of the exponent (for proof, see Flis et al. 2019, Sect. 8). Two of the above-mentioned future directions are discussed in greater depth below.

5.1 Corrective adjustments

As noted in Sect. 3, the "pot and ladle" formula is fairly robust against minor violations of the underlying assumptions. Nevertheless, in some cases the violations are of such a magnitude that the approximation produced by the formula is subject to a substantial error (see fn. 17). To avoid that, it is possible to devise corrective adjustments to the formula that improve the quality of approximation, albeit at the cost of greater complexity. We do not attempt to do so in the present article, as in virtually all cases such adjustments would require additional information as to the spatial distribution of each party's vote (e.g., the covariance of the district-level normalized vote shares and district size) that neither is known before the election nor is commonly reported as a part of nationwide election results or electoral polls. The *regional correction* described in "Appendix 1" is an exception here, but only because it deals with an extreme example of a distributional anomaly where regional parties do not even compete in some districts.

Of course, in some cases it might be possible to infer such additional information from historical patterns or demographic data. One example of an ad hoc correction to the "pot and ladle" formula that uses such an approach is a recent work by Evci and Kaminski (2019), who have developed just such a correction to deal with the distributional anomalies of the Turkish party system (wherein the Kurdish HDP party shares many features of a regional party, with its support being very concentrated in southeastern Turkey, but registers candidate lists in the whole country, thereby ruling out the application of our regional correction). We believe, however, that such inference from historical patterns requires considerable caution, as many parameters of interest are quite volatile and can change not only with major political realignments, but even with ordinary electoral swings. For instance, consider the Polish elections of 2007 and 2011, which were characterized by significant correlations between the turnout and the support for each of the two major parties (positive for the winning PO and negative for the losing PiS). However, in 2015 those correlations virtually disappeared. Besides, inference from historical patterns of vote distributions is of course impossible for new parties, which is an especially significant drawback at a time when party systems in many countries are in a state of flux.

5.2 Extension of the efficiency gap test

The FPTP electoral system used in more than 60 countries (ACE Project 2019), including the United States, the United Kingdom and most of the Commonwealth, can be thought of as a limiting case of the Jefferson–D'Hondt system (or other divisor methods). Let us, therefore, apply our formula to such a system, $m = 1$, with only two relevant parties, i.e., $n = 2$. We get from (2):

$$q_i = 2p_i - \frac{1}{2}. \tag{13}$$

Note that the natural threshold becomes equal to $1/(2m + n) = 1/4$. Some of the assumptions underlying the "pot and ladle" formula also can be simplified in this case: A2 is equivalent to the requirement that the normalized vote shares of both relevant parties be uncorrelated with the number of votes they receive together, and A1 is equivalent to the assumption that both parties' shares of *wasted votes*, defined as the excess of the normalized vote share over $1/2$ for the winning party and the entire normalized vote share for the losing party and, thus, equivalent to the half of our rounding residuals, are equal. Note, that the same condition was postulated by the McGhee–Stephanopoulos *efficiency gap* standard (Stephanopoulos and McGhee 2015), currently one of the most popular tests for the existence of partisan gerrymandering. Moreover, our "pot and ladle" formula for single-member districts, (13), is equivalent to their seats-votes formula. It also implies that the restrictive McGhee–Stephanopoulos assumption that the sum of normalized vote shares of both relevant parties is constant across districts can be relaxed by requiring only an absence of correlation with party vote shares (as we mention above), thus providing a generalization of the efficiency gap test.

Several attempts have been made to extend the efficiency gap test to multi-party elections (McGhee 2017; Stephanopoulos and McGhee 2018; Tapp 2018; Veomett 2018). The main issue arises from the difficulties of providing a natural extension of the definition of wasted votes when multiple parties are competing. We propose that defining wasted votes in terms of rounding residuals provides just such a natural extension, applicable not only to multi-party elections, but also to multi-member districts (Flis et al. 2019, Sect. 7).

6 Conclusion

The "pot and ladle" formula presented in this article has a number of both practical and theoretical applications. First, it facilitates the easy translation of vote shares into seat numbers, which constitute a natural complement of opinion and exit poll results. In such cases, aggregate national vote shares usually are all that is known. Their disaggregation into district-level results can be done only by means of complex and volatile election demographic models. In addition, such models are especially unreliable for new parties, whose territorial support patterns cannot be inferred from earlier elections. Our formula provides a simple alternative that relies only on aggregate results and on the numbers of seats and districts, and yet provides a high degree of accuracy.

Second, the "pot and ladle" formula enables researchers and practitioners to simulate counterfactual election results without relying on restrictive assumptions about the territorial distribution of party votes, making it a useful tool for evaluating political strategies and what-if scenarios as well as for assessing the effects of electoral engineering. The formula

can be used to quantify for each party the expected consequences of electoral reforms that involve changes in the mean district magnitude and the statutory vote threshold. The presentation by one of the authors of this article (J.F.) of such results during the 2017 public debate on local electoral reform in Poland contributed to the governing party's decision to withdraw the controversial proposal to shrink the districts.

Thirdly, we explain how the seat bias under the Jefferson–D'Hondt method depends on the mean district magnitude and the number of parties. While the relationship of the bias to district magnitude, captured by the micromega rule, is well known to students of electoral systems, the relationship of the bias to the number of parties has been somewhat underappreciated outside of purely theoretical studies on election bias. We demonstrate how, and under what conditions, those two strands of electoral system research can be combined to form a complete picture of the conditions determining the magnitude of the seat bias.

Finally, the "pot and ladle" formula provides a consistent normative criterion for the absence of 'skewness' in the Jefferson–D'Hondt variant of proportional voting system. If the election results deviate significantly from the formula, then it must be 'skewed', either as a result of some unnatural correlations (possibly, though not necessarily, caused by malapportionment or gerrymandering), or due to some random numerical artifacts of the system.

Appendix 1: Regional correction

As noted in connection with assumption A3, regional parties pose a significant problem in applying the "pot and ladle" formula, as all of their votes are concentrated in a limited number of districts. Accordingly, their nationwide vote shares are a poor approximation of their actual level of support in those districts. If exact regional vote shares are known, for instance, because separate regional polls are held (as is usually the case in Spain), the problem can be avoided by applying the "pot and ladle" formula for each region independently and then summing over all regions. However, even if no such data are available, seat allocations still can be estimated on the basis of the nationwide distribution of votes, provided that it is known before the election which parties contest which districts and that certain additional assumptions hold.

Let a party that contests fewer than all districts be a *regional party*, and let other parties be *national parties*. Let the set of districts contested by a given regional party be a *region*. We will treat the set of districts with no regional parties as another single region. Using those terms, we can express the assumptions for the regional correction as follows:

R1 There is no partial overlap between any regions;

R2 The voters-to-seats ratio does not vary between regions;

R3 The relative size of the national parties (i.e., the ratios of their respective vote shares) does not vary between regions.

Of those, assumptions R1 and R2 sound natural (although R1 can be quite restrictive—for instance, it prevents the regional correction from being applied in Belgium prior to 2014, as the Flemish and Walloon regions overlapped in the Brussels–Halle–Vilvoorde district). R3 is less obvious, but the effects of its violations tend to cancel each other out on the national scale.

If the votes of a regional party i are concentrated entirely in a single region r (per R1) then, by R2, we can express its regional vote share as simply as

$$P_i^r := P_i \frac{v}{v^r} = P_i \frac{s}{s^r}, \tag{14}$$

where P_i is the i-th party's nationwide vote share, P_i^r is its regional vote share (both are non-renormalized), v and s are, respectively, the national vote and seat counts, and v^r and s^r are, again respectively, the regional vote and seat counts. For regional vote shares to sum up to 1, the vote shares of the *national party j* need to be rescaled (per R3) to

$$P_j^r := P_j \frac{1 - \sum_{l \in R^r} P_l^r}{1 - \sum_{l \in R} P_l}, \tag{15}$$

where R is the set of all regional parties and R^r is the set of regional parties running in region r. In the region with no regional parties, (15) will simplify to

$$P_j^0 := \frac{P_j}{1 - \sum_{l \in R} P_l}. \tag{16}$$

With those approximations of regional vote shares, as well as precise data on the number of seats and districts within each region, the "pot and ladle" formula can be applied for each region without any further modifications.

Appendix 2: Aggregate errors for post-1945 general elections in nine EU countries

Error is defined as the share of misallocated seats, i.e., the total variation distance between the actual and estimated seat share vectors, and modified cube law refers to the seat share estimates obtained by Taagepera's (1986) modified cube law of election results (Table 5).[20]

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[20] Under Taagepera's modified cube law for proportional elections, the expected seat share of the i-th party is given by $q_i = P_i^x/(P_i^x + (\eta - 1)^{1-x}(1 - P_i)^x)$, where $x := (\log v / \log s)^{1/m}$, v is the number of voters, s is the number of seats, P_i is the non-normalized vote share of the i-th party, and η is the effective number of parties, i.e., $\eta := \left(\sum_{i=1}^{n} P_i^2\right)^{-1}$.

Table 5 Aggregate errors for post-1945 general elections in nine EU countries

Country	Year	Seats	Districts	Relevant parties*	Error of the "pot and ladle" formula (%)	Error of the modified cube law (%)
Belgium	2014	150	11	7	3.8	4.7
Croatia	2000	146	11	4	1.1	1.8
	2003	144	11	7.3	1.6	7.5
	2007	145	11	6.3	1.7	8.5
	2011	143	11	3.5	3.1	5.2
	2015	143	11	5.2	1.8	5.6
	2016	143	11	5.4	2.5	3.9
Czech Republic	2002	200	14	4	1.0	0.8
	2006	200	14	5	1.6	3.9
	2010	200	14	5	0.9	1.5
	2013	200	14	7	2.5	2.5
	2017	200	14	9	2.3	5.9
Finland	2003	200	15	6.4	1.8	3.9
	2007	200	15	7.4	2.9	3.9
	2011	200	15	7.4	3.5	4.1
	2015	200	13	7.4	2.8	3.2
Luxembourg	1945	51	4	4.3	2.1	7.5
	1948	26	2	4	4.7	5.0
	1951	26	2	3	1.5	4.2
	1954	52	4	4	2.3	6.5
	1959	52	4	4	2.7	5.6
	1964	56	4	5	1.4	5.9
	1968	56	4	4	2.7	4.9
	1974	59	4	5	1.9	2.9
	1979	59	4	6	2.8	6.8
	1984	64	4	5	3.1	5.4
	1989	60	4	7	3.7	6.7
	1994	60	4	5	1.3	3.0
	1999	60	4	6	3.9	5.0
	2004	60	4	5	1.8	3.8
	2009	60	4	6	1.6	4.5
	2013	60	4	7	2.4	5.6
	2018	60	4	7	4.8	7.4

Table 5 (continued)

Country	Year	Seats	Districts	Relevant parties*	Error of the "pot and ladle" formula (%)	Error of the modified cube law (%)
Netherlands	1948	100	1	8	1.0	1.3
	1952	100	1	8	0.7	1.4
	1956	150	1	7	0.4	0.6
	1959	150	1	8	0.8	1.5
	1963	150	1	10	0.8	1.2
	1967	150	1	11	0.9	1.5
	1971	150	1	14	1.2	2.2
	1972	150	1	14	1.1	1.7
	1977	150	1	11	0.5	1.9
	1981	150	1	10	0.9	1.0
	1982	150	1	12	0.9	1.9
	1986	150	1	9	0.8	1.8
	1989	150	1	9	0.8	1.8
	1994	150	1	12	1.0	2.0
	1998	150	1	9	0.8	1.0
	2002	150	1	10	0.8	1.5
	2003	150	1	9	0.9	1.2
	2006	150	1	10	1.0	1.1
	2010	150	1	10	1.0	0.9
	2012	150	1	11	1.1	1.7
	2017	150	1	13	1.0	1.2
Poland	2005	460	41	6	1.2	3.2
	2007	460	41	4	1.3	3.0
	2011	460	41	5	0.9	4.9
	2015	460	41	5	1.5	5.0
Portugal	1975	250	25	5	2.4	4.4
	1976	263	24	4	1.3	1.7
	1979	250	22	3	0.4	1.5
	1980	250	22	3	0.6	1.0
	1983	250	22	4	1.0	1.6
	1985	250	22	5	1.4	2.9
	1987	250	22	5	2.1	5.0
	1991	230	22	4	2.6	3.3
	1995	230	22	4	1.1	3.4
	1999	230	22	4	2.3	2.7
	2002	230	22	4	2.5	3.3
	2005	230	22	5	1.6	5.0
	2009	230	22	5	2.6	4.9
	2011	230	22	5	2.3	3.8
	2015	230	22	4	1.6	3.2

Table 5 (continued)

Country	Year	Seats	Districts	Relevant parties*	Error of the "pot and ladle" formula (%)	Error of the modified cube law (%)
Spain	1977	350	52	4.3	5.2	7.6
	1979	350	52	4.5	6.2	7.8
	1982	350	52	3.3	3.6	4.7
	1986	350	52	3.5	3.2	3.4
	1989	350	52	4.5	1.5	6.9
	1993	350	52	3.7	1.7	4.9
	1996	350	52	3.4	2.4	3.2
	2000	350	52	2.9	3.2	2.9
	2004	350	52	2.5	2.1	2.2
	2008	350	52	2.3	1.2	1.8
	2011	350	52	2.9	3.5	3.3
	2015	350	52	4.4	4.1	6.0
	2016	350	52	4.4	2.7	4.3

*Averaged across regions

Source Global Elections Database (Brancati 2007) for pre-2007 elections. Constituency-Level Elections Archive (Kollman et al. 2018) and the web sites of the national electoral authorities for subsequent elections

Appendix 3: Statutory thresholds

Let $t := (2m + n)^{-1}$ be the natural vote threshold for some fixed $n \in \mathbb{N}$, and let $\tau > t$ be the statutory threshold. Without loss of generality we can assume that the same threshold applies to all parties, and that $p_1 \geq \cdots \geq p_n$. Assume that $p_{n-1} \geq \tau$ and $p_n \geq t$. Now let us consider the seat share of the i-th party, q_i, where $i = 1, \ldots, n - 1$, as a function of p_n. It can be seen that q_i has a jump discontinuity at τ, and its oscillation at that point, T_i, can easily be obtained from the seat share formula (2):

$$T_i := \lim_{p_n \to \tau^-} q_i - \lim_{p_n \to \tau^+} q_i = \frac{p_i}{2m}\left(\frac{\tau/t - 1}{1 - \tau}\right). \tag{17}$$

Formula (17) can easily be extended to the case of k out of n parties being uncertain to cross the threshold (assuming $i = 1, \ldots, n - k$):

$$T_i(k) = \frac{p_i}{2m}\left(\frac{\tau/t - 1}{1/k - \tau}\right). \tag{18}$$

A party's relative seat gain from the others' failure to cross the threshold can be expressed as:

$$\frac{T_i(k)}{q_i} = \frac{\tau - t}{(1/k - \tau)(1 - t/p_i)} \approx \frac{\tau - t}{(1/k - t)(1 - t/p_i)}. \tag{19}$$

Note that

$$\frac{T_i(k)}{q_i} = \frac{\tau - t}{(1/k - \tau)(1 - t/p_i)} \geq k\frac{\tau - t}{1 - t/p_i} \geq k(\tau - t), \tag{20}$$

and

$$\frac{T_i(k)}{q_i} = \frac{\tau - t}{(1/k - \tau)(1 - t/p_i)} \leq \frac{\tau}{1/k - \tau}.$$ (21)

Accordingly, for $k = 1$ a party's relative seat gain from a single competitor's failure to cross the threshold satisfies $\tau - t \leq T_i/q_i \leq \tau/(1 - \tau)$.

Let us move to another issue introduced in Sect. 4.3, the "swing party" problem. Recall that p_A and p_B are the aggregate vote shares of, respectively, party blocs A and B. Under what condition bloc A would benefit from a swing party's failure to clear the threshold? The positive answer is equivalent to the following inequalities:

$$\frac{p_A}{2m}\left(\frac{\tau/t - 1}{1 - \tau}\right) > \frac{1}{2} - p_A\left(1 + \frac{n}{2m}\right) + \frac{n_A}{2m},$$ (22)

$$p_A \frac{\tau/t - 1}{1 - \tau} > m - p_A(2m + n) + n_A,$$ (23)

$$p_A \frac{1 - t}{t(1 - \tau)} > m + n_A.$$ (24)

As we assume that there are no relevant parties other than bloc A, bloc B, and the swing party C, we have $1 - \tau = p_A + p_B$ and $n - 1 = n_A + n_B$, and thus (24) is equivalent to:

$$p_A > \left(\frac{m + n_A}{2m + n}\right)\left(\frac{1 - \tau}{1 - t}\right) = \left(\frac{m + n_A}{2m + n}\right)\left(\frac{p_A + p_B}{1 - t}\right).$$ (25)

Finally, we obtain:

$$\frac{p_A}{m + n_A} > \frac{p_B}{m + n_B}.$$ (26)

References

ACE Project (2019). *Electoral system (Chamber 1)*. ACE Electoral Knowledge Network. http://aceproject.org/epic-en/CDMap?question=ES005&f=. Accessed 14 May 2019.

Baldini, G., & Pappalardo, A. (2009). *Elections, electoral systems and volatile voters*. Basingstoke: Palgrave Macmillan.

Balinski, M. L., & Young, H. P. (1978a). The Jefferson method of apportionment. *SIAM Review, 20*(3), 278–284. https://doi.org/10.1137/1020040.

Balinski, M. L., & Young, H. P. (1978b). Stability, coalitions and schisms in proportional representation systems. *American Political Science Review, 72*(3), 848–858. https://doi.org/10.2307/1955106.

Balinski, M. L., & Young, H. P. (2001). *Fair representation: Meeting the ideal of one man, one vote*. Washington, DC: Brookings Institution Press.

Barceló, J., & Muraoka, T. (2018). The effect of variance in district magnitude on party system inflation. *Electoral Studies, 54,* 44–55. https://doi.org/10.1016/j.electstud.2018.04.016.

Benoit, K. (2000). Which electoral formula is the most proportional? A new look with new evidence. *Political Analysis, 8*(4), 381–388. https://doi.org/10.1093/oxfordjournals.pan.a029822.

Blau, A. (2001). Partisan bias in British general elections. *British Elections & Parties Review, 11*(1), 46–65. https://doi.org/10.1080/13689880108413053.

Bochsler, D. (2010). Who gains from apparentments under D'Hondt? *Electoral Studies, 29*(4), 617–627. https://doi.org/10.1016/j.electstud.2010.06.001.

Bormann, N. C., & Golder, M. (2013). Democratic electoral systems around the world, 1946–2011. *Electoral Studies, 32*(2), 360–369. https://doi.org/10.1016/j.electstud.2013.01.005.

Brancati, D. (2007). *Global Elections Database*. Available at: http://www.globalelectionsdatabase.com/. Accessed 12 June 2016.

Calvo, E., & Rodden, J. (2015). The Achilles heel of plurality systems: Geography and representation in multiparty democracies. *American Journal of Political Science, 59*(4), 789–805. https://doi.org/10.1111/ajps.12167.

Carey J. M. (2017). Electoral system design in new democracies. In Herron, E., Pekkanen, R., & Shugart, M. S. (Eds.) *Oxford handbook of electoral systems* (pp. 85–111). New York: Oxford. https://doi.org/10.1093/oxfordhb/9780190258658.001.0001.

Carstairs, A. M. (1980). *A short history of electoral systems in Western Europe*. London: George Allen & Unwin.

Chafee, Z. (1929). Congressional reapportionment. *Harvard Law Review, 42*(8), 1015–1047. https://doi.org/10.2307/1331072.

Colomer, J. M. (2004). *The handbook of electoral system choice*. London: Palgrave Macmillan.

D'Hondt, V. (1882). *Système pratique et raisonné de représentation proportionnelle*. Bruxelles: Libraire C. Muquardt. https://doi.org/10.3931/e-rara-39876.

D'Hondt, V. (1883). Formule du minimum dans la représentation proportionnelle. Moyen facile de trouver le diviseur. *Représentation proportionnelle. Revue mensuelle 2*, pp. 117–128, 129–130.

D'Hondt, V. (1885). *Exposé du système pratique de représentation proportionnelle. Adopté par le Comité de l'Association Réformiste Belge*. Gand: Eug. Vanderhaeghen.

Dančišin, V. (2013). Hľadanie volebného deliteľa Victorom D'Hondtom. *European Electoral Studies, 10*(1), 63–70.

Deza, M. M., & Deza, E. (2014). *Encyclopedia of distances*. Heidelberg: Springer.

Drton, M., & Schwingenschlögl, U. (2005). Asymptotic seat bias formulae. *Metrika, 62*(1), 23–31. https://doi.org/10.1007/s001840400352.

Equer, M. (1911). Relation entre la méthode d'Hondt et la proportionnalité. *La Grande Revue, Deuxième série, 31,* 130–137.

Evci, U. J., & Kaminski, M. M. (2019). Electoral engineering in 2018 Turkish parliamentary elections. Unpublished manuscript in the authors' possession.

Flis, J., Słomczyński, W., & Stolicki, D. (2019). *Seat allocation and seat bias under the Jefferson–D'Hondt System*. arXiv: 1805.08291 [physics.soc-ph].

Gfeller, J. (1890). Du transfert des suffrages et de la répartition des sièges complémentaires. *Représentation proportionnelle. Revue mensuelle, 9,* 120–131.

Gudgin, G., & Taylor, J. P. (1979). *Seats, votes, and the spatial organisation of elections*. London: Pion.

Hagenbach-Bischoff, E. (1888). *Die Frage der Einführung einer Proportionalvertretung statt des absoluten Mehres*. Basel: H. Georg.

Hagenbach-Bischoff, E. (1905). *Die Verteilungsrechnung beim Basler Gesetz nach dem Grundsatz der Verhältniswahl*. Basel: Berichthaus.

Happacher, M., & Pukelsheim, F. (1996). Rounding probabilities: Unbiased multipliers. *Statistics & Decisions, 14*(4), 373–382. https://doi.org/10.1524/strm.1996.14.4.373.

Happacher, M., & Pukelsheim, F. (2000). Rounding probabilities: Maximum probability and minimum complexity multipliers. *Journal of Statistical Planning and Inference, 85*(1–2), 145–158. https://doi.org/10.1016/S0378-3758(99)00077-4.

Humphreys, J. H. (1911). *Proportional representation: A study in methods of election*. London: Methuen & Co.

Huntington, E. V. (1921). The mathematical theory of the apportionment of representatives. *Proceedings of the National Academy of Sciences, 7*(4), 123–127.

Huntington, E. V. (1928). The apportionment of representatives in Congress. *Transactions of the American Mathematical Society, 30*(1), 85–110.

Huntington, E. V. (1931). Methods of apportionment in congress. *American Political Science Review, 25*(4), 961–965. https://doi.org/10.2307/1946616.

James, E. J. (1897). The first apportionment of federal representatives in the United States. *Annals of the American Academy of Political and Social Science, 9*(1), 1–41.

Janson, S. (2014). Asymptotic bias of some election methods. *Annals of Operations Research, 215*(1), 89–136. https://doi.org/10.1007/s10479-012-1141-2.

Jefferson, T. (1792). Opinion on apportionment bill. In Oberg, B., & Looney, J. J. (2008) *The papers of Thomas Jefferson, digital edition*. Charlottesville: University of Virginia Press.

Joachim, V. (1917). K otázce poměrného zastoupení. *Správní obzor, 9*(8), 289–298.

Kaminski, M. M. (2001). Coalitional stability of multi-party systems: Evidence from Poland. *American Journal of Political Science, 45*(2), 294–312. https://doi.org/10.2307/2669342.

Kaminski, M. M. (2002). Do parties benefit from electoral manipulation? Electoral laws and heresthetics in Poland. *Journal of Theoretical Politics, 14*(3), 325–359. https://doi.org/10.1177/095169280201400 303.

Kaminski, M. M. (2018). Spoiler effects in proportional representation systems: Evidence from eight Polish parliamentary elections, 1991–2015. *Public Choice, 176*(3–4), 441–460. https://doi.org/10.1007/s1112 7-018-0565-x.

Karpov, A. (2015). Alliance incentives under the D'Hondt method. *Mathematical Social Sciences, 74*(C), 1–7. https://doi.org/10.1016/j.mathsocsci.2014.12.001.

Katz, J. N., & King, G. (1999). A statistical model for multiparty electoral data. *American Political Science Review, 93*(1), 15–32. https://doi.org/10.2307/2585758.

Kollman, K., Hicken, A., Caramani, D., Backer, D., & Lublin, D. (2018). Constituency-Level Elections Archive. Ann Arbor: Center for Political Studies, University of Michigan. Available at: http://www. electiondataarchive.org/. Accessed 12 June 2016.

Leutgäb, P., & Pukelsheim, F. (2009). List apparentements in local elections—A lottery. In Holler, M., & Nurmi, H. (Eds.) *Power, voting, and voting power: 30 years after* (pp. 489–500). Berlin: Springer. https ://doi.org/10.1007/978-3-642-35929-3_7.

Li, Y., & Shugart, M. S. (2016). The Seat Product Model of the effective number of parties: A case for applied political science. *Electoral Studies, 41,* 23–43. https://doi.org/10.1016/j.electstud.2015.10.011.

Lijphart, A. (1990). The political consequences of electoral laws, 1945–85. *American Political Science Review, 84*(2), 481–496. https://doi.org/10.2307/1963530.

Lijphart, A., & Gibberd, R. W. (1977). Thresholds and payoffs in list systems of proportional representation. *European Journal of Political Research, 5*(3), 219–244. https://doi.org/10.1111/j.1475-6765.1977. tb01289.x.

Linzer, D. A. (2012). The relationship between seats and votes in multiparty systems. *Political Analysis, 20*(3), 400–416. https://doi.org/10.1093/pan/mps017.

Marshall, A. W., Olkin, I., & Pukelsheim, F. (2002). A majorization comparison of apportionment methods in proportional representation. *Social Choice and Welfare, 19*(4), 885–900. https://doi.org/10.1007/ s003550200164.

McGhee, E. (2014). Measuring partisan bias in single-member district electoral systems. *Legislative Studies Quarterly, 39*(1), 55–85. https://doi.org/10.1111/lsq.12033.

McGhee, E. (2017). Measuring efficiency in redistricting. *Election Law Journal, 16*(4), 417–442. https:// doi.org/10.1089/elj.2017.0453.

Mora, X. (2013). La regla de Jefferson–D'Hondt i les seves alternatives. *Materials Matemàtics, 4,* 1–34.

Morse, M., Von Neumann, J., & Eisenhart, L. P. (1948). *Report to the President of the National Academy of Sciences*. Washington, DC: National Academy of Sciences.

Palomares, A., & Ramírez, Gonzáles V. (2003). Thresholds of the divisor methods. *Numerical Algorithms, 34*(2), 405–415. https://doi.org/10.1023/B:NUMA.0000005353.82970.ce.

Pavia, J., & García-Cárceles, B. (2016). Estimating representatives from election poll proportions: The Spanish case. *Statistica Applicata—Italian Journal of Applied Statistics, 25*(3), 325–340.

Pólya, G. (1918a). Sur la représentation proportionnelle en matière électorale. *L'Enseignement Mathématique, 20,* 355–379.

Pólya, G. (1918b). Über die Verteilungssysteme der Proportionalwahl. *Zeitschrift für schweizerische Statistik und Volkswirtschaft, 54,* 363–387.

Pólya, G. (1919a). Proportionalwahl und wahrscheinlichkeitsrechnung. *Zeitschrift für die gesamte Staatswissenschaft, 74,* 297–322.

Pólya, G. (1919b). Über die systeme der sitzverteilung bei proportionalwahl. *Wissen und Leben—Schweizerische Halbmonatsschrift, 12*(259–268), 307–312.

Poptcheva, E. M. (2016). *Understanding the D'Hondt method. Allocation of parliamentary seats and leadership positions*. European Parliamentary Research Service Briefing PE 580.901, http://www.europarl. europa.eu/RegData/etudes/BRIE/2016/580901/EPRS_BRI(2016)580901_EN.pdf.

Pukelsheim, F. (2014). *Proportional representation: Apportionment methods and their applications*. Heidelberg: Springer.

Pukelsheim, F. (2017). *Proportional representation: Apportionment methods and their applications* (2nd ed.). Heidelberg: Springer.

Rae, D. W. (1967). *The political consequences of electoral laws*. New Haven: Yale University Press.

Rae, D. W., Hanby, V. J., & Loosemore, J. (1971). Thresholds of representation and thresholds of exclusion. An analytic note on electoral systems. *Comparative Political Studies, 3*(4), 479–488. https://doi. org/10.1177/001041407100300406.

Rokkan, S. (1968). Elections: Electoral systems. In Sills, D. L. (Ed.) *International encyclopaedia of the social sciences* (Vol. 5, pp. 6–21). New York: Crowell-Collier-Macmillan.

Sainte-Laguë, A. (1910). La représentation proportionnelle et la méthode des moindres carrés. *Annales scientifiques de l'École Normale Supérieure, Sér., 3*(27), 529–542. https://doi.org/10.24033/asens.627.

Schuster, K., Pukelsheim, F., Drton, M., & Draper, N. R. (2003). Seat biases of apportionment methods for proportional representation. *Electoral Studies, 22*(4), 651–676. https://doi.org/10.1016/S0261-3794(02)00027-6.

Shugart, M. S., & Taagepera, R. (2017). Electoral system effects on party systems. In Herron, E., Pekkanen, R., & Shugart, M. S. (Eds.) *Oxford handbook of electoral systems* (pp. 41–68). New York: Oxford. https://doi.org/10.1093/oxfordhb/9780190258658.001.0001.

Shugart, M. S., & Taagepera, R. (2017b). *Votes from seats: Logical models of electoral systems*. Cambridge: Cambridge University Press.

Stephanopoulos, N. O., & McGhee, E. M. (2015). Partisan gerrymandering and the efficiency gap. *University of Chicago Law Review, 82*(2), 831–900.

Stephanopoulos, N. O., & McGhee, E. M. (2018). The measure of a metric: The debate over quantifying partisan gerrymandering. *Stanford Law Review, 70*(5), 1503–1568.

Szpiro, G. G. (2010). *Numbers rule: The vexing mathematics of democracy, from Plato to the present*. Princeton: Princeton University Press.

Taagepera, R. (1986). Reformulating the cube law for proportional representation elections. *American Political Science Review, 80*(2), 489–504. https://doi.org/10.2307/1958270.

Taagepera, R. (2007). *Predicting party sizes: The logic of simple electoral systems*. Oxford: Oxford University Press.

Taagepera, R., & Laakso, M. (1980). Proportionality profiles of West European electoral systems. *European Journal of Political Research, 8*(4), 423–446. https://doi.org/10.1111/j.1475-6765.1980.tb00582.x.

Taagepera, R., & Shugart, M. S. (1989). *Seats and votes: The effects and determinants of electoral systems*. New Haven: Yale University Press.

Tapp, K. (2018). *Measuring political gerrymandering*. arXiv: 1801.02541 [physics.soc-ph].

Udina, F., & Delicado, P. (2005). Estimating parliamentary composition through electoral polls. *Journal of the Royal Statistical Society. Series A (Statistics in Society), 168*(2), 387–399.

Veomett, E. (2018). *The efficiency gap, voter turnout, and the efficiency principle*. arXiv: 1801.05301 [physics.soc-ph].