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Potential in the Canonical Formalism of Gravity

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We attempt to derive the gravitational potential from the canonical formalism viewpoint of the gravitational field coupled with matter. The two-body potential in an arbitrary coordinate frame (i.e., gauge) is obtained by solving constraint equations. It is shown, by making a transformation of the time coordinate, that the potential coincides with Hiida's obtained with the aid of the "diagrammatic" quantization. A simple equation to get the static potential to any order is presented. Explicit forms of the static potential are given up to the five-body potential. From the form of potential it is immediately seen that there exists a relation between two-body and many-body static potentials.

§ 1. Introduction

Recently some authors[1] have obtained the equation of motion of Einstein, Infeld and Hoffmann[2] (hereafter referred to as EIH) relying upon the quantum theory of gravity within the framework of the Minkowski space. As is well known the pure graviton (transverse traceless part of the gravitational field) does not contribute to the EIH approximation. The parts other than the transverse traceless part of the gravitational field have no particle property[3] and consequently cannot be quantized properly but are subjected to constraints. Therefore, the potential in the EIH equation should be determined by solving the constraint equations.[4] This means that one should treat the problem in a gauge which corresponds to the Coulomb gauge in quantum electrodynamics.

When we derive the gravitational potential, there appears a problem of whether the explicit form of the potential depends upon the coordinate frame. Hiida and Okamura have shown that the velocity dependent two-body potential depends on the gauge used and further that the static gravitational potential of order G^2 (G: Newton's gravitational constant) also depends on the gauge.[5] In the EIH approximation, the Hamiltonian of Hiida and Okamura is expressed as

$$H_0=\sum_a\{\underset{a}{p_i}\underset{a}{p_i}/2\underset{a}{m}-(\underset{a}{p_i}\underset{a}{p_i})^2/8\underset{a}{m}^3\}-G\underset{1}{m}\underset{2}{m}/r+\tfrac{1}{2}G^2\underset{1}{m}\underset{2}{m}(\underset{1}{m}+\underset{2}{m})/r^2$$
$$-\tfrac{3}{2}G\{(\underset{2}{m}/\underset{1}{m})\underset{1}{p_i}\underset{1}{p_i}+(\underset{1}{m}/\underset{2}{m})\underset{2}{p_i}\underset{2}{p_i}\}/r+4G\underset{1}{p_i}\underset{2}{p_i}/r-\tfrac{1}{2}G\underset{1}{p_i}\underset{2}{p_j}r_{,ij}$$
$$-G(z-1)\underset{1}{p_i}\underset{2}{p_j}r_{,ij}+\tfrac{1}{2}G(z-1)\{(\underset{2}{m}/\underset{1}{m})\underset{1}{p_i}\underset{1}{p_j}+(\underset{1}{m}/\underset{2}{m})\underset{2}{p_i}\underset{2}{p_j}\}r_{,ij}$$

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$$-\tfrac{1}{2}G^2(z-1)\underset{1}{m}\underset{2}{m}(\underset{1}{m}+\underset{2}{m})/r^3, \tag{1·1}$$

in which $(z-1)=\frac{1}{2}(x-1)$ where x is the parameter of Hiida and Okamura. However, they showed that the total potential gives rise to the same value of the advance of the perihelion of the orbit of the relative motion irrespective of the gauge. Their result was obtained by using "diagrammatic" quantization. It is therefore interesting to deal with the problem from the standpoint of the canonical formalism, since the relation between the arbitrariness of the coordinate frame and the gauge freedom is clear in the canonical formalism. This is one of the aims of the present paper.

On the other hand, Hiida and Okamura[6)] have discussed the relation between the two-body and many-body static gravitational potentials and found a simple substitution law to get the two-body potential in order G^{n-1} from the n-body potential obtained from tree diagrams for n-body scattering. Since the transverse traceless part of the gravitational field does not contribute to the static gravitational potential (except for the renormalization effect for G), one should also determine the static potential of higher order in G by solving the constraint equations.

In order to work out the constraint equations it seems to us that the most adequate procedure to adopt is the canonical formulation of the gravitational field developed by Arnowitt, Deser and Misner.[7)] In this paper, the authors show that, by employing their canonical formalism, a simple equation for determining the static potential of higher order in G is found from the constraint equations. Explicit forms of the static gravitational potentials of orders G^3 and G^4 are also given in addition to the EIH potential in general gauge. The two-body potential of Hiida and Okamura[5)] is derived from our potential by making a transformation of the time coordinate. In our description it is easily seen that the total potential is gauge invariant and the substitution law of Hiida and Okamura[6)] holds.

§ 2. Canonical formalism and coordinate conditions

We shall first present a brief sketch of the canonical formalism for the gravitational field. The Lagrangian density L_G for the gravitational field coupled with matter fields whose energy-momentum tensor is $T_\nu{}^\mu$ can be written as (except for the divergence terms)*)

$$L_G=\pi^{ij}g_{ij,0}+N_\mu R^\mu, \tag{2·1}$$

where

$$N_0\equiv(-{}^4g^{00})^{-1/2},\quad N_i\equiv{}^4g_{0i},\quad R^0=g^{-1/2}(gR+\tfrac{1}{2}\pi^2-\pi^{ij}\pi_{ij})-T_0{}^0,$$

*) Latin indices run from 1 to 3, Greek from 0 to 3. Ordinary differentiation is denoted by a comma. We use units $16\pi Gc^{-4}=c=\hbar=1$ and summation convention for all repeated tensor indices.

$$R^i = 2\pi^{ij}{}_{|j} + T^i{}_0, \quad \pi^{ij} = (-{}^4g)^{1/2}({}^4\Gamma^0_{mn} - g_{mn}g^{pq}{}^4\Gamma^0_{pq})g^{im}g^{jn}, \quad \pi = g_{ij}\pi^{ij}.$$

Here and in the following we mark every four-dimensional quantity with the prefix "4", so that all unmarked quantities shoud be understood as three-dimensional. In particular, $g^{ij}(={}^4g^{ij}+N^iN^j/N^2)$ is the matrix inverse to $g_{ij}(\neq {}^4g_{ij})$ and all three-dimensional indices are raised and lowered with respect to g^{ij} and g_{ij}. The R means the three-dimensional curvature scalar formed from g_{ij} and g^{ij}, and the vertical bar " | " in R_i indicates three-dimensional covariant differentiation.

Varying Eq. (2·1), we obtain the constraint equations:

$$g^{-1/2}(gR + \tfrac{1}{2}\pi^2 - \pi_{ij}\pi^{ij}) - T^0{}_0 = 0\,, \tag{2·2}$$

$$2\pi^{ij}{}_{|j} + T^i{}_0 = 0\,. \tag{2·3}$$

We decompose π^{ij} and $\bar{h}_{ij}$ given by

$$\bar{h}_{ij} = g_{ij} + \delta_{ij} \tag{2·4}$$

in the following way

$$f_{ij} = f^{TT}_{ij} + f^T_{ij} + f_{i,j} + f_{j,i}\,, \tag{2·5}$$

where f^{TT}_{ij} is the transverse traceless part of f_{ij} and

$$\begin{aligned} f^T_{ij} &= \tfrac{1}{2}\{f^T\delta_{ij} - (1/\Delta)f^T{}_{,ij}\}, \\ f^T &= f_{ii} - (1/\Delta)f_{ij,ij}\,, \\ f_i &= (1/\Delta)\{f_{ij,j} - \tfrac{1}{2}(1/\Delta)f_{mn,mni}\}, \end{aligned} \tag{2·6}$$

where $\Delta = \nabla^2$.

On inserting the orthogonal decomposition of g_{ij} and π^{ij}, the generator of canonical transformation arising from the Lagrangian (2·1) is expressed as

$$\begin{aligned} G = \int d^3x\{&\pi^{ijTT}\delta\bar{h}^{TT}_{ij} - (-\Delta\bar{h}^T)\delta[-\tfrac{1}{2}(1/\Delta)\{\pi^T + z(1/\Delta)\pi^{mn}{}_{,mn}\}] \\ &+ (-2\pi^{ij}{}_{,j})\delta[\bar{h}_i - \tfrac{1}{4}z(1/\Delta)\bar{h}^T{}_{,i}]\}, \end{aligned} \tag{2·7}$$

in which the constraint equations (2·2) and (2·3) are taken into account. The z in (2·7) is an arbitrary numerical constant and designates gauges. If we impose the following coordinate conditions

$$x^0 = t = -\tfrac{1}{2}(1/\Delta)\{\pi^T + z(1/\Delta)\pi^{mn}{}_{,mn}\}, \tag{2·8}$$

$$x^i = \bar{h}^i - \tfrac{1}{4}z(1/\Delta)\bar{h}^T{}_{,i}\,, \tag{2·9}$$

the generator can be written in a canonical form

$$G = \int d^3x\{\pi^{ijTT}\delta\bar{h}^{TT}_{ij} + \mathcal{T}^\mu{}_0\delta x^\mu\}, \tag{2·10}$$

where

$$\mathcal{T}_0{}^0 = -H = \Delta\bar{h}^T, \tag{2·11}$$

$$\mathcal{T}_0{}^i = -2\pi^{ij}{}_{,j}\,.$$

In order to describe the gravitational field in terms of ordinary variables

$$h_{ij} = g_{ij} - \delta_{ij}\,, \tag{2·12}$$

it is enough to carry out the following replacements,

$$\bar{h}_{ij}^{TT} = h_{ij}^{TT}, \quad \bar{h}_{ij}^{T} = h_{ij}^{T}, \quad \bar{h}^{T} = h^{T}, \quad \bar{h}_i = h_i + x^i. \tag{2·13}$$

The coordinate conditions (2·8) and (2·9) are rewritten as

$$h_i - \tfrac{1}{4} z (1/\Delta) h^T{}_{,i} = 0\,, \tag{2·14}$$

$$\pi^T + z(1/\Delta)\pi^{mn}{}_{,mn} \equiv \pi^{ii} + (z-1)(1/\Delta)\pi^{mn}{}_{,mn} = 0\,. \tag{2·15}$$

From (2·14) it follows that

$$g_{ij} = h_{ij}^{TT} + \delta_{ij}(1 + \tfrac{1}{2}h^T) + \tfrac{1}{2}(z-1)h^T{}_{,ij}. \tag{2·16}$$

On the other hand, the Lagrangian density is reduced to

$$L_G = \pi^{ijTT} h_{ij,0}^{TT} + \Delta h^T, \tag{2·17}$$

in which Δh^T is the solution of constraint equations (2·2) and (2·3). We have then true dynamical variables h_{ij}^{TT} and π^{ijTT} as the canonical variables. The Hamiltonian density describing our system is given by

$$H = -\Delta h^T. \tag{2·18}$$

It should be emphasized that the above $-\Delta h^T$ is the abbreviation for its value in terms of h_{ij}^{TT}, π^{ijTT}, x^μ and dynamical variables of matter fields as obtained from solving the constraint equations. Thus, $-\Delta h^T$ is not in general divergence.

We shall now pick out a preferred coordinate system characterized by

$$z = 1\,. \tag{2·19}$$

In this coordinate system, $\pi^{ijTT} = 0$ means $\pi^{ij} = 0$ provided the $T^i{}_0$ of the matter field is considered to vanish. The metric tensor can be written as

$$g_{ij} = h_{ij}^{TT} + \delta_{ij}(1 + \tfrac{1}{2}h^T) \tag{2·20}$$

and the boundary value condition is

$$h^T \to 0 \quad \text{as } r \to \infty\,. \tag{2·21}$$

The metric tensor becomes isotropic when $\pi^{ijTT} = h_{ij}^{TT} = T^i{}_0 = 0$. Thus, the coordinate system with $z=1$ is the most appropriate for treating the static gravitational potential as will be done in § 4.

§ 3. Two-body potential in general gauge

We shall now find the explicit form of the potential in general gauge within the approximation carried out by EIH where h_{ij}^{TT} does not contribute. In the first, we shall express the constraint equations (2·2) and (2·3) in the form suitable for the present approximation.

We expand $g^{1/2}R$ in general gauge as follows, though it can be simplified in the special gauge $z=1$ (cf. § 4),

$$g^{1/2}R=(g^{1/2}g^{ik}\Gamma^j_{ik})_{,j}-(g^{1/2}g^{ik}\Gamma^j_{ij})_{,k}$$
$$+\tfrac{1}{2}h_{in,m}h_{im,n}-\tfrac{1}{4}h_{mn,j}h_{mn,j}-\tfrac{1}{2}h_{ii,n}h_{mn,m}+\tfrac{1}{4}h_{ii,m}h_{jj,m}$$
$$+\tfrac{1}{4}h_{ii}\{-\tfrac{1}{2}h_{mn,j}h_{mn,j}+\tfrac{1}{2}h_{ii,m}h_{jj,m}+h_{jm,i}h_{im,j}-h_{ij,i}h_{nn,j}\}$$
$$+\tfrac{1}{2}h_{ij}\{\tfrac{1}{2}h_{mn,i}h_{mn,j}-\tfrac{1}{2}h_{mm,i}h_{nn,j}-h_{in,m}h_{jm,n}+h_{mn,m}h_{ij,n}$$
$$+h_{im,n}h_{jm,n}-h_{ij,m}h_{nn,m}+h_{jm,i}h_{mn,m}+h_{mm,i}h_{jn,n}-2h_{mn,i}h_{jn,m}\}, \qquad (3\cdot 1)$$

which can be reduced to

$$g^{1/2}R=-\Delta h^T+\frac{1}{8}\Delta(h^Th^T)+\frac{1}{4}(z-1)\left(h^T{}_{,n}\frac{1}{\Delta}h^T{}_{,mn}\right)_{,n}+\frac{1}{8}h^T{}_{,i}h^T{}_{,i}$$
$$+\frac{1}{16}(z-1)^2\left\{\frac{1}{2}\left(\frac{1}{\Delta}h^T{}_{,jn}\frac{1}{\Delta}h^T{}_{,jn}\right)_{,mm}-\left(h^T{}_{,j}\frac{1}{\Delta}h^T{}_{,jn}\right)_{,n}\right\}-\frac{3}{32}h^Th^T{}_{,m}h^T{}_{,m}$$
$$-\frac{1}{32}(z-1)h^Th^T{}_{,m}h^T{}_{,m}+\frac{1}{64}(z-1)^2\left\{\frac{1}{2}\frac{1}{\Delta}h^T{}_{,mn}\frac{1}{\Delta}h^T{}_{,mn}\Delta h^T\right.$$
$$\left.-\frac{1}{4}h^Th^T\Delta h^T-\frac{1}{\Delta}h^T{}_{,m}h^T{}_{,m}\Delta h^T\right\}. \qquad (3\cdot 2)$$

In the above reduction, we discarded the divergence of cubic terms of h^T which do not contribute to the Hamiltonian $\boldsymbol{H}=\int d^3xH(x)$. On the contrary, it should be noted that the divergences of quadratic terms of h^T give rise to the G^2 static potential. As $T^\mu{}_0$ we take the energy-momentum tensor for the system of two point particles

$$T^i{}_0(x)=\sum_{a=1}^{2}\underset{a}{p^i}\delta(\boldsymbol{x}-\underset{a}{\boldsymbol{x}}), \qquad (3\cdot 3)$$

$$T^0{}_0(x)=\sum_a(\underset{a}{p_i}\underset{a}{p^i}+\underset{a}{m}^2)^{1/2}\delta(\boldsymbol{x}-\underset{a}{\boldsymbol{x}})$$
$$=\sum_a\underset{a}{m}\left[1+\underset{a}{p_i}\underset{a}{p_j}\left\{\left(1-\frac{1}{2}h^T\right)\delta_{ij}-\frac{1}{2}(z-1)\frac{1}{\Delta}h^T{}_{,ij}\right\}\Big/\underset{a}{m}^2\right]^{1/2}\delta(\boldsymbol{x}-\underset{a}{\boldsymbol{x}}) \qquad (3\cdot 4)$$

in which the index a written as $\underset{a}{m}$ is used to distinguish between particles. It follows, in the present approximation, that

$$-\frac{1}{2}\pi^2+\pi^{ij}\pi_{ij}=-\frac{1}{2}\pi^2+\pi^{ij}\frac{1}{2}\left(\delta_{ij}\pi^T-\frac{1}{\Delta}\pi^T{}_{,ij}\right)+2\pi^{ij}\pi_{i,j}$$
$$=(z-1)\frac{1}{\Delta}\pi^{ij}{}_{,ij}\frac{1}{\Delta}\pi^{mn}{}_{,mn}+\frac{1}{2}\frac{1}{\Delta}\pi^{ij}{}_{,ij}\frac{1}{\Delta}\pi^{mn}{}_{,mn}+2\pi^{ij}\pi_{i,j} \qquad (3\cdot 5)$$

in which the coordinate condition (2·15) is used.

To simplify the calculations in some degree we introduce φ in place of h^T by

$$(1+\tfrac{1}{8}\varphi)^4=1+\tfrac{1}{2}h^T. \tag{3·6}$$

Inserting Eqs. (3·3) (3·6) into Eqs. (2·2) and (2·3), we have

$$\begin{aligned}-\varDelta\varphi=&\frac{1}{8}\varphi\varDelta\varphi-\frac{1}{4}(z-1)\left(\varphi_{,n}\frac{1}{\varDelta}\varphi_{,mn}\right)_{,m}-\frac{1}{16}(z-1)^2\left\{\frac{1}{2}\left(\frac{1}{\varDelta}\varphi_{,jn}\frac{1}{\varDelta}\varphi_{,jn}\right)_{,mm}\right.\\&\left.-\left(\varphi_{,j}\frac{1}{\varDelta}\varphi_{,jn}\right)_{,n}\right\}+\frac{1}{32}(z-1)\,\varphi\varphi_{,m}\varphi_{,m}-\frac{1}{64}(z-1)^2\left\{\frac{1}{2}\frac{1}{\varDelta}\varphi_{,mn}\frac{1}{\varDelta}\varphi_{,mn}\varDelta\varphi\right.\\&\left.-\frac{1}{4}\varphi\varphi\varDelta\varphi-\frac{1}{\varDelta}\varphi_{,m}\varphi_{,m}\varDelta\varphi\right\}+\frac{1}{2}(2z-1)\frac{1}{\varDelta}\pi^{ij}{}_{,ij}\frac{1}{\varDelta}\pi^{mn}{}_{,mn}+2\pi^{ij}\pi^{i}{}_{,j}\\&+\sum_a \underset{a}{m}\left\{1+\frac{1}{2}\underset{a}{p_i}\underset{a}{p_i}\left(1-\frac{1}{2}\varphi\right)\Big/\underset{a}{m}^2-\frac{1}{4}(z-1)\underset{a}{p_i}\underset{a}{p_j}\frac{1}{\varDelta}\varphi_{,ij}\Big/\underset{a}{m}^2\right.\\&\left.-\frac{1}{8}(\underset{a}{p_i}\underset{a}{p_i})^2\Big/\underset{a}{m}^4\right\}\delta(\boldsymbol{x}-\underset{a}{\boldsymbol{x}}),\end{aligned} \tag{3·7}$$

$$-2(\varDelta\pi^i+\pi^j{}_{,ij})=\sum_a \underset{a}{p_i}\delta(\boldsymbol{x}-\underset{a}{\boldsymbol{x}}). \tag{3·8}$$

Let us expand φ in power series of (v/c) of particle:*)

$$\varphi={}_2\varphi+{}_4\varphi+{}_6\varphi+\cdots, \tag{3·9}$$

where the indices written as left subscripts indicate the order of (v/c). The π^i and $(1/\varDelta^2)\pi^{mn}{}_{,mni}$ which are of order $(v/c)^3$ are then found to be

$$\pi^i=\sum_a\{\underset{a}{p_i}/8\pi\underset{a}{r}-\underset{a}{p_j}\underset{a}{r}_{,ij}/32\pi\}, \tag{3·10}$$

$$(1/\varDelta^2)\pi^{mn}{}_{,mni}=\sum_a \underset{a}{p_j}\underset{a}{r}_{,ij}/16\pi, \tag{3·11}$$

in which $\underset{a}{r}=\{(x^k-\underset{a}{x}^k)(x^k-\underset{a}{x}^k)\}^{1/2}$ and the relation $(1/\varDelta^2)\delta(\boldsymbol{x}-\underset{a}{\boldsymbol{x}})=-\underset{a}{r}/8\pi$ is used. With the aid of the successive approximation method we obtain

$${}_2\varphi=\sum_a \underset{a}{m}/4\pi\underset{a}{r}, \tag{3·12}$$

$$\begin{aligned}{}_4\varphi=&\sum_a\{-\underset{1}{m}\underset{2}{m}/128\pi^2 r\underset{a}{r}+\underset{a}{p_i}\underset{a}{p_i}/8\pi\underset{a}{m}\underset{a}{r}\}+\frac{1}{4}(z-1)\frac{1}{\varDelta}\left({}_2\varphi_{,n}\frac{1}{\varDelta}{}_2\varphi_{,mn}\right)_{,m}\\&+\frac{1}{16}(z-1)^2\frac{1}{\varDelta}\left\{\frac{1}{2}\varDelta\left(\frac{1}{\varDelta}{}_2\varphi_{,jn}\frac{1}{\varDelta}{}_2\varphi_{,jn}\right)-\left({}_2\varphi_{j}\frac{1}{\varDelta}{}_2\varphi_{,jn}\right)_{,n}\right\},\end{aligned} \tag{3·13}$$

where $r=\{(\underset{1}{x}^k-\underset{2}{x}^k)(\underset{1}{x}^k-\underset{2}{x}^k)\}^{1/2}$. After some straightforward calculation we have

$$\begin{aligned}\frac{1}{8}\int d^3x\varphi\varDelta\varphi=&\frac{1}{8}\int d^3x\{{}_2\varphi\varDelta_2\varphi+{}_4\varphi\varDelta_2\varphi+{}_2\varphi\varDelta_4\varphi\}\\=&-\underset{1}{m}\underset{2}{m}/16\pi r+\underset{1}{m}\underset{2}{m}(\underset{1}{m}+\underset{2}{m})/2(16\pi)^2r^2-\{(\underset{2}{m}/\underset{1}{m})\underset{1}{p_i}\underset{1}{p_i}\end{aligned}$$

*) Following EIH, we regard that $\dot{x}$ as being of order (v/c) and $\underset{a}{m}$ of order $(v/c)^2$.

$$+ (m_1/m_2) p_{2i} p_{2i}\}/2(16\pi)r - \frac{3}{4}(z-1)\, m_1 m_2 (m_1+m_2) \Big/ (16\pi)^2 r^2$$

$$+\frac{1}{64}(z-1)^2 \int d^3x \left\{\frac{1}{2}\,\frac{1}{\Delta}\varphi_{,mn}\,\frac{1}{\Delta}\varphi_{,mn} - \frac{1}{4}\varphi\varphi\Delta\varphi - \frac{1}{\Delta}\varphi_{,m}\varphi_{,m}\Delta\varphi\right\}. \qquad (3\cdot14)$$

From Eqs. (3·7), (3·10) and (3·14), it follows that

$$H = -\int d^3x \Delta h^T = -\int d^3x \Delta\varphi = \sum_a \{p_{ai}p_{ai}/2m_a - (p_{ai}p_{ai})^2/8m_a^3\} - G m_1 m_2/r$$
$$+ G^2 m_1 m_2 (m_1+m_2)/2r^2 - \tfrac{3}{2}G\{(m_2/m_1) p_{1i}p_{1i} + (m_1/m_2) p_{2i}p_{2i}\}/r + 4G p_{1i}p_{2i}/r$$
$$-\tfrac{1}{2}G p_{1i}p_{2j} r_{,ij} + G(z-1) p_{1i}p_{2j} r_{,ij} - \tfrac{1}{2}G(z-1)\{(m_2/m_1) p_{1i}p_{1j}$$
$$+ (m_1/m_2) p_{2i}p_{2j}\} r_{,ij} - \tfrac{1}{2}(z-1) G^2 m_1 m_2 (m_1+m_2)/r^2, \quad (G=1/16\pi) \qquad (3\cdot15)$$

which is nothing but the energy integral in our coordinate system. Our Hamiltonian (3·15) is equal to (1·1) of Hiida and Okamura except for the sign of $G(z-1)$ of the velocity dependent potentials. The difference comes from the fact that our time coordinate depends on z as seen from Eq. (2·8), whereas Hiida's is independent of the gauge parameter. By taking account of this fact, it would be verified that the above Hamiltonian gives rise to the same shift in the perihelion of relative motion of two bodies. In § 5, we shall actually show that this is so.

We shall here reduces (3·15) to (1·1) by making a transformation of time coordinate. From Eqs. (2·9) and (2·16), it follows that the transformation of space coordinates $x^i \to x'^i = x^i - (1/4)(z-1)(1/\Delta) h^T{}_{,i}$ induces the gauge transformation $g_{ij}(x) \to g'_{ij}(x) = g_{ij}(x) + \frac{1}{2}(z-1) h^T{}_{,ij}$ (cf. $g'_{ij}(x) \neq g'_{ij}(x')$). Under such a gauge transformation h^T is invariant and then the energy integral $-\int d^3x \Delta h^T$ remains invariant. Therefore, if the time coordinate is independent of $(z-1)$, the relationship between p_i and $\dot{x}^i$ would also be independent of $(z-1)$. On the other hand, we have from Eq. (3·15).

$$p_{ai} = m_a \dot{x}_a^i + \tfrac{1}{2} m_a \dot{x}_a^i (\dot{\boldsymbol{x}}_a)^2 - 3G m_a \dot{x}_a^i m_b/|\boldsymbol{x}_a - \boldsymbol{x}_b| + 4G m_a m_b \dot{x}_b^i/|\boldsymbol{x}_a - \boldsymbol{x}_b|$$
$$-\tfrac{1}{2}G m_a m_b \dot{x}_b^j |\boldsymbol{x}_a - \boldsymbol{x}_b|_{,ij} + G(z-1) m_a (m_b \dot{x}_b^j - m_b \dot{x}_a^j) |\boldsymbol{x}_b - \boldsymbol{x}_a|_{,ij}. \qquad (3\cdot16)$$

The last term may be considered as caused by the transformation of the time coordinate. Accordingly, if we express the energy integral in terms of the same time coordinate as that of the special isotropic frame $z=1$, we must replace p_{ai} in (3·15) by

$$p_{ai} \to p_{ai} - G(z-1)(m_a p_b^j - m_b p_a^j) |\boldsymbol{x}_a - \boldsymbol{x}_b|_{,ij}. \qquad (3\cdot17)$$

This replacement enables us to reduce (3·15) to (1·1).*) We may conclude

*) By making use of a unitary transformation the Hamiltonian (1·1) with arbitrary z can be transformed into the Hamiltonian of EIH with $z=1$. A detailed discussion is given in Ref. 10).

that our potential is the most general one.

§ 4. Static potential of higher order

When the static gravitational potential is concerned with, one can put

$$h_{ij}^{TT}=\pi^{ijTT}=\pi^{T}=\pi^{i}=0\,. \tag{4·1}$$

The constraint equation (2·2) reduces to

$$g^{1/2}R=T^{0}{}_{0}\,. \tag{4·2}$$

On the other hand, Eq. (2·3) is satisfied automatically, since $T^{i}{}_{0}$ can be put to zero in the static approximation. As was seen in the preceding section, it is useful to take the special coordinate system with $z=1$ to study the potential. We shall therefore refer to this coordinate system. From (2·16), g_{ij} can be written as

$$g_{ij}=\chi^{4}\delta_{ij} \tag{4·3}$$

which enables us to write the constraint equation (4·2) as

$$-8\chi\Delta\chi=T^{0}{}_{0}\,. \tag{4·4}$$

In the above derivation, no approximation is carried out except for Eq. (4·1). The superiority of the isotropic coordinate system with $z=1$ is in the fact that $g^{1/2}R$ can be rigorously expressed as $-8\chi\Delta\chi$.

We shall now work out the constraint equation (4·4). On putting

$$\chi=1+\tfrac{1}{8}\varphi\,, \tag{4·5}$$

we solve the non-linear equation for φ by means of successive approximation. When we proceed the approximation, there arises a problem of non-commutability between $T^{0}{}_{0}(x)$ and $T^{0}{}_{0}(x')$ at equal time. If the $T^{0}{}_{0}$ is constructed by fields with spin 0, $\frac{1}{2}$, 1 but not fields of higher spin, it obeys the commutation relation at equal time[8)]

$$[T^{00}(x),T^{00}(x')]=-i\{T^{0k}(x)+T^{0k}(x')\}\partial_{k}\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{4·6}$$

Accordingly, we are able to regard $T^{0}{}_{0}$ as a commutable quantity as far as the static approximation is concerned. The formal solution for φ is given by

$$\begin{aligned}\varphi&=C[T(1+\tfrac{1}{8}\varphi)^{-1}]\\&=C[T\{1+(-\varphi/8)+(-\varphi/8)^{2}+\cdots\}],\end{aligned} \tag{4·7}$$

where $T=T^{0}{}_{0}$ and

$$C(x,x')=(1/4\pi)1/|\boldsymbol{x}-\boldsymbol{x}'| \tag{4·8}$$

which statisfies $-\Delta C(x,x')=\delta(\boldsymbol{x}-\boldsymbol{x}')$. In Eq. (4·7) we regard $C(x,x')$ as the matrix element of an operator in which states are labeled by space coordinates, that is, Eq. (4·7) means

$$\varphi(x) = (1/4\pi) \int d^3x' C(x, x') T(x') \{1+\tfrac{1}{8}\varphi(x')\}^{-1}.$$

We substitute (4·7) into φ appearing on the right-hand side of Eq. (4·7) and repeat the process. The solution is written as

$$\begin{aligned}\varphi = &CT + (-1/8)CTCT + (-1/8)^2 CT\{(CT)^2 + CTCT\}\\ &+ (-1/8)^3 CT\{(CT)^3 + CT(CT)^2 + 2(CT)CTCT + CTCTCT\}\\ &+ (-1/8)^4 CT\{(CT)^4 + (CTCT)(CTCT) + CT(CT)^3\\ &+ 2(CT)CT(CT)^2 + 2CT(CT)CTCT + 3(CT)^2CTCT\\ &+ 2(CT)CTCTCT + CTCT(CT)^2 + CTCTCTCT\}\\ &+ (-1/8)^5\{\cdots\} + \cdots,\end{aligned} \tag{4·9}$$

in which, for instance,

$$\begin{aligned}CT(CT)^2CTCT = &(1/4\pi)^5 \int\cdots\int d^3x_1 d^3x_2 \cdots d^3x_5 \frac{1}{|\boldsymbol{x}-\boldsymbol{x}_1|} T(x_1) \frac{1}{|\boldsymbol{x}_1-\boldsymbol{x}_2|} T(x_2)\\ &\times \frac{1}{|\boldsymbol{x}_1-\boldsymbol{x}_3|} T(x_3) \frac{1}{|\boldsymbol{x}_1-\boldsymbol{x}_4|} T(x_4) \frac{1}{|\boldsymbol{x}_4-\boldsymbol{x}_5|} T(x_5).\end{aligned}$$

From Eqs. (4·3) and (2·16) with $h_{ij}^{TT}=0$ and $z=1$ it follows that

$$h^T = \varphi + (3/16)\varphi^2 + (1/64)\varphi^3 + 2(1/8)^4\varphi^4. \tag{4·10}$$

The integral of the Hamiltonian density (2·18) is expressed in terms of φ as (cf. below Eq. (3·15))

$$\begin{aligned}\boldsymbol{H} &= -\int d^3x \Delta h^T = -\int d^3x \Delta\varphi\\ &= \int d^3x T^0{}_0(x) + V_2 + V_3 + V_4 + V_5 + \cdots,\end{aligned} \tag{4·11}$$

where

$$V_2 = -\frac{1}{2}G \iint d^3x_1 d^3x_2 T(x_1) \frac{1}{r_{12}} T(x_2), \tag{4·12}$$

$$V_3 = \left(-\frac{1}{2}G\right)^2 2 \iiint d^3x_1 d^3x_2 d^3x_3 T(x_1) \frac{1}{r_{12}} T(x_2) \frac{1}{r_{23}} T(x_3), \tag{4·13}$$

$$\begin{aligned}V_4 = &\left(-\frac{1}{2}G\right)^3 \int\cdots\int d^3x_1 \cdots d^3x_4 \Big[3T(x_1) \frac{1}{r_{12}} T(x_2) \frac{1}{r_{23}} T(x_3)\\ &\times \frac{1}{r_{34}} T(x_4) + 2T(x_1) \frac{1}{r_{12}} T(x_2) \frac{1}{r_{23}} \frac{1}{r_{24}} T(x_3) T(x_4)\Big],\end{aligned} \tag{4·14}$$

$$V_5=\left(-\frac{1}{2}G\right)^4\int\cdots\int d^3x_1\cdots d^3x_5\left[4T(x_1)\frac{1}{r_{12}}T(x_2)\frac{1}{r_{23}}T(x_3)\frac{1}{r_{34}}T(x_4)\right.$$

$$\times\frac{1}{r_{45}}T(x_5)+8T(x_1)\frac{1}{r_{12}}T(x_2)\frac{1}{r_{23}}T(x_3)\frac{1}{r_{34}}\frac{1}{r_{35}}T(x_4)T(x_5)$$

$$\left.+2T(x_1)\frac{1}{r_{12}}T(x_2)\frac{1}{r_{23}}\frac{1}{r_{24}}\frac{1}{r_{25}}T(x_3)T(x_4)T(x_5)\right], \qquad (4\cdot15)$$

where $r_{mn}=|\boldsymbol{x}_m-\boldsymbol{x}_n|$. Graphically, the above potential is represented by Fig. 1.

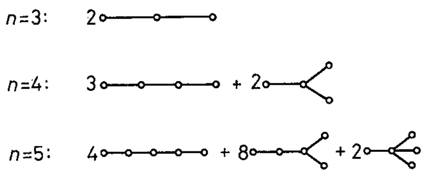

Fig. 1. Graphical representation of many-body potentials. The small circles represent the source T and the lines the Green function $C(x,x')$.

In general, V_n can be regarded as n-body static gravitational potential. The potential, which has no branch such as the first terms in V_4 and V_5, is given by

$$\overline{V}_n=\left(-\frac{1}{2}G\right)^{n-1}(n-1)\int\cdots\int d^3x_1\cdots d^3x_n T(x_1)\frac{1}{r_{12}}T(x_2)$$

$$\times\frac{1}{r_{23}}T(x_3)\cdots\frac{1}{r_{(n-1)n}}T(x_n). \qquad (4\cdot16)$$

To verify Eq. (4·16), we note the following relation. Equation (4·7) can also be expressed as

$$-\Delta\varphi=T\{1+(-1/8)CT+(-1/8)^2CTCT+\cdots\}$$

$$\times\{1+(-1/8)^2\varphi^2+(-1/8)^3\varphi^3+\cdots\}. \qquad (4\cdot17)$$

The terms which come from the first term of the second curly bracket evidently have no branch. The terms which come from $\varphi^n(n\geq3)$ have always branches. However, from the term in the second curly bracket there appears branchless tree diagrams by combining T and two branchless tree diagrams of φ. From Eq. (4·17) it follows that the term $T(-1/8)^2\varphi^2$ produces $(n-2)$ branchless trees of $\overline{V}_n$ and then

$$\text{total numbers of branchless trees}=1+(n-2)=n-1\,, \qquad (4\cdot18)$$

which proves (4·16).

Though V_n is considered as the n-body static potential, it can also be regarded as two-body static potential of order G^{n-1}. For instance, if we put in Eq. (4·16) for branchless diagram

$$T(x_m)=\underset{1}{m}\delta(\boldsymbol{x}_m-\underset{1}{\boldsymbol{x}}) \quad \text{for } m=\text{odd},$$

$$T(x_m)=\underset{2}{m}\delta(\boldsymbol{x}_m-\underset{2}{\boldsymbol{x}}) \quad \text{for } m=\text{even},$$

and vice versa, we obtain the two-body static potential $\overline{V}_2^{n-1}$ of order G^{n-1}. Therefore, the relation between the n-body static potential V_n and the two-body potential V_2^{n-1} of order G^{n-1} is evident. The relation holds also when the two-body potential is replaced by the one between two celestial bodies A, B consisting of many point particles.[6] In this case, we take $T(x_m)$ as

$$T(x_m)=\sum_c \underset{c}{m}\delta(\boldsymbol{x}_m-\underset{c}{\boldsymbol{x}}),$$

where the summation of c is carried out over A or B.

§ 5. Discussion

Standing on the canonical formalism of gravity, we have obtained the two-body potential in general gauge and the many-body static gravitational potential.

The relation between the coordinate frame and the gauge is apparent in our formalism. To study the relation more explicitly we see where the gauge parameter z appears when the shift in the perihelion is calculated. For this purpose it is appropriate to follow the method of Infeld and Plebanski.[9] They started from the simple coordinate system in which the Newtonian center of mass is at rest, i.e.,

$$\underset{1}{m}\underset{1}{\dot{x}}^k+\underset{2}{m}\underset{2}{\dot{x}}^k=0,$$

and made an appropriate transformation both in time and the relative space coordinates $\eta^k=\underset{1}{x}^k-\underset{2}{x}^k$:

$$dt=dt'(1+Gp\mu_0/r')\beta, \tag{5·1}$$

$$\eta^k=\eta'^k(1+Gs\mu/r')\alpha, \tag{5·2}$$

where $\mu_0=\underset{1}{m}+\underset{2}{m}$, $\mu=\underset{1}{m}\underset{2}{m}/\mu_0$, $r'^2=\eta'^k\eta'^k$ and p, s, α, β are constants chosen so that the starting Lagrangian for the two-body case takes the form of the Lagrangian for a single particle

$$L'=\tfrac{1}{2}\dot{\eta}'^k\dot{\eta}'^k+G\mu_0/r'+G^2 3\mu_0^2/(1-3\mu/\mu_0)r'^2. \tag{5·3}$$

In the derivation of L' the relation $L'dt'=Ldt$ is used and an arbitrary constant multiplied by the Lagrangian is discarded. When we apply this method to our energy integral given by (3·15), we must take account of the fact that our time coordinate depends on the parameter z. From Eq. (2·8), it is seen that our time coordinate t relates with t_0 of the special case of $z=1$ by

$$t=t_0-\frac{1}{2}(z-1)\frac{1}{\Delta^2}\pi^{mn}{}_{,mn}$$

$$=t_0+\tfrac{1}{2}G(z-1)\sum_b \underset{b}{p}_j r_{,j}\,. \qquad (5\cdot4)$$

The Hamiltonian in the time coordinate system t is written as

$$\boldsymbol{H}=-\int d^3x(\Delta h_0{}^T+\delta\Delta h_0{}^T)\left\{t_0-\frac{1}{2}(z-1)\frac{1}{\Delta^2}\pi^{mn}{}_{,mn}\right\}_{,0}.$$

By taking account of the relation $\delta\Delta h_0{}^T=-(\Delta h_0{}^T)_{,0}(t-t_0)$, the Hamiltonian in the system t_0 is given by

$$\boldsymbol{H}_0=\boldsymbol{H}-(z-1)\int d^3x\Delta h_0{}^T\left(\frac{1}{\Delta^2}\pi^{mn}{}_{,mn}\right)_{,0}$$

$$=\boldsymbol{H}-2G(z-1)\,[\underset{1}{p}_i\underset{2}{p}_j r_{,ij}-\tfrac{1}{2}\{(\underset{2}{m}/\underset{1}{m})\underset{1}{p}_i\underset{1}{p}_j+(\underset{1}{m}/\underset{2}{m})\underset{2}{p}_i\underset{2}{p}_j\}r_{,ij}]. \qquad (5\cdot5)$$

Making the transformations (5·1) and (5·2) after (5·4), we get the same Lagrangian as Eq. (5·3) under the conditions for p, s, α, β:

$$\alpha=1-3\mu/\mu_0,\quad \beta=(1-3\mu/\mu_0)^{3/2},\quad p=1+(3+2\mu/\mu_0)/(1-3\mu/\mu_0),$$
$$s=\tfrac{1}{2}\{\mu+(z-1)\mu_0\}/\mu(1-3\mu/\mu_0). \qquad (5\cdot6)$$

The constants p, α, β are the same as those of Infeld and Plebanski. The factor $(z-1)$ appears only in s which takes part in the transformation of the space coordinates (5·2). The arbitrariness of z is absorbed in the transformation of space-time coordinates and does not affect the value of the shift in the perihelion.*) Such a fact corresponds to the arbitrariness of our coordinate condition given by Eqs. (2·8), (2·9), (2·13), (2·14) and (2·15).

An equation for deriving the static gravitational potential can be found within the framework of general relativity. Since the static potential is a classical non-relativistic concept, it may be conjectured that such a type of equation can be derived from the Newtonian theory. In fact, L. N. Cooper**) has shown that such an equation can be obtained by applying the equivalence principle for the one-body problem. When the bare mass distribution is ρ_0, the Poisson equation $\Delta\phi=4\pi G\rho_0=\frac{1}{4}\rho_0$ for static potential should be modified so that the true mass density ρ may include the particle's gravitational self-energy $\frac{1}{2}\rho\phi$, i.e., $\Delta\phi=\frac{1}{4}\rho=\frac{1}{4}(\rho_0+\frac{1}{2}\rho\phi)$. By eliminating ρ, the equation $\Delta\phi=\frac{1}{4}\rho_0(1-\frac{1}{2}\phi)^{-1}$ is obtained. The arguments may be extended to the case of many-body by replacing ρ_0 with $T^0{}_0$

$$4\Delta\phi=T^0{}_0(1-\tfrac{1}{2}\phi)^{-1}. \qquad (5\cdot7)$$

If we put $\phi=-\varphi/4$, Eq. (5·7) is nothing but Eq. (4·4). Though h^T is not equal to ϕ but $h^T=-4\phi+3\phi^2-\phi^3+(1/8)\phi^4$, it is interesting that the Hamiltonian $\boldsymbol{H}=-\int d^3x\Delta h^T$ gives the potential energy of the same type as derived from the equation for 4ϕ.

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*) The perihelion motion has been discussed through the medium of Eq. (5·5). We may immediately perform the reduction to a form of (5·3) by making a generalization of the transformation (5·1) if the constant term $(\underset{1}{m}+\underset{2}{m})$ is retained in the Lagrangian.[10]

**) See the footnote "6" in Ref. 11).

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