POTENTIAL OPERATORS AND CONSERVATIVE SYSTEMS*

Giovanni Romano**

SOMMARIO: Si applica la teoria degli operatori potenziali negli spazi di Hilbert per formulare una definizione rigorosa di carico conservativo.

Ciò consente di dimostrare in modo corretto una ben nota condizione di conservatività usualmente introdotta con una argomentazione inesatta.

I risultati generali ottenuti sono applicati al caso particolare di carico-pressione.

L'analisi è condotta nel campo delle grandi deformazioni ottenendo una condizione generale, necessaria e sufficiente, affinchè il carico-pressione sia conservativo.

SUMMARY: The theory of potential operators in Hilbert spaces is applied to a rigorous definition of conservative loading.

This approach allows correct proof of a well-known condition of conservativeness, usually introduced with a misleading argument. The special case of pressure loading is then examined as an application of the previous results. The analysis is performed in the large (finite deformations) getting a general condition for the conservativeness of pressure loading, not previously found to the author's knowledge.

Introduction.

Conservative systems play a specially important role among mechanical systems in general and the energetic approach provides a simple and fascinating interpretation of their behaviour. However, in spite of its basic interest, the notion of conservativeness has been often introduced in an intuitive form in the context of continuum mechanics.

This circumstance has often caused the related topics to be treated in a restrictive and sometimes misleading way. A rigorous approach to the subject is presented in this paper.

It is founded upon the mathematical theory of potential operators that is the natural extension to the continuum of the well-known theory of differential forms of ordinary calculus. The basic definitions and results of the theory of potential operators are reported in the special case of real Hilbert spaces that suffices for the usual purposes and allows us to simplify the exposition. The usual definition of potential operator is slightly generalized to include the important case of non-homogeneous boundary conditions.

The general theory is then applied to the definition of conservative loading.

A necessary and sufficient condition is found to be the selfadjointness of the Frèchet differential of the load operator. It is worth noting that this condition has often been referred to in the literature on the linear theory of elastic stability but its proof has been attributed, with a misleading argument, to a recourse to the well-known Betti reciprocity theorem of classical elasticity [2].

The special case of pressure loading is then examined in full generality getting, as an application of the previous results, a necessary and sufficient condition for its conservativeness.

It is the first example, to the author's knowledge, of such an analysis performed "in the large" (finite deformations). For the sake of completeness an appendix provides some results of surface deformation theory that are used in the paper.

1. Potential Operators.

In this section we shall briefly present the fundamental results of potential operators theory in real Hilbert spaces.

We would point out that an analogous theory can be developed in (more general) Banach spaces, but restriction to the special case of real Hilbert spaces suffices for our purposes and greatly simplifies the presentation.

It may be noticed moreover that from a rigorous point of view some of our assumptions could here and there be relaxed to some extent but again this would result in a formally more involved treatment of the subject. However, this paper is not addressed to mathematicians, who may find elsewhere [1] a more general and thoroughly developed theory, but is intended to carry over into the engineering field some useful concepts and results of the mathematical theory of potential operators.

a) *Basic Definitions.*

Let H be a real Hilbert space. The scalar product of two elements will be denoted by:

$$\langle \mathbf{u}, \mathbf{w} \rangle \qquad \mathbf{u}, \mathbf{w} \in H$$

Let A be an operator defined on a subset $D \subseteq H$. We shall assume that D is a coset of a subspace H_D of H, i. e.:

$$\forall\, \mathbf{u}, \mathbf{w} \in D \quad \Rightarrow \quad \mathbf{u} - \mathbf{w} \in H_D .$$

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** Istituto di Scienza delle Costruzioni, Facoltà di Ingegneria, Università di Napoli.

In the sequel we shall denote by:

$$A'(\mathbf{u})\mathbf{h} \qquad \mathbf{u} \in D \qquad \mathbf{h} \in H_D$$

the Frèchet differential (1) of A at the point $\mathbf{u}$ with increment $\mathbf{h}$.

We give the following:

Definition.

An operator A is said to be potential on D if there exists a Frechet differentiable functional ϕ on D such that:

$$\phi'(\mathbf{u})\mathbf{h} = \langle A(\mathbf{u}), \mathbf{h}\rangle \qquad \forall\ \mathbf{u} \in D,\ \mathbf{h} \in H_D .$$

We shall briefly write $A = \text{grad}\,\phi$.

A will be called the gradient of ϕ and ϕ the potential of A on D.

This definition is a slight generalization of the usual one which requires the linearity of D.

Allowance for the nonlinearity of D is basic in most applications to include the case of non homogeneous boundary conditions.

b) *Basic Theorems.*

If the domain D of the operator A is simply connected the following theorem gives a necessary and sufficient condition for the operator A to be potential on D.

Theorem 1.1.

Let A be a continuous operator defined on an simply connected domain $D \subseteq H$. In order that A be a potential operator on D it is necessary and sufficient that the curvilinear integral

$$\oint \langle A(\mathbf{u}), d\mathbf{u}\rangle \tag{1.2'}$$

be equal to zero around any closed curve which lies in D.

Evidently this last condition is equivalent to saying that the curvilinear integral:

$$\int_L \langle A(\mathbf{u}), d\mathbf{u}\rangle \tag{1.2''}$$

is path independent, i. e. for any curve L lying in D, it does not depend on the shape of the curve but only on its endpoints.

If each of the conditions (1.2) is satisfied the potential ϕ of A is uniquely determined, to within an additive constant, and is given by:

$$\phi(\mathbf{u}) = \phi(\mathbf{u}_0) + \int_0^1 \langle A(\mathbf{u}_0 + t(\mathbf{u} - \mathbf{u}_0)), (\mathbf{u} - \mathbf{u}_0)\rangle\, dt . \tag{1.3}$$

(1) We recall that an operator A is said to be Frèchet differentiable at $\mathbf{u} \in D$ if there exists a linear operator $A'(\mathbf{u})$ such that

$$A(\mathbf{u} + \mathbf{h}) - A(\mathbf{u}) = A'(\mathbf{u})\mathbf{h} + \Omega(\mathbf{u}, \mathbf{h})$$

with

$$\lim_{\|\mathbf{h}\| \to 0} \frac{\|\Omega(\mathbf{u}, \mathbf{h})\|}{\|\mathbf{h}\|} = 0 \quad \forall\ \mathbf{h} \in H_D$$

there $\|\cdot\|$ is the norm induced by the scalar product.

If the domain D is linear, taking $\mathbf{u}_0 = \mathbf{0}$, (1.3) may be simplified to:

$$\phi(\mathbf{u}) = \phi(0) + \int_0^1 \langle A(t\mathbf{u}), \mathbf{u}\rangle\, dt . \tag{1.4}$$

If operator A is supposed to be Frèchet differentiable on D a convenient test for the potentialness of A is given by the following:

Theorem 1.2.

Let A be a Frèchet differentiable operator on the simply connected domain $D \subseteq H$. For A to be potential on D it is necessary and sufficient that:

$$\langle A'(\mathbf{u})\mathbf{h}_1, \mathbf{h}_2\rangle = \langle A'(\mathbf{u})\mathbf{h}_2, \mathbf{h}_1\rangle \qquad \mathbf{u} \in D \qquad \mathbf{h}_1, \mathbf{h}_2 \in H_D \tag{1.5}$$

i. e. that its Frèchet derivative be a symmetric operator.

It is worth noting that a linear operator is potential if and only if it is selfadjoint, since it coincides with its Frèchet derivative. If L is a linear selfadjoint operator on $D \subseteq H$ its potential will then be given by:

$$\phi(\mathbf{u}) = \phi(\mathbf{u}_0) + \int_0^1 \langle L(\mathbf{u}_0 + t(\mathbf{u} - \mathbf{u}_0)), (\mathbf{u} - \mathbf{u}_0)\rangle\, dt =$$

$$= \phi(\mathbf{u}_0) + \frac{1}{2}\langle L(\mathbf{u} - \mathbf{u}_0), (\mathbf{u} - \mathbf{u}_0)\rangle + \langle L\mathbf{u}_0, \mathbf{u} - \mathbf{u}_0\rangle$$

which, if D is linear, may be simplified to:

$$\phi(\mathbf{u}) = \phi(0) + \frac{1}{2}\langle L\mathbf{u}, \mathbf{u}\rangle . \tag{1.6}$$

It is interesting to show how the condition (1.5) reduces to a familiar form when the space is Euclidean (finite dimensional).

In fact let $\{\mathbf{e}_i\}$ $i = 1, 2, \ldots, n$ be an orthonormal basis in the Euclidean vector space E_n, and set (2):

$$\mathbf{h}_1 = \alpha_{1i}\mathbf{e}_i \qquad \mathbf{h}_2 = \alpha_{2k}\mathbf{e}_k \qquad k, i = 1, 2, \ldots, n$$

then condition (1.5) may be written as:

$$\langle A'(\mathbf{u})\alpha_{1i}\mathbf{e}_i, \alpha_{2k}\mathbf{e}_k\rangle = \langle A'(\mathbf{u})\alpha_{2k}\mathbf{e}_k, \alpha_{1i}\mathbf{e}_i\rangle . \tag{1.7}$$

By the arbitrariness of the components α_{1i} and α_{2k} condition (1.7) reduces to:

$$\langle A'(\mathbf{u})\mathbf{e}_i, \mathbf{e}_k\rangle = \langle A'(\mathbf{u})\mathbf{e}_k, \mathbf{e}_i\rangle . \tag{1.8}$$

Now setting $A(\mathbf{u}) = A_j(\mathbf{u})\mathbf{e}_j$ and $\mathbf{u} = \mathbf{u}_i\mathbf{e}_i$ and noting that:

$$A'(\mathbf{u})\mathbf{e}_i = \frac{\partial A_j(\mathbf{u})}{\partial u_i}\mathbf{e}_j$$

we have:

$$\langle A'(\mathbf{u})\mathbf{e}_i, \mathbf{e}_k\rangle = \langle \frac{\partial A_j(\mathbf{u})}{\partial u_i}\mathbf{e}_j, \mathbf{e}_k\rangle = \frac{\partial A_k(\mathbf{u})}{\partial u_i}$$

(2) We adopt the usual summation convention with respect to repeated indices (Einstein convention).

so that condition (1.5) becomes:

$$\frac{\partial A_i(\mathbf{u})}{\partial u_k} = \frac{\partial A_k(\mathbf{u})}{\partial u_i} \qquad i, k = 1, 2, ..., n$$

that is the usual condition of zero rotation of the vector field $A(\mathbf{u})$.

2. Conservative loading.

We shall now apply the previous theory to give an appropriate definition of conservative loading and to formulate the necessary and sufficient condition for a load to be conservative. To this end let us first introduce some background notions.

Let us consider a body B embedded in the Euclidean point space and denote by B_χ the domain occupied by B in the configuration χ and by ∂B_χ its boundary.

Let us choose a reference configuration $\mathbf{k}$ and denote by $\mathbf{x}$ and $\mathbf{X}$ respectively the position vectors of the same particle of B in the configurations χ and $\mathbf{k}$ (Fig. 1).

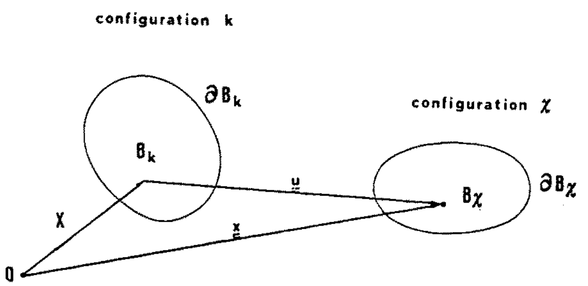

Fig. 1.

The displacement vector from $\mathbf{k}$ is then defined by:

$$\mathbf{u} = \mathbf{x} - \mathbf{X}.$$

If we denote by t_χ the tractions in the configuration χ we define the equivalent tractions $\mathbf{t}_k$ per unit area in the reference configuration by:

$$\int_{\partial P_\chi} \mathbf{t}_\chi ds = \int_{\partial P_k} \mathbf{t}_k ds$$

where ∂P is the boundary of an arbitrary part P of B.

Now let $\{\partial B_{kp}, \partial B_{ku}\}$ be a partition of the boundary of B. Mixed boundary conditions of place and traction are defined if we assign the traction $\mathbf{t}_k$ on ∂B_{kp} and the displacement $\mathbf{u}$ on ∂B_{ku} i. e. (Fig. 2):

$$\mathbf{t}_k = \mathbf{p}_k(\mathbf{X}, t) \qquad \mathbf{X} \in \partial B_{kp}$$
$$\mathbf{u} = \eta(\mathbf{X}, t) \qquad \mathbf{X} \in \partial B_{ku}.$$

In the sequel we shall be interested in the special case in which the tractions on ∂B_{kp} depend directly only on the displacements (positional loading):

$$\mathbf{t}_k = \mathbf{p}_k[\mathbf{u}(\mathbf{X}, t)] \qquad \mathbf{X} \in \partial B_{kp} \qquad (2.1')$$
$$\mathbf{u} = \eta(\mathbf{X}, t) \qquad \mathbf{X} \in \partial B_{ku}. \qquad (2.1'')$$

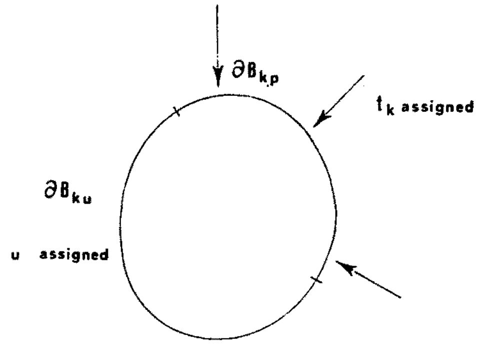

Fig. 2.

Now if we denote by:

$$\mathbf{v} \cdot \mathbf{w}$$

the scalar product between two vectors of the 3-dimensional Euclidean vector space, we may introduce an Hilbert space on the set $L_2(\partial B_{kp})$ of square integrable vector functions on ∂B_{kp} defined by:

$$\mathbf{v} = v(\mathbf{X}) \qquad \mathbf{X} \in \partial B_{kp}$$

with the scalar product:

$$\langle v, w \rangle = \int_{\partial B kp} \mathbf{v} \cdot \mathbf{w} ds. \qquad (2.2)$$

Let us now introduce the load operator p defined on the set D of admissible displacement functions, i. e. satisfying (2.1″)

$$\mathbf{p}_k = p(\mathbf{u}).$$

Since it is trivially verified that the condition of linearity of the set H_D is satisfied, we may give the following:

Definition 2.1.

A load distribution $p(\mathbf{u})$ is said to be conservative if the work done by the load distribution vanishes around any closed curve in the space of admissible configurations (defined by the displacement function $\mathbf{u}$). In mathematical terms this condition is written as:

$$\oint \langle p(\mathbf{u}), d\mathbf{u} \rangle = 0 \qquad (2.3)$$

where the curvilinear integral can be taken around any closed curve in the domain of p, i. e. in the set D of admissible displacement functions.

Condition (2.3) may alternatively expressed stating that the operator $p(\mathbf{u})$ is potential on D.

The functional P (defined to within an additive constant) such that:

$$p = \text{grad } P$$

will be called the load potential.

If we assume $p(\mathbf{u})$ to be Frèchet differentiable, a necessary and sufficient condition for the existence of the load potential will be:

$$\int_{\partial B kp} p'(\mathbf{u})\mathbf{h}_1 \cdot \mathbf{h}_2 ds = \int_{\partial B kp} p'(\mathbf{u})\mathbf{h}_2 \cdot \mathbf{h}_1 ds \qquad (2.4)$$
$$\mathbf{u} \in D \qquad \mathbf{h}_1, \mathbf{h}_2 \in H_D.$$

If the symmetry condition (2.4) is satisfied the potential will be:

$$P(\mathbf{u}) = P(\mathbf{u}_0) + \int_0^1 dt \int_{\partial B k p} p(\mathbf{u}_0 + t(\mathbf{u} - \mathbf{u}_0)) \cdot (\mathbf{u} - \mathbf{u}_0) ds \tag{2.5}$$

which in the case of linearity of D may be simplified to:

$$P(\mathbf{u}) = P(\mathbf{0}) + \int_0^1 dt \int_{\partial B k p} p(t\mathbf{u}) \cdot \mathbf{u}\, ds. \tag{2.6}$$

It is apparent here that the equivalence between the symmetry condition (2.4) and the defining condition (2.3) is a general mathematical result.

And yet surprisingly in the literature on the theory of linear elastic stability this equivalence has been attributed to the same argument that proves the well-known Betti reciprocity theorem of classical elasticity [2]. This seems to be a major shortcoming of the lack of a rigorous approach to the theory of conservative systems.

3. A special case: Pressure loading.

As an example of application of the previous theory we shall now derive the necessary and sufficient condition for the conservativeness of a special, interesting case of loading, i. e. pressure loading.

Let us first state:

Definition 3.1.

A surface load distribution is said to be a pressure loading if its intensity is constant during the deformation of the body while its direction remains parallel to the unit normal to the boundary surface (assumed to be regular).

Let us consider a closed curve C on $\partial B_{\chi p}$ and denote its interior by $\partial B_{\chi pc}$ (Fig. 3). We shall consider the pressure loading defined, in the generic configuration $\boldsymbol{\chi}$ by:

$$\begin{aligned} &\mathbf{p}_{\chi}(\mathbf{x}) = p\mathbf{n}_{\chi}(\mathbf{x}) && \mathbf{x} \in \partial B_{\chi pc} \\ &\mathbf{p}_{\chi}(\mathbf{x}) = \mathbf{0} && \mathbf{x} \in \partial B_{\chi p} - \partial B_{\chi pc} \end{aligned} \tag{3.1}$$

where $\mathbf{n}_{\chi}(\mathbf{x})$ is the unit normal to $\partial B_{\chi p}$ at $\mathbf{x}$ and p is a constant that measures the intensity of the load distribution.

Now by a formula of surface deformation theory (formula (4.7) of the appendix):

$$\int_{\partial B_{\chi}} \mathbf{n}_{\chi} ds = \int_{\partial B k} \partial_F \det \mathbf{F}\, \mathbf{n}_k ds \tag{3.2}$$

where the tensor $\mathbf{F}$ is the deformation gradient from $\mathbf{k}$ to $\boldsymbol{\chi}$ and $\mathbf{n}_k$ the unit normal to the surface ∂B_k in the reference configuration $\mathbf{k}$.

By (3.2) the load distribution in the reference configuration $\mathbf{k}$ equivalent to (3.1) on ∂B_{kpc} will be:

$$\mathbf{p}_k = p(\mathbf{u}) = p\partial_F \det \mathbf{F}\, \mathbf{n}_k.$$

Let us now evaluate the Frèchet differential of $p(\mathbf{u})$ along a direction $\mathbf{u}^* \in H_D$. If we choose an orthonormal basis $\{\mathbf{e}_i\}$ in the Euclidean space by formula (4.5) of the appendix we have:

$$p'(\mathbf{u})\overset{*}{\mathbf{u}} = p(e_{ijk}e_{pqr}F_{kr}\overset{*}{u}_{j,q}n_p)\mathbf{e}_i \tag{3.3}$$ (3)

where:

$$\mathbf{n}_k = n_p \mathbf{e}_p.$$

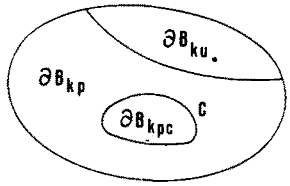

Fig. 3.

The symmetry condition (2.4) may now be written as:

$$\int_{\partial B k p} e_{ijk}e_{pqr}F_{kr}u'_{j,q}u''_i n_p ds = \int_{\partial B k p} e_{ijk}e_{pqr}F_{kr}u''_{j,q}u'_i n_p ds$$

$$\mathbf{u}', \mathbf{u}'' \in H_D$$

and hence:

$$\int_{\partial B k p} e_{ijk}e_{pqr}F_{kr}(u'_j u''_i)_{,q}\, n_p\, ds = 0. \tag{3.4}$$

Now since $e_{pqr}F_{kr,q} = e_{pqr}(\delta_{kr} + u_{k,r})_{,q} = 0$ condition (3.4) becomes:

$$\int_{\partial B k p} e_{ijk}e_{pqr}(F_{kr}u'_j u''_i)_{,q}\, n_p\, ds = 0$$

and in vector notation:

$$\int_{\partial B k p} \operatorname{rot}\, [\mathbf{F}^T(\mathbf{u}' \times \mathbf{u}'')] \cdot \mathbf{n}_k\, ds = 0$$

where $\mathbf{u}' \times \mathbf{u}''$ denotes the vector product of $\mathbf{u}'$ by $\mathbf{u}''$. Hence by Stokes' formula:

$$\oint_C \mathbf{F}^T(\mathbf{u}' \times \mathbf{u}'') \cdot \boldsymbol{\tau}\, dl = 0 \tag{3.5}$$

where $\boldsymbol{\tau}$ is the unit vector tangent to the closed curve C.

The simplest case in which (3.5) is satisfied is when on the closed curve C the displacement $\mathbf{u}$ is assigned and hence $\mathbf{u}' = \mathbf{u}'' = \mathbf{0}$.

Condition (3.5) is quite general and to the author's knowledge has not been found before. Two special cases of (3.5) have been given by Pearson [3] and Bolotin [2], the former for infinitesimal deformations and the latter for pressure loading on shells.

If we consider the case of hydrostatic pressure, i. e. pressure loading on the whole surface ∂B_k with constant intensity, the load potential assumes a particularly simple form.

(3) The comma (,) as usual, denotes the derivative operator.

$$P(\mathbf{u}) = \int_0^1 dt \int_{\partial Bk} p\partial_F \det \mathbf{F}(t\mathbf{u})\mathbf{n}_k \cdot \mathbf{u}\, ds =$$

$$= \int_0^1 dt \int_{\partial Bk} p(\partial_F \det \mathbf{F}(t\mathbf{u}))^T \mathbf{u} \cdot \mathbf{n}_k ds =$$

$$= p \int_0^1 dt \int_{Bk} \nabla \cdot \{(\partial_F \det \mathbf{F})^T \mathbf{u}\}\, dv =$$

$$= p \int_0^1 dt \left[\int_{Bk} TR\{(\partial_F \det \mathbf{F}(t\mathbf{u}))^T \nabla \mathbf{u}\}\, dv + \right.$$

$$\left. + \int_{Bk} \{\nabla \cdot \partial_F \det \mathbf{F}(t\mathbf{u})\} \cdot \mathbf{u}\, dv \right. . \qquad (3.6)\,(4)$$

Now the last integral in (3.6) is equal to zero since:

$$\nabla \cdot \partial_F \det \mathbf{F} = \frac{1}{2} e_{ijk} e^{pqr} (F^{.j}_{.q} F^{.k}_{.r}),\ {}_p\mathbf{e}^i = \mathbf{0}$$

by the symmetry of the second derivatives of the displacement components. Finally noting that:

$$\nabla \mathbf{u} = \frac{d}{dt} \mathbf{F}(t\mathbf{u})$$

we have:

$$TR \left\{ [\partial_F \det \mathbf{F}(t\mathbf{u})]^T \frac{d}{dt} \mathbf{F}(t\mathbf{u}) \right\} = \frac{d}{dt} \det \mathbf{F}(t\mathbf{u})$$

and hence:

$$P(\mathbf{u}) = p \int_0^1 dt \int_{Bk} \frac{d}{dt} \det \mathbf{F}(t\mathbf{u})\, dv =$$

$$= p \int_{Bk} \{\det \mathbf{F}(\mathbf{u}) - \det \mathbf{F}(\mathbf{0})\}\, dv = p\{V_\chi - V_k\}$$

where V_χ and V_k denote respectively the volume occupied in the current and in the reference configurations.

Appendix.

Surface Deformation.

Let us consider a body B in the Euclidean space and choose an arbitrary regular surface S_k in the reference configuration **k**. Let the parametric equations of S_k be:

$$\mathbf{X} = \mathbf{X}(\eta^1, \eta^2) .$$

In an arbitrary configuration χ of B the deformed surface will be represented by:

$$\mathbf{x} = \mathbf{x}(\eta^1, \eta^2) .$$

(4) The ∇ denotes the symbolic derivative operator defined with reference to the basis $\mathbf{e}^k$ by:

$$\nabla = {}_{,k}\, \mathbf{e}^k$$

where $_{,k}$ is the covariant derivative operator.

Hence $\nabla\, \mathbf{a}$ denotes the gradient of vector $\mathbf{a}$ as a function of $\mathbf{X}$ and $\nabla \cdot \mathbf{a}$ and $\nabla \cdot \mathbf{T}$ the divergence of the vector $\mathbf{a}$ and tensor $\mathbf{T}$ respectively. The symbol TR $\{\mathbf{A}\}$ denotes the trace of the tensor $\mathbf{A}$ and $\mathbf{A}^T$ the transpose of $\mathbf{A}$.

Now if we set:

$$\mathbf{X}_\alpha = \frac{\partial \mathbf{X}}{\partial \eta_\alpha} \qquad \mathbf{x}_\beta = \frac{\partial \mathbf{x}}{\partial \eta_\beta} \qquad \alpha, \beta = 1, 2$$

by the chain rule of differential calculus we have:

$$\mathbf{x}_\alpha = \mathbf{F}\mathbf{X}_\alpha \qquad (4.1)$$

where $\mathbf{F}$ is the deformation gradient from $\mathbf{k}$ to χ.

The area of the surface S_k will be given by:

$$\int_{Sk} ds = \iint [\det(\mathbf{X}_\alpha \cdot \mathbf{X}_\beta)]^{\frac{1}{2}}\, d\eta^1 d\eta^2 =$$

$$= \iint \|\mathbf{X}_1 \times \mathbf{X}_2\| d\eta^1 d\eta^2 \qquad (4.2)$$

where a dot ($\cdot$) is the symbol for the scalar product in the Euclidean vector space, $\|\mathbf{v}\|$ is the norm of the vector $\mathbf{v}$ and the vector product has been denoted by $\times$. Moreover we have explicitly:

$$\det(\mathbf{X}_\alpha \cdot \mathbf{X}_\beta) = \begin{vmatrix} \mathbf{X}_1 \cdot \mathbf{X}_1 & \mathbf{X}_1 \cdot \mathbf{X}_2 \\ \mathbf{X}_2 \cdot \mathbf{X}_1 & \mathbf{X}_2 \cdot \mathbf{X}_2 \end{vmatrix}$$

i. e. the Gram determinant of the vectors $\mathbf{X}_1$ and $\mathbf{X}_2$. and the equality in (4.2) follows from the well-known formula:

$$\det(\mathbf{X}_\alpha \cdot \mathbf{X}_\beta) = \|\mathbf{X}_1 \times \mathbf{X}_2\|^2.$$

Now the unit normal to S_k is given by

$$\mathbf{n}_k = \frac{\mathbf{X}_1 \times \mathbf{X}_2}{\|\mathbf{X}_1 \times \mathbf{X}_2\|} . \qquad (4.3)$$

If we set $\mathbf{V} = \mathbf{X}_1 \times \mathbf{X}_2$, $\mathbf{v} = \mathbf{x}_1 \times \mathbf{x}_2$ and choose an arbitrary basis in the Euclidean space we have:

$$V_p = \frac{1}{2} e_{pqr} e^{\alpha\beta} X^q_\alpha X^r_\beta \qquad (4.4)$$

where $e_{\alpha\beta}$ is the alternating tensor on the surface [4]. From (4.4) it follows that:

$$e^{\alpha\beta} X^q_\alpha X^r_\beta = e^{pqr} V_p .$$

Now we have:

$$v_i = \frac{1}{2} e_{ijk} e^{\alpha\beta} x^j_\alpha x^k_\beta = \frac{1}{2} e_{ijk} e^{\alpha\beta} F^{.j}_{.q} F^{.k}_{.r} X^q_\alpha X^r_\beta =$$

$$= \frac{1}{2} e_{ijk} e^{pqr} F^{.j}_{.q} F^{.k}_{.r} V_p \qquad (4.5)$$

and

$$F^{.i}_{.p} v_i = \frac{1}{2} e_{ijk} F^{.i}_{.p} F^{.j}_{.q} F^{.k}_{.r} e^{\alpha\beta} X^q_\alpha X^r_\beta =$$

$$= \frac{1}{2} e_{pqr} e^{\alpha\beta} X^q_\alpha X^r_\beta \det \mathbf{F} = X_p \det \mathbf{F}$$

and in vector notation:

$$\mathbf{F}^T(\mathbf{x}_1 \times \mathbf{x}_2) = (\mathbf{X}_1 \times \mathbf{X}_2) \det \mathbf{F}$$

whence, by (4.2) and (4.3):

$$\int_{S\chi} \mathbf{n}_\chi ds = \int_{Sk} (\det \mathbf{F})\mathbf{F}^{-T}\mathbf{n}_k ds$$

or, noting that (5):

$$\partial_F \det \mathbf{F} = (\det \mathbf{F})\mathbf{F}^{-T}, \tag{4.6}$$

$$\int_{S\chi} \mathbf{n}_\chi ds = \int_{Sk} (\partial_F \det \mathbf{F})\mathbf{n}_k ds \tag{4.7}$$

(5) A proof of (4.6) may be found in Ref. [5].

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REFERENCES

[1] M. M. Vainberg, *Variational methods for the study of non-linear operators*, Holden-Day, 1964.
[2] V. V. Bolotin, *Nonconservative problems of the theory of elastic stability*, Pergamon Press, 1963.
[3] C. E. Pearson, *General theory of elastic stability*, Quart. Appl. Math. 14, 133, 1956.
[4] I. S. Sokolnikoff, *Tensor analysis*, II Ed., Wiley, 1967.
[5] D. C. Leigh, *Nonlinear continuum mechanics*, McGraw-Hill, 1968.