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Power comparison of non-parametric tests: Small-sample properties from Monte Carlo experiments*

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SUMMARY *Non-parametric tests that deal with two samples include scores tests (such as the Wilcoxon rank sum test, normal scores test, logistic scores test, Cauchy scores test, etc.) and Fisher's randomization test. Because the non-parametric tests generally require a large amount of computational work, there are few studies on small-sample properties, although asymptotic properties with regard to various aspects were studied in the past. In this paper, the non-parametric tests are compared with the* t*-test through Monte Carlo experiments. Also, we consider testing structural changes as an application in economics.*

1 Introduction

There are many kinds of non-parametric test (distribution-free tests), such as scores tests, Fisher's test, etc. However, almost all the studies in the past have related to asymptotic properties. In this paper, we examine small-sample properties of non-parametric two-sample tests by Monte Carlo experiments.

One of the features of non-parametric tests is that we do not have to impose any assumption on the underlying distribution. With no restriction on the distribution, it can be expected that non-parametric tests are less powerful than the conventional parametric tests, such as the *t*-test. However, Hodges and Lehman (1956) and Chernoff and Savage (1958) showed that the Wilcoxon rank sum test is as powerful as the *t*-test under the location shift alternatives and, moreover, that the Wilcoxon test is sometimes much more powerful than the *t*-test. In particular, the remarkable

*This paper is an extention of Diebold *et al.* (1992), where the Wilcoxon test, Fisher test and *t*-test were compared with respect to sample power. In this paper, more non-parametric tests are examined.

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fact about the Wilcoxon test is that it is about 95% as powerful as the usual *t*-test for normal data. Chernoff and Savage (1958) proved that Pitman's asymptotic relative efficiency of the normal scores test relative to the *t*-test is greater than unity under the location shift alternatives.[1] This implies that the power of the normal scores test is always greater than that of the *t*-test. According to Mehta and Patel (1992), the normal scores test is less powerful than the Wilcoxon test if the tails of the underlying distributions are diffuse.

Fisher's test statistic is the difference between two sample means, so it is asymptotically equivalent to the *t*-test (Bradley, 1968).[2] The scores tests are similar to the Fisher test, except that the test statistic of the scores test is the sum of scores, while the Fisher test statistic is the difference between two sample means. Both test statistics are discretely distributed and we have to obtain all the possible combinations for the tests.

It is quite difficult to obtain all the possible combinations, and the computational time is also quite long. Mehta and Patel (1983, 1986a) and Mehta *et al.* (1984, 1985, 1988) carried out a program on the Fisher permutation test (a generalization of the Fisher two-sample test treated in this paper, i.e. independence test by $r \times c$ contingency table) using a network algorithm.[3]

In this paper, we consider small-sample properties of two-sample non-parametric tests (i.e. the scores tests and the Fisher test) by comparing with the *t*-test, which is the usual parametric test. Finally, the test of structural change is examined as an application in economics.

2 Overview of non-parametric tests

It is well known for testing two-sample means that the *t*-test gives us a uniform powerful test under the normality assumption, but not under non-normality. We consider a distribution-free test in this paper, which is also called a non-parametric test. The normal scores test, Wilcoxon (1945) rank sum test and Fisher (1935) test are famous non-parametric tests, which are similar tests. We have two sample groups. We test if two samples are generated from the same distribution. Let x_1, x_2, ..., x_{n1} be mutually independently distributed as $F(x)$, and let y_1, ..., y_{n2} be mutually independently distributed as $G(y)$. $F(x)$ and $G(y)$ are continuous distribution functions. Under the assumptions, we consider the null hypothesis of no difference between two sample means. The null hypothesis H_0 is represented by

$$H_0: F(x) = G(x)$$

The scores tests and the Fisher test are usually applied under the alternative of location shift.[4] One possible alternative hypothesis H_1 is given by

$$H_1: F(x) = G(x - \mu), \qquad \mu > 0$$

where a shift in the location parameter μ is tested.

Let $n1$ be the sample size of group 1 and let $n2$ be that of group 2. We consider randomly taking $n1$ samples out of $n1 + n2$ samples, and mixing the two groups. Then, we have ${}_{n1+n2}C_{n1}$ combinations. Each event of ${}_{n1+n2}C_{n1}$ combinations occurs with equal probability $1/{}_{n1+n2}C_{n1}$. For the scores tests and the Fisher test, all the possible combinations are compared with the original two samples.

2.1 Scores tests

For the scores tests, the two samples $\{x_i\}_{i=1}^{n1}$ and $\{y_j\}_{j=1}^{n2}$ are converted into the data ranked by size. Let $\{Rx_i\}_{i=1}^{n1}$ and $\{Ry_j\}_{j=1}^{n2}$ be the ranked samples that correspond to $\{x_i\}_{i=1}^{n1}$ and $\{y_j\}_{j=1}^{n2}$. The scores test statistic s_0 is represented by

$$s_0 = \sum_{i=1}^{n1} a(Rx_i) \tag{1}$$

where $a(\cdot)$ is a function to be specified.

For all the possible combinations of taking $n1$ samples out of $n1+n2$ samples (i.e. ${}_{n1+n2}C_{n1}$ combinations), we compute the sum of the scores. Let the scores sum be s_m, $m = 1, 2, \ldots, {}_{n1+n2}C_{n1}$. (Note that at least one of s_m, $m = 1, 2, \ldots, {}_{n1+n2}C_{n1}$, is equal to s_0.) s_m occurs with equal probability (i.e. $1/{}_{n1+n2}C_{n1}$) for all the combinations. Comparing s_0 and s_m, the following probabilities can be computed. We have

$$\mathrm{Prob}(s < s_0) = \frac{\text{Number of combinations less than } s_0 \text{ out of } s_m,\ m = 1, 2, \ldots, {}_{n1+n2}C_{n1}}{{}_{n1+n2}C_{n1} \text{ (Number of all the possible combinations)}}$$

$$\mathrm{Prob}(s = s_0) = \frac{\text{Number of combinations equal to } s_0 \text{ out of } s_m,\ m = 1, 2, \ldots, {}_{n1+n2}C_{n1}}{{}_{n1+n2}C_{n1} \text{ (Number of all the possible combinations)}}$$

$$\mathrm{Prob}(s > s_0) = \frac{\text{Number of combinations greater than } s_0 \text{ out of } s_m,\ m = 1, 2, \ldots, {}_{n1+n2}C_{n1}}{{}_{n1+n2}C_{n1} \text{ (Number of all the possible combinations)}}$$

where s is taken as a random variable generated from the scores test statistic.

If $\mathrm{Prob}(s < s_0)$ is small enough, then s_0 is located at the right tail of the distribution, which implies that $F(x) < G(x)$ for all x. Similarly, if $\mathrm{Prob}(s > s_0)$ is small enough, then s_0 is located at the left tail of the distribution, which implies that $F(x) > G(x)$ for all x. Therefore, in the case of the null hypothesis H_0: $F(x) = G(x)$ and the alternative H_1: $F(x) \neq G(x)$, the null hypothesis is rejected at the 10% significance level when $\mathrm{Prob}(s < s_0) \leqslant 0.05$ or $\mathrm{Prob}(s > s_0) \leqslant 0.05$.

We can consider various scores tests by specifying the function of $a(\cdot)$. The scores tests examined in this paper are the Wilcoxon rank sum test, the normal scores test, the logistic scores test and the Cauchy scores test.

2.1.1 Wilcoxon rank sum test. One of the most famous non-parametric tests is the Wilcoxon rank sum test. The Wilcoxon test statistic w_0 is the scores test defined as $a(Rx_i) = Rx_i$, which is given by

$$w_0 = \sum_{i=1}^{n1} Rx_i \tag{2}$$

In the past, it was too difficult to obtain the exact distribution of w, from a computational point of view. Therefore, under the null hypothesis, we have tested utilizing the fact that w has approximately normal distribution, with mean $E(w)$ and variance $\mathrm{Var}(w)$. We have

$$E(w) = \frac{n1(n1+n2+1)}{2}$$

$$\mathrm{Var}(w) = \frac{n1\, n2(n1+n2+1)}{12}$$

Therefore, in the past, the following statistic was used for the Wilcoxon test statistic:

$$aw_0 = \frac{w_0 - E(w)}{[\mathrm{Var}(w)]^{1/2}} \tag{3}$$

This is called the asymptotic Wilcoxon test statistic in this paper. aw is asymptotically distributed as a standard normal random variable. Mann and Whitney (1947) demonstrated that the normal approximation is quite accurate when $n1$ and $n2$ are larger than 7 (see Mood *et al.*, 1974).

Hodges and Lehman (1956) showed that Pitman's asymptotic relative efficiency of the Wilcoxon test relative to the t-test is quite good. They obtained the result that the asymptotic relative efficiency is greater than 0.864 under the null hypothesis of location shift. This result implies that the Wilcoxon test does not perform too poorly compared with the t-test and, moreover, that the Wilcoxon test may be much better than the t-test. In particular, they showed that the relative efficiency of the Wilcoxon test is 1.33 when the density function $f(x)$ takes the form[5]

$$f(x) = \frac{x^2 \exp(-x)}{\Gamma(3)} \tag{4}$$

where $\Gamma(3)$ is a gamma function with parameter 3. In general, for the distributions with large tails, the Wilcoxon test is more powerful than the t-test.

All the past studies are concerned with asymptotic properties. In Section 3, we examine the small-sample cases, i.e. $n1 = n2 = 5, 7, 9$.

2.1.2 Normal scores test. The normal scores test statistic ns_0 is

$$ns_0 = \sum_{i=1}^{n1} \Phi^{-1}\left(\frac{Rx_i}{n1 + n2 + 1}\right) \tag{5}$$

where $\Phi(\cdot)$ is a standard normal distribution. The scores test that $a(\cdot)$ in equation (1) is assumed to be

$$a(x) = \Phi^{-1}\left(\frac{x}{n1 + n2 + 1}\right)$$

is called the normal scores test.[6]

Chernoff and Savage (1958) proved that the asymptotic relative efficiency of the normal scores test relative to the t-test is greater than or equal to unity, i.e. that the normal scores test is equivalent to the t-test under the normality assumption and that the power of the normal scores test is greater than that of the t-test otherwise.

2.1.3 Logistic scores test. The logistic scores test statistic ls_0 is given by

$$ls_0 = \sum_{i=1}^{n1} F^{-1}\left(\frac{Rx_i}{n1 + n2 + 1}\right) \tag{6}$$

where

$$F(x) = \frac{1}{1 + e^{-x}}$$

which is a logistic distribution.

2.1.4 Cauchy scores test. The Cauchy scores test statistic cs_0 is represented by

$$cs_0 = \sum_{i=1}^{n1} F^{-1}\left(\frac{Rx_i}{n1 + n2 + 1}\right) \tag{7}$$

where $F(x) = 1/2 + (1/\pi)\tan^{-1} x$, which is a Cauchy distribution.

By specifying a functional form for $a(\cdot)$, various scores tests can be constructed. In this paper, the four scores tests discussed above and the Fisher test in the following section are compared.

2.2 Fisher's two-sample test

While the Wilcoxon test statistic is the rank sum of the two samples, the Fisher test statistic uses the difference between two sample means, i.e. $\bar{x} - \bar{y}$ for the two samples $\{x_i\}_{i=1}^{n1}$ and $\{y_j\}_{j=1}^{n2}$. Thus, the test statistic is given by

$$f_0 = \bar{x} - \bar{y} \tag{8}$$

where $\bar{x} = (1/n1)\Sigma x_i$ and $\bar{y} = (1/n2)\Sigma y_i$. For all the possible combinations (i.e. ${}_{n1+n2}C_{n1}$ combinations taking $n1$ out of $n1 + n2$), we compute the difference between the sample means. Let f_m, $m = 1, 2, \ldots, {}_{n1+n2}C_{n1}$, be the difference between the two samples means for all the possible combinations. (Note that at least one out of f_m, $m = 1, 2, \ldots, {}_{n1+n2}C_{n1}$, is equal to f_0.) For all m, f_m occurs with equal probability (i.e. $1/{}_{n1+n2}C_{n1}$). Comparing f_0 and f_m, we can compute Prob($f < f_0$), Prob($f = f_0$) and Prob($f > f_0$), where f is a random variable generated from the Fisher test statistic.

Fisher's two-sample test is of the same type as the scores tests, in the sense of the use of all the possible combinations, but the Fisher test uses more information than do the scores tests, because the scores tests utilize the ranked data as the test statistics, while the Fisher test uses the original data. It might be expected that the Fisher test is more powerful than the scores tests. However, Bradley (1968) stated that the Fisher test and the t-test are asymptotically equivalent, because they both use the difference between two sample means as the test statistic.[7] Therefore, it can be shown that the asymptotic relative efficiency of the Fisher test is sometimes better or worse, compared with the Wilcoxon test.

3 Power comparison (small-sample properties)

In Section 2, we have introduced the four scores tests and the Fisher test. In this section, we examine the small-sample properties by Monte Carlo experiments. Assuming a specific distribution for group 1 samples (i.e. $\{x_i\}_{i=1}^{n1}$) and group 2 samples (i.e. $\{y_j\}_{j=1}^{n2}$), and generating random draws, we compare the non-parametric tests and t-test with respect to sample power. The t-test is compared with the non-parametric tests introduced in this paper.

The t-test statistic is represented as

$$t_0 = \frac{\bar{x} - \bar{y}}{s(1/n1 + 1/n2)^{1/2}}$$

where the degree of freedom is $n1 + n2 - 2$. $\bar{x}$, $\bar{y}$ and s are given by

$$\bar{x} = \sum_i x_i, \qquad \bar{y} = \sum_j y_j, \qquad s^2 = \frac{(n1 - 1)s_x^2 + (n2 - 1)s_y^2}{n1 + n2 - 2}$$

$$s_x^2 = \frac{1}{n1 - 1}\sum_i (x_i - \bar{x})^2, \qquad s_y^2 = \frac{1}{n2 - 1}\sum_j (y_j - \bar{y})^2$$

In the case of $\mathrm{Var}(x) = \mathrm{Var}(y)$, the t-test provides a uniformly very powerful test. Otherwise, the test statistic t_0 does not follow a t-distribution, so the t-test is meaningless.

The null hypothesis is H_0: $F(x) = G(x)$. We compare the sample powers for shifts not only in the location parameter (i.e. μ) but also in the scale parameter (i.e. σ). Therefore, the alternative hypothesis is given by

$$H_1: F(x) = G\left(\frac{x - \mu}{\sigma}\right)$$

We perform Monte Carlo experiments in the following cases: $n1 = n2 = 5, 7, 9$, $\mu = 0.0, 0.5, 1.0$, and $\sigma = 1.0, 1.5, 2.0$ for each of the significance levels, i.e. $\alpha = 0.10, 0.05, 0.01$. The underlying distributions of $\{x_i\}$ and $\{y_j\}$ are normal in Table 1, uniform in Table 2, logistic in Table 3, Cauchy in Table 4 and $\chi^2(6)/2$ in Table 5.

First, we generate normal random draws as $x_i \sim N(0, 1)$ for $i = 1, \ldots, n1$, and $y_i \sim N(\mu, \sigma^2)$ for $j = 1, \ldots, n2$. Theoretically, in the case of $\sigma = 1$, the t-test is more powerful than any other test, because the underlying distribution is normal with equal variance; in the case of $\sigma \neq 1$, the non-parametric tests generally are more powerful than the t-test. (However, because the studies by Hodges and Lehman (1956) and Chernoff and Savage (1958) can be applied only to a shift in the location parameter, it is not appropriate to conclude that the non-parametric tests are more powerful than the t-test in the case of $\sigma \neq 1$.) The results are in Table 1.[8] t, w, aw, ns, ls, cs and f represent the t-test, Wilcoxon test, asymptotic Wilcoxon test, normal scores test, logistic scores test and Cauchy scores test respectively. Even in the case of $\sigma = 1$, the Wilcoxon test exhibits greater power than the Fisher test and t-test. ns, ls and cs are not as powerful as t. Also aw does not perform as well as w, although aw approaches w as the sample size increases. The Fisher two-sample test uses more information than does the Wilcoxon test, in that the Fisher test utilizes the original data, while the Wilcoxon test utilizes the ranked data. Therefore, it is expected that the small-sample Fisher test is more powerful than the Wilcoxon test. However, the Monte Carlo experiment shows that f performs between t and w but f is close to t.

In Table 2, the uniform random draws are generated as follows: $x_i \sim U(-5, 5)$ for $i = 1, \ldots, n1$, and $y_j = \mu + \sigma v_j$, where $v_j \sim U(-5, 5)$, for $j = 1, \ldots, n2$. For a large shift in the location parameter (i.e. large μ), the Cauchy scores test is the most powerful test when $\sigma = 1$, μ increases, and $n1$ and $n2$ are large. w is the best test for almost all the cases.

TABLE 1. Normal distribution

		$n1=n2=5$			$n1=n2=7$			$n1=n2=9$		
μ		$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$	$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$	$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$
(*a*) $\alpha=0.10$										
0.0	*t*	0.1061	0.1065	0.1084	0.1022	0.1029	0.1067	0.1055	0.1035	0.1054
	f	0.1086	0.1086	0.1100	0.1026	0.1029	0.1056	0.1061	0.1031	0.1041
	w	0.1150°°	0.1137°°	0.1220°°°	0.1049	0.1111°°	0.1138°°	0.1140°°	0.1196°°°	0.1204°°°
	aw	0.1150°°	0.1137°°	0.1220°°°	0.1049	0.1111°°	0.1138°°	0.0986**	0.1003*	0.1051
	ns	0.1071	0.1053	0.1074	0.1011	0.1012	0.0972***	0.1041	0.1010	0.0970**
	ls	0.1086	0.1018*	0.0942***	0.1018	0.0998*	0.0933***	0.1050	0.1000*	0.0906***
	cs	0.1086	0.1018*	0.0942***	0.1020	0.0918***	0.0724***	0.1054	0.0868***	0.0605***
0.5	*t*	0.3018	0.2551	0.2251	0.3629	0.2918	0.2493	0.4095	0.3307	0.2808
	f	0.3085°	0.2598°	0.2286	0.3627	0.2916	0.2474	0.4100	0.3305	0.2789
	w	0.3187°°°	0.2703°°°	0.2401°°°	0.3588	0.2952	0.2508	0.4208°°	0.3436°°	0.2937°°
	aw	0.3187°°°	0.2703°°°	0.2401°°°	0.3588	0.2952	0.2508	0.3867***	0.3116***	0.2654***
	ns	0.3050	0.2526	0.2127**	0.3500**	0.2777***	0.2249***	0.3986**	0.3124***	0.2520***
	ls	0.3069	0.2429**	0.1930***	0.3479***	0.2727***	0.2141***	0.3987**	0.3073**	0.2409***
	cs	0.3069°	0.2429**	0.1930***	0.3381***	0.2506***	0.1727***	0.3689***	0.2582***	0.1581***
1.0	*t*	0.5898	0.4572	0.3773	0.7080	0.5618	0.4577	0.7865	0.6403	0.5224
	f	0.5986°	0.4650°	0.3839°	0.7083	0.5601	0.4556	0.7867	0.6400	0.5193
	w	0.6021°°	0.4718°°	0.3958°°°	0.6932***	0.5501**	0.4495*	0.7861	0.6397	0.5244
	aw	0.6021°°	0.4718°°	0.3958°°°	0.6932***	0.5501**	0.4495*	0.7611***	0.6056***	0.4894***
	ns	0.5818*	0.4482*	0.3647**	0.6879***	0.5349***	0.4152***	0.7722***	0.6097***	0.4746***
	ls	0.5734***	0.4298***	0.3247***	0.6869***	0.5286***	0.4034***	0.7702***	0.6045***	0.4579***
	cs	0.5734***	0.4298***	0.3247***	0.6556***	0.4789***	0.3236***	0.7126***	0.5060***	0.3121***

TABLE 1—(*Continued*)

μ		$n1=n2=5$			$n1=n2=7$			$n1=n2=9$		
		$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$	$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$	$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$
(*b*) $\alpha=0.05$										
0.0	*t*	0.0518	0.0551	0.0587	0.0515	0.0526	0.0552	0.0551	0.0540	0.0550
	f	0.0538	0.0577	0.0598	0.0513	0.0524	0.0554	0.0554	0.0538	0.0550
	w	0.0798$^{\circ\circ\circ}$	0.0815$^{\circ\circ\circ}$	0.0866$^{\circ\circ\circ}$	0.0646$^{\circ\circ\circ}$	0.0683$^{\circ\circ\circ}$	0.0721$^{\circ\circ\circ}$	0.0599$^{\circ\circ}$	0.0635$^{\circ\circ\circ}$	0.0667$^{\circ\circ\circ}$
	aw	0.0495*	0.0521*	0.0523**	0.0503	0.0523	0.0542	0.0506*	0.0540	0.0570
	ns	0.0583$^{\circ\circ}$	0.0586°	0.0578	0.0508	0.0506	0.0503**	0.0519*	0.0529	0.0524*
	ls	0.0583$^{\circ\circ}$	0.0586°	0.0578	0.0513	0.0504	0.0491**	0.0519*	0.0515*	0.0494**
	cs	0.0544°	0.0537	0.0524**	0.0511	0.0482*	0.0422***	0.0529	0.0469***	0.0374***
0.5	*t*	0.1861	0.1495	0.1281	0.2279	0.1747	0.1479	0.2725	0.2094	0.1678
	f	0.1897	0.1533°	0.1311	0.2278	0.1744	0.1482	0.2731	0.2101	0.1686
	w	0.2437$^{\circ\circ\circ}$	0.2022$^{\circ\circ\circ}$	0.1767$^{\circ\circ\circ}$	0.2623$^{\circ\circ\circ}$	0.2051$^{\circ\circ\circ}$	0.1723$^{\circ\circ\circ}$	0.2856$^{\circ\circ\circ}$	0.3234$^{\circ\circ\circ}$	0.1865$^{\circ\circ\circ}$
	aw	0.1770**	0.1384***	0.1168***	0.2180**	0.1708*	0.1425*	0.2563***	0.1943***	0.1629*
	ns	0.1963$^{\circ\circ}$	0.1564°	0.1278	0.2230*	0.1675*	0.1300***	0.2665*	0.1986**	0.1497***
	ls	0.1963$^{\circ\circ}$	0.1564°	0.1278	0.2233*	0.1658**	0.1282***	0.2655*	0.1932***	0.1392***
	cs	0.1819*	0.1430*	0.1161***	0.2161**	0.1551***	0.1143***	0.2532***	0.1697***	0.1055***
1.0	*t*	0.4219	0.3171	0.2542	0.5523	0.4081	0.3094	0.6559	0.4895	0.3707
	f	0.4269°	0.3218°	0.2585	0.5524	0.4104	0.3106	0.6563	0.4893	0.3712
	w	0.4982$^{\circ\circ}$	0.3827$^{\circ\circ\circ}$	0.3190$^{\circ\circ\circ}$	0.5848$^{\circ\circ\circ}$	0.4406$^{\circ\circ\circ}$	0.3445$^{\circ\circ\circ}$	0.6583	0.4946°	0.3832$^{\circ\circ}$
	aw	0.3969***	0.2908***	0.2247***	0.5252***	0.3860***	0.2956**	0.6206***	0.4563***	0.3510***
	ns	0.4288°	0.3143	0.2432**	0.5340***	0.3809***	0.2740***	0.6393***	0.4619***	0.3312***
	ls	0.4288°	0.3143	0.2432**	0.5345***	0.3795***	0.2696***	0.6368***	0.4538***	0.3154***
	cs	0.4055***	0.2912***	0.2188***	0.5206***	0.3569***	0.2382***	0.6047***	0.3985***	0.2430***

TABLE 1—(*Continued*)

μ		$n1=n2=5$			$n1=n2=7$			$n1=n2=9$		
		$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$	$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$	$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$
(c) $\alpha=0.01$										
0.0	t	0.0092	0.0098	0.0114	0.0103	0.0103	0.0116	0.0117	0.0122	0.0133
	f	0.0117$^{\circ\circ}$	0.0121$^{\circ\circ}$	0.0142$^{\circ\circ}$	0.0097	0.0108	0.0127°	0.0113	0.0123	0.0136
	w	0.0165$^{\circ\circ\circ}$	0.0171$^{\circ\circ\circ}$	0.0188$^{\circ\circ\circ}$	0.0133$^{\circ\circ}$	0.0158$^{\circ\circ\circ}$	0.0176$^{\circ\circ\circ}$	0.0141$^{\circ\circ}$	0.0139°	0.0170$^{\circ\circ\circ}$
	aw	0.0076*	0.0083*	0.0099*	0.0097	0.0096	0.0121	0.0111	0.0113	0.0130
	ns	0.0165$^{\circ\circ\circ}$	0.0171$^{\circ\circ\circ}$	0.0188$^{\circ\circ\circ}$	0.0111	0.0124$^{\circ\circ}$	0.0136°	0.0126	0.0109*	0.0116*
	ls	0.0165$^{\circ\circ\circ}$	0.0171$^{\circ\circ\circ}$	0.0188$^{\circ\circ\circ}$	0.0105	0.0121°	0.0130°	0.0124	0.0107*	0.0110**
	cs	0.0165$^{\circ\circ\circ}$	0.0171$^{\circ\circ\circ}$	0.0188$^{\circ\circ\circ}$	0.0110	0.0120°	0.0132°	0.0120	0.0108*	0.0098***
0.5	t	0.0495	0.0401	0.0359	0.0706	0.0505	0.0416	0.0953	0.0671	0.0509
	f	0.0552$^{\circ\circ}$	0.0458$^{\circ\circ}$	0.0430$^{\circ\circ\circ}$	0.0697	0.0512	0.0427	0.0946	0.0668	0.0522
	w	0.0711$^{\circ\circ\circ}$	0.0593$^{\circ\circ\circ}$	0.0514$^{\circ\circ\circ}$	0.0842$^{\circ\circ\circ}$	0.0610$^{\circ\circ\circ}$	0.0536$^{\circ\circ\circ}$	0.1082$^{\circ\circ\circ}$	0.0737$^{\circ\circ}$	0.0603$^{\circ\circ\circ}$
	aw	0.0396***	0.0322***	0.0300***	0.0620***	0.0450**	0.0360**	0.0881**	0.0606**	0.0487*
	ns	0.0711$^{\circ\circ\circ}$	0.0593$^{\circ\circ\circ}$	0.0514$^{\circ\circ\circ}$	0.0701	0.0520	0.0418	0.0947	0.0601**	0.0452**
	ls	0.0711$^{\circ\circ\circ}$	0.0593$^{\circ\circ\circ}$	0.0514$^{\circ\circ\circ}$	0.0684	0.0489	0.0396*	0.0949	0.0586***	0.0429***
	cs	0.0711$^{\circ\circ\circ}$	0.0593$^{\circ\circ\circ}$	0.0514$^{\circ\circ\circ}$	0.0685	0.0490	0.0393*	0.0905*	0.0550***	0.0372***
1.0	t	0.1661	0.1070	0.0829	0.2604	0.1659	0.1163	0.3573	0.2257	0.1509
	f	0.1798$^{\circ\circ\circ}$	0.1210$^{\circ\circ\circ}$	0.0940$^{\circ\circ\circ}$	0.2629	0.1671	0.1226°	0.3569	0.2267	0.1561°
	w	0.2104$^{\circ\circ\circ}$	0.1470$^{\circ\circ\circ}$	0.1120$^{\circ\circ\circ}$	0.2820$^{\circ\circ\circ}$	0.1874$^{\circ\circ\circ}$	0.1329$^{\circ\circ\circ}$	0.3731$^{\circ\circ\circ}$	0.2400$^{\circ\circ\circ}$	0.1680$^{\circ\circ\circ}$
	aw	0.1320***	0.0890***	0.0717*	0.2315***	0.1427***	0.0977***	0.3359***	0.2083***	0.1433**
	ns	0.2104$^{\circ\circ\circ}$	0.1470$^{\circ\circ\circ}$	0.1120$^{\circ\circ\circ}$	0.2529**	0.1605*	0.1074**	0.3449**	0.2069***	0.1313***
	ls	0.2104$^{\circ\circ\circ}$	0.1470$^{\circ\circ\circ}$	0.1120$^{\circ\circ\circ}$	0.2483**	0.1540***	0.1017***	0.3423**	0.2014***	0.1233**
	cs	0.2104$^{\circ\circ\circ}$	0.1470$^{\circ\circ\circ}$	0.1120$^{\circ\circ\circ}$	0.2470***	0.1541***	0.1020***	0.3281***	0.1888***	0.1091***

TABLE 2. Uniform distribution

μ		$n1=n2=5$			$n1=n2=7$			$n1=n2=9$		
		$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$	$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$	$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$
(*a*) $\alpha=0.10$										
0.0	*t*	0.0985	0.1004	0.1037	0.1018	0.1002	0.1015	0.0966	0.1023	0.1046
	f	0.1038°	0.1051°	0.1089°	0.1031	0.1010	0.1029	0.0979	0.1030	0.1053
	w	0.1127$^{\circ\circ\circ}$	0.1171$^{\circ\circ\circ}$	0.1234$^{\circ\circ\circ}$	0.1076°	0.1127$^{\circ\circ\circ}$	0.1187$^{\circ\circ\circ}$	0.1097$^{\circ\circ\circ}$	0.1187$^{\circ\circ\circ}$	0.1262$^{\circ\circ\circ}$
	aw	0.1127$^{\circ\circ\circ}$	0.1171$^{\circ\circ\circ}$	0.1234$^{\circ\circ\circ}$	0.1076°	0.1127$^{\circ\circ\circ}$	0.1187$^{\circ\circ\circ}$	0.0940	0.1028	0.1106°
	ns	0.1042°	0.1043°	0.1044	0.1030	0.0975	0.0947**	0.1014°	0.0949**	0.0940***
	ls	0.1035°	0.0954*	0.0823***	0.1027	0.0931**	0.0886***	0.1009°	0.0897***	0.0827***
	cs	0.1035°	0.0954*	0.0823***	0.1030	0.0805***	0.0586***	0.1019°	0.0614***	0.0345***
0.5	*t*	0.1463	0.1376	0.1340	0.1624	0.1456	0.1366	0.1764	0.1590	0.1485
	f	0.1535$^{\circ\circ}$	0.1435°	0.1385°	0.1646	0.1479	0.1373	0.1786	0.1605	0.1497
	w	0.1660$^{\circ\circ\circ}$	0.1551$^{\circ\circ\circ}$	0.1527$^{\circ\circ\circ}$	0.1712$^{\circ\circ}$	0.1548$^{\circ\circ}$	0.1502$^{\circ\circ\circ}$	0.1934$^{\circ\circ\circ}$	0.1733$^{\circ\circ\circ}$	0.1653$^{\circ\circ\circ}$
	aw	0.1660$^{\circ\circ\circ}$	0.1551$^{\circ\circ\circ}$	0.1527$^{\circ\circ\circ}$	0.1712$^{\circ\circ}$	0.1548$^{\circ\circ}$	0.1502$^{\circ\circ\circ}$	0.1696*	0.1509**	0.1444*
	ns	0.1594$^{\circ\circ\circ}$	0.1393	0.1307	0.1722$^{\circ\circ}$	0.1378**	0.1241***	0.1919$^{\circ\circ\circ}$	0.1414***	0.1240***
	ls	0.1618$^{\circ\circ\circ}$	0.1285**	0.1039***	0.1747$^{\circ\circ\circ}$	0.1301***	0.1130***	0.1962$^{\circ\circ\circ}$	0.1359***	0.1128***
	cs	0.1618$^{\circ\circ\circ}$	0.1285**	0.1039***	0.1826$^{\circ\circ\circ}$	0.1146***	0.0762***	0.2130$^{\circ\circ\circ}$	0.0937***	0.0487***
1.0	*t*	0.2098	0.1840	0.1676	0.2425	0.2063	0.1830	0.2792	0.2302	0.1996
	f	0.2172°	0.1916°	0.1730°	0.2471	0.2088	0.1847	0.2821	0.2319	0.2008
	w	0.2367$^{\circ\circ\circ}$	0.1971$^{\circ\circ\circ}$	0.1825$^{\circ\circ\circ}$	0.2555$^{\circ\circ\circ}$	0.2097	0.1894°	0.3045$^{\circ\circ\circ}$	0.2378°	0.2110$^{\circ\circ}$
	aw	0.2367$^{\circ\circ\circ}$	0.1971$^{\circ\circ\circ}$	0.1825$^{\circ\circ\circ}$	0.2555$^{\circ\circ\circ}$	0.2097	0.1894°	0.2683**	0.2122***	0.1881**
	ns	0.2299$^{\circ\circ\circ}$	0.1786*	0.1574**	0.2686$^{\circ\circ\circ}$	0.1943**	0.1580***	0.3135$^{\circ\circ\circ}$	0.2033**	0.1664***
	ls	0.2402$^{\circ\circ\circ}$	0.1686***	0.1271***	0.2740$^{\circ\circ\circ}$	0.1850***	0.1452***	0.3224$^{\circ\circ\circ}$	0.1953***	0.1486***
	cs	0.2402$^{\circ\circ\circ}$	0.1686***	0.1271***	0.2945$^{\circ\circ\circ}$	0.1635***	0.0983***	0.3616$^{\circ\circ\circ}$	0.1453***	0.0685***

TABLE 2—*(Continued)*

μ		$n1=n2=5$			$n1=n2=7$			$n1=n2=9$		
		$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$	$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$	$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$
(*b*) $\alpha=$ 0.05										
0.0	*t*	0.0513	0.0549	0.0603	0.0470	0.0506	0.0535	0.0485	0.0537	0.0565
	f	0.0540°	0.0550	0.0595	0.0470	0.0503	0.0533	0.0489	0.0537	0.0568
	w	0.0759°°°	0.0818°°°	0.0882°°°	0.0623°°°	0.0686°°°	0.0729°°°	0.0876°°°	0.0628°°°	0.0706°°°
	aw	0.0486*	0.0495**	0.0491***	0.0452	0.0525	0.0572°	0.0469	0.0525	0.0590°
	ns	0.0565°°	0.0572°	0.0543**	0.0490	0.0501	0.0466***	0.0505	0.0498*	0.0487***
	ls	0.0565°°	0.0572°	0.0543**	0.0497°	0.0493	0.0455***	0.0498	0.0467***	0.0421***
	cs	0.0504	0.0538	0.0510***	0.0495°	0.0478*	0.0388***	0.0500	0.0394***	0.0250***
0.5	*t*	0.0799	0.0749	0.0743	0.0881	0.0807	0.0757	0.0939	0.0880	0.0824
	f	0.0821	0.0761	0.0739	0.0883	0.0806	0.0754	0.0940	0.0882	0.0822
	w	0.1151°°°	0.1094°°°	0.1110°°°	0.1128°°°	0.0987°°°	0.0940°°°	0.1070°°°	0.0949°°	0.0952°°°
	aw	0.0752*	0.0691**	0.0649***	0.0848*	0.0773*	0.0753	0.0893*	0.0808**	0.0796*
	ns	0.0882°°°	0.0788°	0.0720	0.0929°	0.0733**	0.0638***	0.1036°°°	0.0791***	0.0673***
	ls	0.0882°°°	0.0788°	0.0720	0.0938°°	0.0726**	0.0625***	0.1057°°°	0.0736***	0.0600***
	cs	0.0826	0.0738	0.0673**	0.0954°°	0.0711***	0.0532***	0.1144°°°	0.0609***	0.0356***
1.0	*t*	0.1194	0.1030	0.0961	0.1458	0.1158	0.1053	0.1689	0.1367	0.1182
	f	0.1215	0.1033	0.0956	0.1454	0.1159	0.1044	0.1687	0.1365	0.1174
	w	0.1667°°°	0.1428°°°	0.1364°°°	0.1762°°°	0.1363°°°	0.1200°°°	0.1824°°°	0.1400	0.1257°°
	aw	0.1156*	0.0923***	0.0807***	0.1409*	0.1083**	0.0983**	0.1596**	0.1195***	0.1096**
	ns	0.1343°°°	0.1063°	0.0911*	0.1548°°	0.1037***	0.0836***	0.1839°°°	0.1181***	0.0930***
	ls	0.1343°°°	0.1063°	0.0911*	0.1569°°°	0.1025***	0.0813***	0.1903°°°	0.1143***	0.0830***
	cs	0.1273°°	0.0998*	0.0846***	0.1643°°°	0.1029***	0.0689***	0.2126°°°	0.0967***	0.0510***

TABLE 2—(*Continued*)

μ		$n1=n2=5$			$n1=n2=7$			$n1=n2=9$		
		$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$	$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$	$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$
(*c*) $\alpha=0.01$										
0.0	*t*	0.0138	0.0148	0.0174	0.0099	0.0128	0.0153	0.0121	0.0127	0.0159
	f	0.0132	0.0163°	0.0179	0.0090	0.0114*	0.0138*	0.0105*	0.0120	0.0145*
	w	0.0170°°	0.0203°°°	0.0226°°°	0.0121°°	0.0149°	0.0187°°	0.0130	0.0152°°	0.0193°°
	aw	0.0090***	0.0105***	0.0133***	0.0084*	0.0094***	0.0114***	0.0100*	0.0121	0.0145*
	ns	0.0170°°	0.0203°°°	0.0226°°°	0.0094	0.0119	0.0137*	0.0105*	0.0122	0.0119***
	ls	0.0170°°	0.0203°°°	0.0226°°°	0.0089*	0.0114*	0.0130*	0.0105*	0.0112*	0.0105***
	cs	0.0170°°	0.0203°°°	0.0226°°°	0.0088*	0.0116*	0.0131*	0.0115	0.0112*	0.0104***
0.5	*t*	0.0202	0.0222	0.0254	0.0197	0.0201	0.0218	0.0251	0.0239	0.0233
	f	0.0205	0.0222	0.0245	0.0175*	0.0182*	0.0207	0.0229*	0.0221*	0.0220
	w	0.0264°°°	0.0279°°°	0.0300°°	0.0229°°	0.0240°°	0.0248°°	0.0257	0.0251	0.0261°
	aw	0.0139***	0.0147***	0.0179***	0.0155***	0.0158***	0.0166***	0.0208**	0.0198**	0.0211*
	ns	0.0264°°°	0.0279°°°	0.0300°°	0.0184	0.0195	0.0185**	0.0227*	0.0194**	0.0170***
	ls	0.0264°°°	0.0279°°°	0.0300°°	0.0182*	0.0190	0.0179**	0.0229*	0.0191***	0.0154***
	cs	0.0264°°°	0.0279°°°	0.0300°°	0.0178*	0.0193	0.0180**	0.0242	0.0187***	0.0143***
1.0	*t*	0.0322	0.0311	0.0320	0.0357	0.0321	0.0303	0.0465	0.0387	0.0361
	f	0.0314	0.0292*	0.0318	0.0319**	0.0290*	0.0291	0.0431*	0.0362*	0.0345
	w	0.0396°°°	0.0374°°°	0.0387°°°	0.0415°°°	0.0358°°	0.0338°°	0.0518°°	0.0416°	0.0380°
	aw	0.0221***	0.0221***	0.0224***	0.0284***	0.0249***	0.0223***	0.0416**	0.0340**	0.0317**
	ns	0.0396°°°	0.0374°°°	0.0387°°°	0.0364	0.0292*	0.0259**	0.0463	0.0338**	0.0260***
	ls	0.0396°°°	0.0374°°°	0.0387°°°	0.0363	0.0289*	0.0247***	0.0469	0.0337**	0.0245***
	cs	0.0396°°°	0.0374°°°	0.0387°°°	0.0359	0.0289*	0.0250***	0.0530°°°	0.0326***	0.0225***

TABLE 3. Logistic distribution

		$n1=n2=5$			$n1=n2=7$			$n1=n2=9$		
μ		$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$	$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$	$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$
(*a*) $\alpha=0.10$										
0.0	*t*	0.1007	0.1015	0.1056	0.1056	0.1021	0.1031	0.1003	0.1019	0.1048
	f	0.1036	0.1057°	0.1092°	0.1046	0.1022	0.1010	0.0992	0.1007	0.1029
	w	0.1127°°°	0.1153°°°	0.1176°°°	0.1076	0.1071°	0.1114°°	0.1097	0.1166°°°	0.1212°°°
	aw	0.1127°°°	0.1153°°°	0.1176°°°	0.1076	0.1071°	0.1114°°	0.0940**	0.1001	0.1045
	ns	0.1042°	0.1059°	0.1046	0.1030	0.1009	0.0960**	0.1014	0.0991	0.0965**
	ls	0.1035	0.1011	0.0928***	0.1027	0.0988*	0.0940**	0.1009	0.0985*	0.0918***
	cs	0.1035	0.1011	0.0928***	0.1030	0.0925***	0.0794***	0.1019	0.0865***	0.0662***
0.5	*t*	0.2014	0.1806	0.1659	0.2242	0.1940	0.1766	0.2459	0.2099	0.1895
	f	0.2054	0.1830	0.1675	0.2225	0.1917	0.1745	0.2447	0.2081	0.1874
	w	0.2132°°	0.1913°°	0.1810°°°	0.2296°	0.1982	0.1872°°	0.2686°°°	0.2293°°°	0.2113°°°
	aw	0.2132°°	0.1913°°	0.1810°°°	0.2296°	0.1982°	0.1872°°	0.2394*	0.2049*	0.1868
	ns	0.2003	0.1771	0.1625	0.2242	0.1871*	0.1648***	0.2447	0.2027*	0.1747***
	ls	0.1971*	0.1701**	0.1465***	0.2224	0.1820***	0.1575***	0.2433	0.1996**	0.1648***
	cs	0.1971*	0.1701**	0.1465***	0.2117**	0.1664***	0.1321***	0.2224***	0.1677***	0.1149***
1.0	*t*	0.3432	0.2790	0.2412	0.4056	0.3265	0.2781	0.4591	0.3660	0.3070
	f	0.3496°	0.2860°	0.2445	0.4054	0.3242	0.2747	0.4574	0.3643	0.3032
	w	0.3589°°°	0.3004°°°	0.2623°°°	0.4168°°	0.3369°°	0.2882°°	0.4907	0.3929°°°	0.3301°°°
	aw	0.3589°°°	0.3004°°°	0.2623°°°	0.4168°°	0.3369	0.2882°°	0.4532*	0.3561**	0.2977**
	ns	0.3436	0.2791	0.2367*	0.4014	0.3187*	0.2604***	0.4556	0.3524**	0.2846***
	ls	0.3370*	0.2678**	0.2122***	0.3979*	0.3102***	0.2499***	0.4530*	0.3437***	0.3699***
	cs	0.3570*	0.2678**	0.2122***	0.3692***	0.2761***	0.2012***	0.3966***	0.2788***	0.1849***

TABLE 3—(*Continued*)

μ		$n1 = n2 = 5$			$n1 = n2 = 7$			$n1 = n2 = 9$		
		$\sigma = 1.0$	$\sigma = 1.5$	$\sigma = 2.0$	$\sigma = 1.0$	$\sigma = 1.5$	$\sigma = 2.0$	$\sigma = 1.0$	$\sigma = 1.5$	$\sigma = 2.0$
(b) $\alpha = 0.05$										
0.0	*t*	0.0493	0.0514	0.0540	0.0473	0.0504	0.0531	0.0508	0.0513	0.0546
	f	0.0525°	0.0538°	0.0577°	0.0476	0.0516	0.0544	0.0513	0.0514	0.0556
	w	0.0759$^{\circ\circ\circ}$	0.0796$^{\circ\circ\circ}$	0.0833$^{\circ\circ\circ}$	0.0623$^{\circ\circ\circ}$	0.0664$^{\circ\circ\circ}$	0.0695$^{\circ\circ\circ}$	0.0576$^{\circ\circ\circ}$	0.0604$^{\circ\circ\circ}$	0.0644$^{\circ\circ\circ}$
	aw	0.0486	0.0493	0.0512*	0.0452	0.0497	0.0527	0.0468*	0.0504	0.0543
	ns	0.0565$^{\circ\circ\circ}$	0.0563$^{\circ\circ}$	0.0568°	0.0490	0.0511	0.0505*	0.0505	0.0520	0.0502*
	ls	0.0565$^{\circ\circ\circ}$	0.0563$^{\circ\circ}$	0.0568°	0.0497°	0.0510	0.0501*	0.0495	0.0508	0.0467***
	cs	0.0504	0.0506	0.0501*	0.0495°	0.0487	0.0452***	0.0500	0.0454**	0.0390***
0.5	*t*	0.1091	0.0965	0.0878	0.1295	0.1074	0.0965	0.1453	0.1200	0.1083
	f	0.1131°	0.0998°	0.0929°	0.1305	0.1096	0.0976	0.1461	0.1206	0.1087
	w	0.1547$^{\circ\circ\circ}$	0.1377$^{\circ\circ\circ}$	0.1318$^{\circ\circ\circ}$	0.1606$^{\circ\circ\circ}$	0.1351$^{\circ\circ\circ}$	0.1219$^{\circ\circ\circ}$	0.1607$^{\circ\circ\circ}$	0.1385$^{\circ\circ\circ}$	0.1246$^{\circ\circ\circ}$
	aw	0.1045*	0.0895**	0.0825*	0.1256*	0.1048	0.0966	0.1379**	0.1173	0.1063
	ns	0.1202$^{\circ\circ\circ}$	0.1013°	0.0923°	0.1291	0.1049	0.0915*	0.1429	0.1188	0.0980***
	ls	0.1202$^{\circ\circ\circ}$	0.1013°	0.0923°	0.1289	0.1041*	0.0899**	0.1419	0.1152°	0.0925***
	cs	0.1114	0.0939	0.0830*	0.1225**	0.0982**	0.0806***	0.1341***	0.0998***	0.0708***
1.0	*t*	0.2103	0.1658	0.1413	0.2650	0.2002	0.1640	0.3135	0.2326	0.1877
	f	0.2190$^{\circ\circ}$	0.1722°	0.1466°	0.2663	0.2032	0.1675	0.3154	0.2347	0.1889
	w	0.2811$^{\circ\circ\circ}$	0.3224$^{\circ\circ\circ}$	0.1965$^{\circ\circ\circ}$	0.3076$^{\circ\circ\circ}$	0.2377$^{\circ\circ\circ}$	0.2012$^{\circ\circ\circ}$	0.3407$^{\circ\circ\circ}$	0.2536$^{\circ\circ\circ}$	0.2104$^{\circ\circ\circ}$
	aw	0.2013**	0.1550**	0.1310**	0.2625	0.1966	0.1639	0.3088*	0.2225**	0.1848
	ns	0.2236$^{\circ\circ\circ}$	0.1716°	0.1429	0.2621	0.1930*	0.1530***	0.3129	0.2235**	0.1716***
	ls	0.2236$^{\circ\circ\circ}$	0.1716°	0.1429	0.2619	0.1926*	0.1493***	0.3071*	0.2179***	0.1608***
	cs	0.2078	0.15805*	0.1281***	0.2493***	0.1787***	0.1302***	0.2801***	0.1864***	0.1238***

TABLE 3—(Continued)

μ		$n1 = n2 = 5$			$n1 = n2 = 7$			$n1 = n2 = 9$		
		$\sigma = 1.0$	$\sigma = 1.5$	$\sigma = 2.0$	$\sigma = 1.0$	$\sigma = 1.5$	$\sigma = 2.0$	$\sigma = 1.0$	$\sigma = 1.5$	$\sigma = 2.0$
(c) $\alpha = 0.01$										
0.0	*t*	0.0090	0.0093	0.0098	0.0079	0.0082	0.0103	0.0104	0.0104	0.0115
	f	0.0130°°°	0.0149°°°	0.0157°°°	0.0095	0.0102	0.0129°°	0.0104	0.0113	0.0129°
	w	0.0170°°°	0.0185°°°	0.0197°°°	0.0121°°°	0.0130°°°	0.0162°°°	0.0130	0.0139°°°	0.0149°°°
	aw	0.0090	0.0092	0.0113°	0.0084	0.0080	0.0096	0.0100	0.0112	0.0129°
	ns	0.0170°°°	0.0185°°°	0.0197°°°	0.0094°	0.0102°°	0.0118°	0.0105	0.0113	0.0122
	l*s*	0.0170°°°	0.0185°°°	0.0197°°°	0.0089°	0.0097°	0.0111	0.0105	0.0110	0.0113
	cs	0.0170°°°	0.0185°°°	0.0197°°°	0.0088	0.0097°	0.0111	0.0115°	0.0101	0.0094*
0.5	*t*	0.0263	0.0217	0.0207	0.0277	0.0249	0.0223	0.0392	0.0324	0.0291
	f	0.0317°°°	0.0284°°°	0.0289°°°	0.0299°	0.0285°°	0.0269°°°	0.0411	0.0337	0.0318°
	w	0.0405°°°	0.0354°°°	0.0345°°°	0.0379°°°	0.0341°°°	0.0323°°°	0.0461°°°	0.0392°°°	0.0375°°°
	aw	0.0222**	0.0209	0.0209	0.0270	0.0229*	0.0216	0.0373	0.0309	0.0292
	ns	0.0405°°°	0.0354°°°	0.0345°°°	0.0323°°	0.0271°	0.0251°	0.0387	0.0324	0.0280
	ls	0.0405°°°	0.0354°°°	0.0345°°°	0.0309°	0.0265°	0.0242	0.0378	0.0317	0.0270*
	cs	0.0405°°°	0.0354°°°	0.0345°°°	0.0311°	0.0264	0.0238°	0.0381	0.0303*	0.0231***
1.0	*t*	0.0586	0.0450	0.0384	0.0857	0.0596	0.0487	0.1102	0.0769	0.0606
	f	0.0722°°°	0.0549°°°	0.0481°°°	0.0927°°	0.0651°°	0.0537°°	0.1158°	0.0820°	0.0646°
	w	0.0879°°°	0.0687°°°	0.0581°°°	0.1068°°°	0.0742°°°	0.0622°°°	0.1307°°°	0.0948°°°	0.0737°°°
	aw	0.0516**	0.0402**	0.0364*	0.0771***	0.0540**	0.0441**	0.1084	0.0773	0.0609
	ns	0.0879°°°	0.0687°°°	0.0581°°°	0.0896°	0.0611	0.0486	0.1133	0.0759	0.0566*
	ls	0.0879°°°	0.0687°°°	0.0581°°°	0.0863	0.0593	0.0466	0.1128	0.0750	0.0548**
	cs	0.0879°°°	0.0687°°°	0.0581°°°	0.0867	0.0593	0.0463*	0.1057*	0.0695**	0.0492***

TABLE 4. Cauchy distribution

μ		$n1=n2=5$			$n1=n2=7$			$n1=n2=9$		
		$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$	$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$	$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$
(*a*) $\alpha=0.10$										
0.0	*t*	0.0853	0.0863	0.0857	0.0846	0.0848	0.0859	0.0886	0.0887	0.0890
	f	0.1027$^{\circ\circ\circ}$	0.1040$^{\circ\circ\circ}$	0.1047$^{\circ\circ\circ}$	0.0999$^{\circ\circ\circ}$	0.0978$^{\circ\circ\circ}$	0.1014$^{\circ\circ\circ}$	0.1001$^{\circ\circ\circ}$	0.1002$^{\circ\circ\circ}$	0.1012$^{\circ\circ\circ}$
	w	0.1127$^{\circ\circ\circ}$	0.1126$^{\circ\circ\circ}$	0.1158$^{\circ\circ\circ}$	0.1076$^{\circ\circ\circ}$	0.1059$^{\circ\circ\circ}$	0.1089$^{\circ\circ\circ}$	0.1097$^{\circ\circ\circ}$	0.1138$^{\circ\circ\circ}$	0.1188$^{\circ\circ\circ}$
	aw	0.1127$^{\circ\circ\circ}$	0.1126$^{\circ\circ\circ}$	0.1158$^{\circ\circ\circ}$	0.1076$^{\circ\circ\circ}$	0.1059$^{\circ\circ\circ}$	0.1089$^{\circ\circ\circ}$	0.0940°	0.0968$^{\circ\circ}$	0.1036$^{\circ\circ\circ}$
	ns	0.1042$^{\circ\circ\circ}$	0.1044$^{\circ\circ\circ}$	0.1061$^{\circ\circ\circ}$	0.1030$^{\circ\circ\circ}$	0.1000$^{\circ\circ\circ}$	0.1016$^{\circ\circ\circ}$	0.1014$^{\circ\circ\circ}$	0.1019$^{\circ\circ\circ}$	0.1013$^{\circ\circ\circ}$
	ls	0.1035$^{\circ\circ\circ}$	0.1018$^{\circ\circ\circ}$	0.1001$^{\circ\circ\circ}$	0.1027$^{\circ\circ\circ}$	0.0991$^{\circ\circ\circ}$	0.0994$^{\circ\circ\circ}$	0.1009$^{\circ\circ\circ}$	0.1028$^{\circ\circ\circ}$	0.1009$^{\circ\circ\circ}$
	cs	0.1035$^{\circ\circ\circ}$	0.1018$^{\circ\circ\circ}$	0.1001$^{\circ\circ\circ}$	0.1030$^{\circ\circ\circ}$	0.0968$^{\circ\circ\circ}$	0.0924$^{\circ\circ}$	0.1019$^{\circ\circ\circ}$	0.0988$^{\circ\circ\circ}$	0.0903
0.5	*t*	0.1467	0.1323	0.1242	0.1483	0.1335	0.1265	0.1502	0.1381	0.1305
	f	0.1777$^{\circ\circ\circ}$	0.1634$^{\circ\circ\circ}$	0.1524$^{\circ\circ\circ}$	0.1761$^{\circ\circ\circ}$	0.1583$^{\circ\circ\circ}$	0.1482$^{\circ\circ\circ}$	0.1737$^{\circ\circ\circ}$	0.1601$^{\circ\circ\circ}$	0.1496$^{\circ\circ\circ}$
	w	0.2073$^{\circ\circ\circ}$	0.1864$^{\circ\circ\circ}$	0.1762$^{\circ\circ\circ}$	0.2196$^{\circ\circ\circ}$	0.1955$^{\circ\circ\circ}$	0.1795$^{\circ\circ\circ}$	0.2588$^{\circ\circ\circ}$	0.2232$^{\circ\circ\circ}$	0.2053$^{\circ\circ\circ}$
	aw	0.2073$^{\circ\circ\circ}$	0.1864$^{\circ\circ\circ}$	0.1762$^{\circ\circ\circ}$	0.2196$^{\circ\circ\circ}$	0.1955$^{\circ\circ\circ}$	0.1795$^{\circ\circ\circ}$	0.2278$^{\circ\circ\circ}$	0.1975$^{\circ\circ\circ}$	0.1826$^{\circ\circ\circ}$
	ns	0.1923$^{\circ\circ\circ}$	0.1709$^{\circ\circ\circ}$	0.1617$^{\circ\circ\circ}$	0.2037$^{\circ\circ\circ}$	0.1783$^{\circ\circ\circ}$	0.1636$^{\circ\circ\circ}$	0.2204$^{\circ\circ\circ}$	0.1904$^{\circ\circ\circ}$	0.1738$^{\circ\circ\circ}$
	ls	0.1811$^{\circ\circ\circ}$	0.1607$^{\circ\circ\circ}$	0.1481$^{\circ\circ\circ}$	0.1994$^{\circ\circ\circ}$	0.1736$^{\circ\circ\circ}$	0.1599$^{\circ\circ\circ}$	0.2145$^{\circ\circ\circ}$	0.1858$^{\circ\circ\circ}$	0.1667$^{\circ\circ\circ}$
	cs	0.1811$^{\circ\circ\circ}$	0.1607$^{\circ\circ\circ}$	0.1481$^{\circ\circ\circ}$	0.1777$^{\circ\circ\circ}$	0.1550$^{\circ\circ\circ}$	0.1382$^{\circ\circ\circ}$	0.1759$^{\circ\circ\circ}$	0.1521$^{\circ\circ\circ}$	0.1328
1.0	*t*	0.2248	0.1919	0.1700	0.2326	0.1976	0.1753	0.2313	0.2002	0.1809
	f	0.2733$^{\circ\circ\circ}$	0.2338$^{\circ\circ\circ}$	0.2094$^{\circ\circ\circ}$	0.2676$^{\circ\circ\circ}$	0.2299$^{\circ\circ\circ}$	0.2049$^{\circ\circ\circ}$	0.2677$^{\circ\circ\circ}$	0.2285$^{\circ\circ\circ}$	0.2061$^{\circ\circ\circ}$
	w	0.3323$^{\circ\circ\circ}$	0.2848$^{\circ\circ\circ}$	0.2508$^{\circ\circ\circ}$	0.3795$^{\circ\circ\circ}$	0.3149$^{\circ\circ\circ}$	0.2752$^{\circ\circ\circ}$	0.4449$^{\circ\circ\circ}$	0.3701$^{\circ\circ\circ}$	0.3201$^{\circ\circ\circ}$
	aw	0.3323$^{\circ\circ\circ}$	0.2848$^{\circ\circ\circ}$	0.2508$^{\circ\circ\circ}$	0.3795$^{\circ\circ\circ}$	0.3149$^{\circ\circ\circ}$	0.2752$^{\circ\circ\circ}$	0.4094$^{\circ\circ\circ}$	0.3339$^{\circ\circ\circ}$	0.2866$^{\circ\circ\circ}$
	ns	0.3071$^{\circ\circ\circ}$	0.2602$^{\circ\circ\circ}$	0.2270$^{\circ\circ\circ}$	0.3372$^{\circ\circ\circ}$	0.2808$^{\circ\circ\circ}$	0.2447$^{\circ\circ\circ}$	0.3758$^{\circ\circ\circ}$	0.3080$^{\circ\circ\circ}$	0.2661$^{\circ\circ\circ}$
	ls	0.2755$^{\circ\circ\circ}$	0.2332$^{\circ\circ\circ}$	0.2019$^{\circ\circ\circ}$	0.3305$^{\circ\circ\circ}$	0.2741$^{\circ\circ\circ}$	0.2369$^{\circ\circ\circ}$	0.3577$^{\circ\circ\circ}$	0.2935$^{\circ\circ\circ}$	0.2569$^{\circ\circ\circ}$
	cs	0.2755$^{\circ\circ\circ}$	0.2332$^{\circ\circ\circ}$	0.2019$^{\circ\circ\circ}$	0.2655$^{\circ\circ\circ}$	0.2230$^{\circ\circ\circ}$	0.1898$^{\circ\circ\circ}$	0.2540$^{\circ\circ\circ}$	0.2146$^{\circ\circ\circ}$	0.1838

TABLE 4—(*Continued*)

μ		$n1=n2=5$			$n1=n2=7$			$n1=n2=9$		
		$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$	$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$	$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$
(*b*) $\alpha=0.05$										
0.0	*t*	0.0274	0.0264	0.0283	0.0279	0.0285	0.0301	0.0305	0.0321	0.0326
	f	0.0536°°°	0.0527°°°	0.0529°°°	0.0488°°°	0.0515°°°	0.0535°°°	0.0502°°°	0.0513°°°	0.0526°°°
	w	0.0759°°°	0.0777°°°	0.0794°°°	0.0623°°°	0.0631°°°	0.0676°°°	0.0576°°°	0.0586°°°	0.0617°°°
	aw	0.0486°°°	0.0497°°°	0.0498°°°	0.0452°°°	0.0477°°°	0.0503°°°	0.0469°°°	0.0489°°°	0.0511°°°
	ns	0.0565°°°	0.0564°°°	0.0562°°°	0.0490°°°	0.0492°°°	0.0516°°°	0.0505°°°	0.0510°°°	0.0527°°°
	ls	0.0565°°°	0.0564°°°	0.0562°°°	0.0497°°°	0.0498°°°	0.0818°°°	0.0495°°°	0.0502°°°	0.0506°°°
	cs	0.0504°°°	0.0499°°°	0.0481°°°	0.0495°°°	0.0491°°°	0.0479°°°	0.0500°°°	0.0500°°°	0.0464°°°
0.5	*t*	0.0605	0.0531	0.0503	0.0632	0.0544	0.0503	0.0670	0.0583	0.0548
	f	0.1038°°°	0.0917°°°	0.0870°°°	0.1061°°°	0.0919°°°	0.0850°°°	0.1057°°°	0.0927°°°	0.0865°°°
	w	0.1505°°°	0.1362°°°	0.1283°°°	0.1529°°°	0.1306°°°	0.1187°°°	0.1520°°°	0.1336°°°	0.1206°°°
	aw	0.0993°°°	0.0872°°°	0.0799°°°	0.1215°°°	0.1044°°°	0.0956°°°	0.1292°°°	0.1143°°°	0.1022°°°
	ns	0.1112°°°	0.0982°°°	0.0889°°°	0.1184°°°	0.1012°°°	0.0884°°°	0.1292°°°	0.1118°°°	0.0989°°°
	ls	0.1112°°°	0.0982°°°	0.0889°°°	0.1180°°°	0.0999°°°	0.0870°°°	0.1260°°°	0.1087°°°	0.0949°°°
	cs	0.1008°°°	0.0888°°°	0.0792°°°	0.1074°°°	0.0899°°°	0.0764°°°	0.1097°°°	0.0916°°°	0.0775°°°
1.0	*t*	0.1159	0.0944	0.0802	0.1202	0.0938	0.0808	0.1220	0.0992	0.0852
	f	0.1799°°°	0.1499°°°	0.1310°°°	0.1813°°°	0.1477°°°	0.1281°°°	0.1818°°°	0.1499°°°	0.1304°°°
	w	0.2607°°°	0.2165°°°	0.1900°°°	0.2792°°°	0.2247°°°	0.1908°°°	0.3027°°°	0.2401°°°	0.2018°°°
	aw	0.1791°°°	0.1457°°°	0.1281°°°	0.2333°°°	0.1877°°°	0.1565°°°	0.2708°°°	0.2124°°°	0.1773°°°
	ns	0.1917°°°	0.1585°°°	0.1367°°°	0.2137°°°	0.1712°°°	0.1421°°°	0.2496°°°	0.1976°°°	0.1621°°°
	ls	0.1917°°°	0.1585°°°	0.1367°°°	0.2084°°°	0.1699°°°	0.1415°°°	0.2385°°°	0.1877°°°	0.1524°°°
	cs	0.1695°°°	0.1409°°°	0.1176°°°	0.1817°°°	0.1459°°°	0.1178°°°	0.1792°°°	0.1450°°°	0.1168°°°

TABLE 4—(*Continued*)

μ		$n1=n2=5$			$n1=n2=7$			$n1=n2=9$		
		$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$	$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$	$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$
(*c*) $\alpha=0.01$										
0.0	*t*	0.0019	0.0019	0.0022	0.0020	0.0018	0.0024	0.0031	0.0030	0.0033
	f	0.0129^{ooo}	0.0141^{ooo}	0.0148^{ooo}	0.0094^{ooo}	0.0097^{ooo}	0.0111^{ooo}	0.0106^{ooo}	0.0105^{ooo}	0.0116^{ooo}
	w	0.0170^{ooo}	0.0180^{ooo}	0.0183^{ooo}	0.0121^{ooo}	0.0125^{ooo}	0.0135^{ooo}	0.0130^{ooo}	0.0134^{ooo}	0.0143^{ooo}
	aw	0.0090^{ooo}	0.0089^{ooo}	0.0104^{ooo}	0.0084^{ooo}	0.0077^{ooo}	0.0086^{ooo}	0.0100^{ooo}	0.0101^{ooo}	0.0117^{ooo}
	ns	0.0170^{oo}	0.0180^{ooo}	0.0183^{ooo}	0.0094^{ooo}	0.0096^{ooo}	0.0101^{ooo}	0.0105^{ooo}	0.0105^{ooo}	0.0118^{ooo}
	ns	0.0170^{ooo}	0.0180^{ooo}	0.0183^{ooo}	0.0089^{ooo}	0.0089^{ooo}	0.0094^{ooo}	0.0105^{ooo}	0.0110^{ooo}	0.0112^{ooo}
	cs	0.0170^{ooo}	0.0180^{ooo}	0.0183^{ooo}	0.0088^{ooo}	0.0088^{ooo}	0.0095^{ooo}	0.0115^{ooo}	0.0106^{ooo}	0.0098^{ooo}
0.5	*t*	0.0077	0.0060	0.0054	0.0079	0.0069	0.0061	0.0100	0.0089	0.0083
	f	0.0336^{ooo}	0.0289^{ooo}	0.0269^{ooo}	0.0281^{ooo}	0.0258^{ooo}	0.0235^{ooo}	0.0341^{ooo}	0.0278^{ooo}	0.0244^{ooo}
	w	0.0418^{ooo}	0.0363^{ooo}	0.0327^{ooo}	0.0387^{ooo}	0.0337^{ooo}	0.0298^{ooo}	0.0455^{ooo}	0.0385^{ooo}	0.0338^{ooo}
	aw	0.0240^{ooo}	0.0212^{ooo}	0.0201^{ooo}	0.0262^{ooo}	0.0239^{ooo}	0.0206^{ooo}	0.0375^{ooo}	0.0313^{ooo}	0.0266^{ooo}
	ns	0.0418^{ooo}	0.0363^{ooo}	0.0327^{ooo}	0.0309^{ooo}	0.0272^{ooo}	0.0233^{ooo}	0.0360^{ooo}	0.0311^{ooo}	0.0261^{ooo}
	ls	0.0418^{ooo}	0.0363^{ooo}	0.0327^{ooo}	0.0295^{ooo}	0.0387^{ooo}	0.0220^{ooo}	0.0349^{ooo}	0.0305^{ooo}	0.0251^{ooo}
	cs	0.0418^{ooo}	0.0363^{ooo}	0.0327^{ooo}	0.0292^{ooo}	0.0256^{ooo}	0.0218^{ooo}	0.0332^{ooo}	0.0277^{ooo}	0.0228^{ooo}
1.0	*t*	0.0227	0.0161	0.0120	0.0281	0.0178	0.0132	0.0303	0.0210	0.0172
	f	0.0738^{ooo}	0.0574^{ooo}	0.0476^{ooo}	0.0743^{ooo}	0.0563^{ooo}	0.0475^{ooo}	0.0758^{ooo}	0.0581^{ooo}	0.0487^{ooo}
	w	0.0863^{ooo}	0.0670^{ooo}	0.0561^{ooo}	0.1001^{ooo}	0.0730^{ooo}	0.0607^{ooo}	0.1168^{ooo}	0.0857^{ooo}	0.0715^{ooo}
	aw	0.0548^{ooo}	0.0430^{ooo}	0.0367^{ooo}	0.0719^{ooo}	0.0524^{ooo}	0.0427^{ooo}	0.0967^{ooo}	0.0709^{ooo}	0.0579^{ooo}
	ns	0.0863^{ooo}	0.0670^{ooo}	0.0561^{ooo}	0.0790^{ooo}	0.0571^{ooo}	0.0464^{ooo}	0.0911^{ooo}	0.0663^{ooo}	0.0542^{ooo}
	ls	0.0863^{ooo}	0.0670^{ooo}	0.0561^{ooo}	0.0727^{ooo}	0.0540^{ooo}	0.0449^{ooo}	0.0875^{ooo}	0.0631^{ooo}	0.0521^{ooo}
	cs	0.0863^{ooo}	0.0670^{ooo}	0.0561^{ooo}	0.0722^{ooo}	0.0530^{ooo}	0.0444^{ooo}	0.0734^{ooo}	0.0539^{ooo}	0.0438^{ooo}

TABLE 5. $\chi^2(6)/2$ distribution

μ		$n1=n2=5$			$n1=n2=7$			$n1=n2=9$		
		$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$	$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$	$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$
(*a*) $\alpha=0.10$										
0.0	*t*	0.0941	0.0827	0.0771	0.1020	0.0842	0.0759	0.1006	0.0871	0.0819
	f	0.0974°	0.0858°	0.0799°	0.1014	0.0820	0.0739	0.0987	0.0864	0.0798
	w	0.1044°°°	0.0824	0.0790	0.1043	0.0744***	0.0658***	0.1085°°	0.0732***	0.0649***
	aw	0.1044°°°	0.0824	0.0790	0.1043	0.0744***	0.0658***	0.0928**	0.0622***	0.0553***
	ns	0.0979°	0.0738***	0.0656***	0.0994	0.0629***	0.0498***	0.0973*	0.0588***	0.0464***
	ls	0.0969	0.0692***	0.0545***	0.0994	0.0600***	0.0444***	0.0974*	0.0548***	0.0412***
	cs	0.0969	0.0692***	0.0545***	0.1008	0.0502***	0.0292***	0.0989	0.0408***	0.0199***
0.5	*t*	0.2089	0.1508	0.1264	0.2362	0.1775	0.1454	0.2519	0.1923	0.1557
	f	0.2178°°	0.1532	0.1281	0.2351	0.1750	0.1404*	0.2506	0.1888	0.1514*
	w	0.2296°°°	0.1532	0.1221*	0.2506°°°	0.1540***	0.1177***	0.2869°°°	0.1682***	0.1256***
	aw	0.2296°°°	0.1532	0.1221*	0.2506°°°	0.1540***	0.1177***	0.2563°	0.1484***	0.1081***
	ns	0.2167°	0.1415**	0.1081***	0.2482°°	0.1404***	0.0959***	0.2725°°°	0.1435***	0.0961***
	ls	0.2214°°°	0.1349***	0.0922***	0.2497°°°	0.1382***	0.0901***	0.2762°°°	0.1394***	0.0896***
	cs	0.2214°°°	0.1349***	0.0922***	0.2480°°	0.1237***	0.0640***	0.2722°°°	0.1118***	0.0496***
1.0	*t*	0.3630	0.2578	0.1947	0.4304	0.3164	0.2458	0.4710	0.3520	0.2709
	f	0.3732°°	0.2634°	0.1970	0.4319	0.3144	0.2429	0.4700	0.3484	0.2673
	w	0.3965°°°	0.2592	0.1889*	0.4652°°°	0.2938***	0.2040***	0.5249°°°	0.3344***	0.2213***
	aw	0.3965°°°	0.2592	0.1889*	0.4652°°°	0.2938***	0.2040***	0.4898°°°	0.3003***	0.1957***
	ns	0.3789°°°	0.2437***	0.1685***	0.4609°°°	0.3796***	0.1754***	0.5132°°°	0.3057***	0.1786***
	ls	0.3773°°	0.2386***	0.1513***	0.4598°°°	0.2751***	0.1654***	0.5155°°°	0.3007***	0.1672***
	cs	0.3773°°	0.2386***	0.1513***	0.4439°°	0.2568***	0.1287***	0.4832°°	0.2607***	0.1044***

TABLE 5—(*Continued*)

μ		$n1=n2=5$			$n1=n2=7$			$n1=n2=9$		
		$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$	$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$	$\sigma=1.0$	$\sigma=1.5$	$\sigma=2.0$
(*b*) $\alpha=0.05$										
0.0	*t*	0.0453	0.0347	0.0318	0.0490	0.0375	0.0325	0.0500	0.0401	0.0359
	f	0.0483°	0.0368°	0.0347	0.0503	0.0382	0.0343°	0.0505	0.0410	0.0362
	w	$0.0724^{\circ\circ\circ}$	$0.0556^{\circ\circ\circ}$	$0.0523^{\circ\circ\circ}$	$0.0639^{\circ\circ\circ}$	$0.0433^{\circ\circ\circ}$	$0.0366^{\circ\circ}$	$0.0574^{\circ\circ\circ}$	0.0384	$0.0337^{\star}$
	aw	0.0460	$0.0318^{\star}$	$0.0275^{\star\star}$	0.0491	$0.0321^{\star\star}$	$0.0284^{\star\star}$	0.0482	$0.0322^{\star\star\star}$	$0.0278^{\star\star\star}$
	ns	$0.0533^{\circ\circ\circ}$	0.0371°	0.0322	0.0508	$0.0302^{\star\star\star}$	$0.0232^{\star\star\star}$	0.0514	$0.0287^{\star\star\star}$	$0.0217^{\star\star\star}$
	ls	$0.0533^{\circ\circ\circ}$	0.0371°	0.0322	0.0504	$0.0297^{\star\star\star}$	$0.0223^{\star\star\star}$	0.0507	$0.0265^{\star\star\star}$	$0.0183^{\star\star\star}$
	cs	$0.0496^{\circ\circ}$	0.0341	$0.0298^{\star}$	0.0510	$0.0268^{\star\star\star}$	$0.0168^{\star\star\star}$	0.0498	$0.0228^{\star\star\star}$	$0.0120^{\star\star\star}$
0.5	*t*	0.1078	0.0750	0.0597	0.1349	0.0858	0.0665	0.1511	0.0990	0.0789
	f	0.1138°	$0.0818^{\circ\circ}$	$0.0660^{\circ\circ}$	0.1392°	0.0898°	0.0709°	0.1532	0.1014	0.0806
	w	$0.1622^{\circ\circ\circ}$	$0.1083^{\circ\circ\circ}$	$0.0895^{\circ\circ\circ}$	$0.1734^{\circ\circ\circ}$	$0.0993^{\circ\circ\circ}$	$0.0719^{\circ\circ}$	$0.1729^{\circ\circ\circ}$	$0.0959^{\star}$	$0.0690^{\star\star\star}$
	aw	0.1130°	$0.0685^{\star\star}$	$0.0526^{\star\star}$	0.1406°	$0.0787^{\star\star}$	$0.0574^{\star\star\star}$	0.1482	$0.0827^{\star\star\star}$	$0.0593^{\star\star\star}$
	ns	$0.1288^{\circ\circ\circ}$	0.0782°	0.0593	$0.1444^{\circ\circ}$	$0.0739^{\star\star\star}$	$0.0482^{\star\star\star}$	$0.1632^{\circ\circ\circ}$	$0.0795^{\star\star\star}$	$0.0499^{\star\star\star}$
	ls	$0.1288^{\circ\circ\circ}$	0.0782	0.0593	$0.1449^{\circ\circ}$	$0.0739^{\star\star\star}$	$0.0476^{\star\star\star}$	$0.1633^{\circ\circ\circ}$	$0.0771^{\star\star\star}$	$0.0455^{\star\star\star}$
	cs	$0.1191^{\circ\circ\circ}$	0.0752	$0.0535^{\star\star}$	$0.1461^{\circ\circ\circ}$	$0.0696^{\star\star\star}$	$0.0394^{\star\star\star}$	$0.1622^{\circ\circ\circ}$	$0.0683^{\star\star\star}$	$0.0296^{\star\star\star}$
1.0	*t*	0.2250	0.1360	0.1016	0.2882	0.1850	0.1264	0.3348	0.2185	0.1500
	f	$0.2369^{\circ\circ}$	$0.1483^{\circ\circ\circ}$	$0.1115^{\circ\circ\circ}$	0.2937°	0.1916°	0.1324°	0.3375	0.2220	0.1535
	w	$0.3062^{\circ\circ\circ}$	$0.1885^{\circ\circ\circ}$	$0.1418^{\circ\circ\circ}$	$0.3543^{\circ\circ\circ}$	$0.2062^{\circ\circ\circ}$	$0.1331^{\circ\circ}$	$0.3848^{\circ\circ\circ}$	0.2149	$0.1345^{\star\star\star}$
	aw	0.2258	$0.1322^{\star}$	$0.0914^{\star\star\star}$	0.3011°	$0.1660^{\star\star\star}$	$0.1057^{\star\star\star}$	$0.3483^{\circ\circ}$	$0.1858^{\star\star\star}$	$0.1172^{\star\star\star}$
	ns	$0.2530^{\circ\circ\circ}$	$0.1475^{\circ\circ\circ}$	0.1013	$0.3133^{\circ\circ\circ}$	$0.1664^{\star\star\star}$	$0.0944^{\star\star\star}$	$0.3665^{\circ\circ\circ}$	$0.1888^{\star\star\star}$	$0.1039^{\star\star\star}$
	ls	$0.2530^{\circ\circ\circ}$	$0.1475^{\circ\circ\circ}$	0.1013	$0.3151^{\circ\circ\circ}$	$0.1676^{\star\star\star}$	$0.0932^{\star\star\star}$	$0.3643^{\circ\circ\circ}$	$0.1853^{\star\star\star}$	$0.0962^{\star\star\star}$
	cs	$0.2385^{\circ\circ\circ}$	0.1379	$0.0934^{\star\star}$	$0.3091^{\circ\circ\circ}$	$0.1599^{\star\star\star}$	$0.0803^{\star\star\star}$	$0.3542^{\circ\circ\circ}$	$0.1696^{\star\star\star}$	$0.0728^{\star\star\star}$

TABLE 5—(*Continued*)

μ		$n1 = n2 = 5$			$n1 = n2 = 7$			$n1 = n2 = 9$		
		$\sigma = 1.0$	$\sigma = 1.5$	$\sigma = 2.0$	$\sigma = 1.0$	$\sigma = 1.5$	$\sigma = 2.0$	$\sigma = 1.0$	$\sigma = 1.5$	$\sigma = 2.0$
(c) $\alpha = 0.01$										
0.0	*t*	0.0084	0.0066	0.0067	0.0082	0.0054	0.0051	0.0089	0.0064	0.0057
	f	$0.0115^{\circ\circ\circ}$	$0.0102^{\circ\circ\circ}$	$0.0095^{\circ\circ\circ}$	0.0094	0.0065	0.0065°	0.0095	$0.0084^{\circ\circ}$	0.0070°
	w	$0.0146^{\circ\circ\circ}$	$0.0125^{\circ\circ\circ}$	$0.0117^{\circ\circ\circ}$	$0.0125^{\circ\circ\circ}$	0.0081	$0.0076^{\circ\circ\circ}$	$0.0123^{\circ\circ\circ}$	0.0074°	0.0065°
	lw	0.0089	0.0066	0.0061	0.0082	0.0050	0.0045	0.0096	0.0062	0.0053
	ns	$0.0146^{\circ\circ\circ}$	$0.0125^{\circ\circ\circ}$	0.0117°	$0.0103^{\circ\circ}$	0.0060	0.0051	0.0101°	0.0057	$0.0044^{\star}$
	ls	$0.0146^{\circ\circ\circ}$	$0.0125^{\circ\circ\circ}$	$0.0117^{\circ\circ\circ}$	0.0099°	0.0055	0.0048	0.0104°	0.0057	$0.0043^{\star}$
	cs	$0.0146^{\circ\circ\circ}$	$0.0125^{\circ\circ\circ}$	$0.0117^{\circ\circ\circ}$	0.0099°	0.0060	0.0048	0.0105°	$0.0056^{\star}$	$0.0038^{\star\star}$
0.5	*t*	0.0232	0.0143	0.0116	0.0328	0.0175	0.0111	0.0411	0.0203	0.0151
	f	$0.0304^{\circ\circ\circ}$	$0.0186^{\circ\circ\circ}$	$0.0170^{\circ\circ\circ}$	$0.0374^{\circ\circ}$	$0.0214^{\circ\circ}$	$0.0147^{\circ\circ\circ}$	0.0440°	$0.0254^{\circ\circ\circ}$	$0.0185^{\circ\circ}$
	w	$0.0401^{\circ\circ\circ}$	$0.0237^{\circ\circ\circ}$	$0.0203^{\circ\circ\circ}$	$0.0465^{\circ\circ\circ}$	$0.0243^{\circ\circ\circ}$	$0.0170^{\circ\circ\circ}$	$0.0516^{\circ\circ\circ}$	$0.0272^{\circ\circ\circ}$	0.0168°
	w	$0.0212^{\star}$	0.0132	0.0112	0.0328	$0.0156^{\star}$	0.0104	0.0405	0.0213	$0.0133^{\star}$
	ls	$0.0401^{\circ\circ\circ}$	$0.0237^{\circ\circ\circ}$	$0.0203^{\circ\circ\circ}$	$0.0387^{\circ\circ\circ}$	0.0189°	0.0122°	$0.0460^{\circ\circ}$	0.0214	$0.0114^{\star\star\star}$
	cs	$0.0401^{\circ\circ\circ}$	$0.0237^{\circ\circ\circ}$	$0.0203^{\circ\circ\circ}$	$0.0381^{\circ\circ}$	0.0180	0.0116	$0.0467^{\circ\circ}$	0.0211	$0.0107^{\star\star\star}$
	cs	$0.0401^{\circ\circ\circ}$	$0.0237^{\circ\circ\circ}$	$0.0203^{\circ\circ\circ}$	$0.0384^{\circ\circ\circ}$	0.0180	0.0116	$0.0476^{\circ\circ\circ}$	0.0206	$0.0096^{\star\star\star}$
1.0	*t*	0.0606	0.0302	0.0200	0.0939	0.0424	0.0262	0.1225	0.0589	0.0345
	f	$0.0784^{\circ\circ\circ}$	$0.0430^{\circ\circ\circ}$	$0.0300^{\circ\circ\circ}$	$0.1022^{\circ\circ}$	$0.0818^{\circ\circ\circ}$	$0.0336^{\circ\circ\circ}$	$0.1295^{\circ\circ}$	$0.0674^{\circ\circ\circ}$	$0.0420^{\circ\circ\circ}$
	w	$0.0969^{\circ\circ\circ}$	$0.0545^{\circ\circ\circ}$	$0.0373^{\circ\circ\circ}$	$0.1249^{\circ\circ\circ}$	$0.0587^{\circ\circ\circ}$	$0.0350^{\circ\circ\circ}$	$0.1555^{\circ\circ\circ}$	$0.0687^{\circ\circ\circ}$	$0.0409^{\circ\circ\circ}$
	w	$0.0553^{\star\star}$	0.0289	0.0197	0.0931	0.0417	$0.0239^{\star}$	$0.1302^{\circ\circ}$	$0.0563^{\star}$	0.0333
	ns	$0.0969^{\circ\circ\circ}$	$0.0545^{\circ\circ\circ}$	$0.0373^{\circ\circ\circ}$	$0.1062^{\circ\circ\circ}$	$0.0480^{\circ\circ}$	0.0266	$0.1403^{\circ\circ\circ}$	0.0586	$0.0312^{\star}$
	ls	$0.0969^{\circ\circ\circ}$	$0.0545^{\circ\circ\circ}$	$0.0373^{\circ\circ\circ}$	$0.1053^{\circ\circ\circ}$	$0.0468^{\circ\circ}$	0.0255	$0.1392^{\circ\circ\circ}$	0.0588	$0.0297^{\star\star}$
	cs	$0.0969^{\circ\circ\circ}$	$0.0545^{\circ\circ\circ}$	$0.0373^{\circ\circ\circ}$	$0.1062^{\circ\circ\circ}$	$0.0467^{\circ\circ}$	0.0259	$0.1356^{\circ\circ\circ}$	$0.0553^{\star}$	$0.0262^{\star\star\star}$

Furthermore, in Tables 3–5, we take three examples of the underlying distributions, such that the Wilcoxon test has a larger asymptotic relative efficiency than that of the t-test; the examples are the logistic distribution, Cauchy distribution and $\chi^2(6)/2$ distribution (i.e. equation (4)).

In Table 3, we generate random draws as follows: $x_i \sim \exp(-x_i)/[1+\exp(-x_i)]^2$ for $i=1,\ldots,n1$, and $y_i=\mu+\sigma v_j$, where $v_j \sim \exp(-v_j)/[1+\exp(-v_j)]^2$, for $j=1,\ldots,n2$. In the case of the logistic distribution, it is known that the asymptotic relative efficiency of the Wilcoxon test to the normal scores test is 1.047 (see Kendall & Stuart, 1979). For large $n1$ and $n2$, w exhibits the best performance, despite α. ns, ls and cs have the best small-sample powers when $\sigma \neq 1$.

When the underlying distribution is Cauchy, it is known that the asymptotic relative efficiency of the Wilcoxon test to the normal scores test is 1.413 (see Kendall & Stuart, 1979). Thus, in Table 4, we consider the following random draws: $x_i \sim 1/\pi(1+x_i^2)$ for $i=1,\ldots,n1$, and $y_j=\mu+\sigma v_j$, where $v_j \sim 1/\pi(1+v_j^2)$, for $j=1,\ldots,n2$. All the non-parametric tests perform better than the t-test. Overall, the order of greatest power is given by the Wilcoxon test, Fisher test and the t-test.

When the underlying distribution is given by equation (4), i.e. $\chi^2(6)/2$, the asymptotic relative efficiency of the Wilcoxon test to the t-test is 1.33 (see Hodges & Lehman, 1956). In Table 5, we generate random draws as follows: $x_i=u_i/2-3$, where $u_i \sim \chi^2(6)$, for $i=1,\ldots,n1$, and $y_i=\mu+\sigma(v_j/2-3)$, where $v_j \sim \chi^2(6)$, for $j=1,\ldots,n2$.[9] When α is small, the non-parametric tests perform better, especially when $\sigma=1$. For small μ and large α, t and f perform quite well.

When the tails of distribution are large, the asymptotic property that the Wilcoxon tests has more asymptotic relative efficiency than the t-test holds in the case of small samples from the Monte Carlo experiments in Tables 1–5. Intuitively, it is expected from an amount of information included in the test statistics that the Fisher test performs better than the Wilcoxon test in the sample power. However, judging from the results obtained in Tables 3–5, the Wilcoxon test is more powerful. The following three facts contribute to this:

(1) both the t-test and the Fisher test take the difference between the two sample means as the test statistic;
(2) both the Fisher test and the Wilcoxon test are non-parametric tests based on all the possible combinations;
(3) for a distribution with fat tails, the Wilcoxon test exhibits more asymptotic relative efficiency than does the t-test.

From these three facts, we can consider that the Fisher test performs between the t-test and the Wilcoxon test, with regard to the sample power. Therefore, the order of the three sample powers is given by $\hat{p}_t \leqslant \hat{p}_f \leqslant \hat{p}_w$ for a distribution with fat tails, and $\hat{p}_w \leqslant \hat{p}_f \leqslant \hat{p}_t$ otherwise.

The theorem proved by Chernoff and Savage (1958), i.e. that the asymptotic relative efficiency of the normal scores test to the t-test is more than unity under the alternative hypothesis of a shifting location parameter, holds in the case of small samples. In the case of $\sigma=1$ in Table 1, the t-test sometimes performs better than the normal scores test. In Tables 2–5, the normal scores test performs better than the t-test for almost all the cases. However, the normal scores test is less powerful than the Wilcoxon test, which is consistent with the results obtained by Mehta and Patel (1992). Thus, the t-test and the normal scores test are similar, but the normal scores test performs slightly better than the t-test.

Generally, in the case of small samples, it might be concluded from Tables 1–5 that we have the following inequality: $\hat{p}_f \leqslant \hat{p}_{ns} \leqslant \hat{p}_w$.

4 Example: Testing structural changes

In a regression analysis, the disturbance term is usually assumed to be normal and we perform testing of a hypothesis. However, sometimes, the normality assumption is too strong. In this paper, 'loosening' the normality assumption, we test a structural change without assuming any distribution for the disturbance term. Consider the standard regression model

$$y_t = x_t\beta + u_t, \qquad t = 1, \dots, T$$

where y_t, x_t, β and u_t respectively are a dependent variable at time t, a $1 \times k$ vector of independent variables at time t, a $k \times 1$ unknown parameter vector to be estimated, and the disturbance term at time t with mean zero and variance σ^2. The sample size is T. Let us define $X_{t-1} = (x_1' x_2' \dots x_{t-1}')'$ and $Y_{t-1} = (y_1 y_2 \dots y_{t-1})'$. β_{t-1} denotes the ordinary least-squares (OLS) estimate of β using the data up to time $t-1$, i.e. $\beta_{t-1} = (X_{t-1}' X_{t-1})^{-1} X_{t-1}' Y_{t-1}$. The recursive residual, i.e.

$$\omega_t = \frac{(y_t - x_t\beta_{t-1})}{[1 + x_t(X_{t-1}'X_{t-1})^{-1}x_t]^{1/2}}, \qquad t = k+1, \dots, T$$

can be computed by recursive OLS estimation, which is distributed with mean zero and variance σ^2 The recursive residuals ω_t, $t = k+1, \dots, T$, are mutually independently distributed and normalized to mean zero and variance σ^2.

Based on the recursive residual ω_t, we perform testing at the structural change.[10] We can accept the structural change if the structure of the recursive residuals changes in a period. Dividing the sample into two groups, we test if both $\{\omega_t\}_{t=k+1}^{n1}$ and $\{\omega_t\}_{t=n1+1}^{T}$ are generated from the same distribution, where $T = n1 + n2$. The null hypothesis is represented by H_0: $F(\omega) = G(\omega)$, while the alternative is H_1: $F(\omega) \neq G(\omega)$. Let $F(\cdot)$ be the distribution of the first $n1$ recursive residuals, and let $G(\cdot)$ be that of the last $n2$ recursive residuals.

We take an example of a Japanese import function. Annual data from *Annual Report on National Accounts* (Economic Planning Agency, Government of Japan) are used. Let GDP_t be the gross domestic product (1985 price, billions of Japanese yen), let M_t be the imports of goods and services (1985 price, billions of Japanese yen), let P_t be the terms of trade index, which is given by the imports of goods and services implicit price deflator (1985 = 1.00) divided by the gross domestic product implicit price deflator (1985 = 1.00).

The following two import functions are estimated and the recursive residuals are computed.

$$\log M_t = \beta_0 + \beta_1 \log GDP_t + \beta_2 \log P_t \tag{7}$$

$$\log M_t = \beta_0 + \beta_1 \log GDP_t + \beta_2 \log P_t + \beta_3 \log M_{t-1} \tag{8}$$

where β_0, β_1, β_2 and β_3 are the unknown parameters to be estimated.[11]

For each regression equation, we compute the recursive residuals from 1961 to 1991, divide the period into two groups, and test if the recursive residuals in the first period are the same as those in the last period. The results are in Tables 6 and 7. In the tables, t, Fisher, Wilcoxon, Asy Wil, Normal, Logistic and Cauchy denote the t-test, Fisher test, Wilcoxon test, asymptotic Wilcoxon test, normal scores test,

TABLE 6. Testing structural change by non-parametric tests: equation (7)

	t		Fisher		Wilcoxon		Asy Wil		Normal		Logistic		Cauchy		Chow	
Year	t_0	p-val	f_0	p-val	w_0	p-val	aw_0	p-val	ns_0	p-val	ls_0	p-val	cs_0	p-val	F_0	p-val
1961	0.9503	0.8254	0.0848	0.8182	28	0.8182	1.1552	0.8760	0.9661	0.8182	1.6094	0.8182	1.7321	0.8182	1.5534	0.7788
1962	0.4816	0.6833	0.0312	0.6742	40	0.6553	0.4527	0.6746	0.5754	0.6591	0.9837	0.6591	1.2165	0.6742	1.8903	0.8475
1963	0.2695	0.6053	0.0145	0.5882	53	0.5352	0.1252	0.5498	0.2673	0.5634	0.4890	0.5645	0.8162	0.5867	1.8224	0.8356
1964	0.3144	0.6223	0.0149	0.6083	72	0.5738	0.2206	0.5873	0.4208	0.5874	0.7327	0.5861	1.0089	0.5806	2.0108	0.8664
1965	0.3106	0.6209	0.0134	0.6086	88	0.5484	0.1506	0.5599	0.3455	0.5654	0.6113	0.5652	0.9134	0.5594	2.5013	0.9216
1966	0.4048	0.6558	0.0162	0.6462	108	0.5988	0.2801	0.6103	0.5756	0.6007	0.9790	0.5966	1.2070	0.5659	3.5358	0.9736
1967	0.8039	0.7862	0.0302	0.7837	137	0.7752	0.7927	0.7860	1.6741	0.7585	2.8248	0.7473	3.3967	0.6506	4.0381	0.9841
1968	1.0030	0.8382	0.0357	0.8382	160	0.8348	1.0082	0.8433	2.1474	0.8051	3.5870	0.7902	4.0394	0.6573	5.2710	0.9951
1969	1.2675	0.8928	0.0431	0.8949	186	0.9037	1.3339	0.9089	2.8961	0.8688	4.8108	0.8518	5.1935	0.6773	7.5633	0.9993
1970	1.7732	0.9570	0.0571	0.9602	217	0.9660	1.8411	0.9672	4.3286	0.9496	7.3122	0.9399	9.3155	0.7485	8.2567	0.9996
1971	2.0532	0.9757	0.0634	0.9782	244	0.9853	2.1768	0.9853	5.1810	0.9732	8.7175	0.9656	10.7198	0.7574	9.5572	0.9999
1972	2.1446	0.9800	0.0646	0.9820	265	0.9890	2.2829	0.9888	5.4891	0.9777	9.2122	0.9706	11.1202	0.7541	11.9225	1.0000
1973	3.0382	0.9976	0.0847	0.9982	298	0.9981	2.8370	0.9977	7.6592	0.9978	13.3866	0.9975	32.1127	0.9889	10.3183	0.9999
1974	3.8916	0.9998	0.1002	0.9999	330	0.9998	3.3512	0.9996	9.3546	0.9998	16.4311	0.9998	39.0679	0.9997	11.4613	1.0000
1975	3.6195	0.9995	0.0946	0.9996	341	0.9993	3.1093	0.9991	8.8813	0.9995	15.6690	0.9995	38.4252	0.9995	10.4418	0.9999
1976	3.6761	0.9996	0.0953	0.9997	359	0.9994	3.1339	0.9991	8.9566	0.9996	15.7904	0.9996	38.5207	0.9995	10.4135	0.9999
1977	3.7292	0.9996	0.0962	0.9997	376	0.9994	3.1339	0.9991	8.9566	0.9996	15.7904	0.9996	38.5207	0.9995	10.2567	0.9999
1978	3.9506	0.9998	0.1004	0.9998	398	0.9997	3.3263	0.9996	9.3473	0.9998	16.4161	0.9998	39.0363	0.9997	10.3053	0.9999
1979	4.4222	0.9999	0.1087	1.0000	422	0.9999	3.6062	0.9998	9.9066	0.9999	17.3218	0.9999	39.8227	0.9999	17.8416	1.0000
1980	4.0266	0.9998	0.1036	0.9998	429	0.9997	3.2791	0.9995	9.054	2.9997	15.9164	0.9997	38.4184	0.9996	19.6243	1.0000
1981	3.7299	0.9996	0.1000	0.9996	438	0.9991	3.0313	0.9988	8.4047	0.9994	14.8578	0.9994	37.4649	0.9994	20.8868	1.0000
1982	3.3324	0.9989	0.0941	0.9986	443	0.9963	2.6351	0.9958	7.3062	0.9975	13.0120	0.9977	35.2752	0.9980	17.7729	1.0000
1983	2.7212	0.9947	0.0826	0.9936	447	0.9860	2.1937	0.9859	6.0581	0.9905	10.8804	0.9913	32.3859	0.9934	12.5561	1.0000
1984	2.4844	0.9907	0.0791	0.9887	455	0.9704	1.8999	0.9713	5.3094	0.9823	9.6566	0.9845	31.2318	0.9914	11.1331	1.0000
1985	1.8828	0.9654	0.0646	0.9614	458	0.9122	1.3863	0.9172	3.8769	0.9421	7.1552	0.9488	27.1097	0.9731	8.1768	0.9996
1986	0.7506	0.7707	0.0282	0.7736	459	0.7620	0.7486	0.7730	1.7068	0.7628	2.9808	0.7590	6.1172	0.7361	2.7787	0.9418
1987	−0.1333	0.4474	−0.0054	0.4598	461	0.5272	0.0934	0.5372	0.0114	0.5021	−0.0637	0.4936	−0.8380	0.4540	0.8033	0.4981
1988	−0.4836	0.3160	−0.0209	0.3251	467	0.3217	−0.4519	0.3257	−0.9547	0.3246	−1.6732	0.3257	−2.5700	0.3448	0.5065	0.3192
1989	−0.1691	0.4334	−0.0080	0.4502	492	0.4682	−0.0552	0.4780	−0.3052	0.4371	−0.6146	0.4277	−1.6165	0.3752	0.5910	0.3743
1990	0.4115	0.6582	0.0222	0.6783	522	0.7544	0.7515	0.7738	0.9429	0.7122	1.5171	0.6974	1.2728	0.6296		
1991	0.3710	0.6434	0.0241	0.6723	536	0.7273	0.6790	0.7514	0.7128	0.6951	1.1493	0.6856	0.9791	0.6458		
1992	0.1119	0.5442	0.0101	0.5455	546	0.5455	0.2100	0.5832	0.1535	0.5455	0.2436	0.5455	0.1927	0.5455		

TABLE 7. Testing structural change by non-parametric tests: equation (8)

	t		Fisher		Wilcoxon		Asy Wil		Normal		Logistic		Cauchy		Chow	
Year	t_0	p-val	f_0	p-val	w_0	p-val	aw_0	p-val	ns_0	p-val	ls_0	p-val	cs_0	p-val	F_0	p-val
1961	0.9736	0.8311	0.0631	0.8182	28	0.8182	1.1552	0.8760	0.9661	0.8182	1.6094	0.8182	1.7321	0.8182	4.5203	0.9942
1962	0.3779	0.6460	0.0178	0.6572	41	0.6799	0.5281	0.7013	0.6580	0.6799	1.1147	0.6780	1.3317	0.6875	6.0978	0.9989
1963	0.2691	0.6052	0.0106	0.6212	60	0.6932	0.5636	0.7135	0.8115	0.6851	1.3584	0.6782	1.5244	0.6510	5.8350	0.9986
1964	0.0156	0.5062	0.0005	0.5216	74	0.6150	0.3310	0.6297	0.5814	0.6199	0.9906	0.6159	1.2308	0.5969	5.3262	0.9976
1965	0.1731	0.5682	0.0054	0.5805	94	0.6606	0.4519	0.6743	0.8115	0.6505	1.3584	0.6430	1.5244	0.5969	6.4682	0.9992
1966	−0.1613	0.4365	−0.0047	0.4478	103	0.5091	0.0467.	0.5186	0.1620	0.5286	0.2998	0.5300	0.5709	0.5315	6.0486	0.9989
1967	−0.0083	0.4967	−0.0002	0.5064	124	0.5768	0.2202	0.5871	0.4701	0.5778	0.7945	0.5742	0.9713	0.5464	6.0117	0.9988
1968	−0.0569	0.4775	−0.0015	0.4857	142	0.5896	0.2521	0.5995	0.5454	0.5860	0.9158	0.5813	1.0668	0.5452	6.0758	0.9989
1969	−0.3322	0.3710	−0.0084	0.3769	152	0.4763	−0.0404	0.4839	−0.0139	0.4979	0.0101	0.5009	0.2804	0.5108	5.4878	0.9180
1970	−0.0496	0.4804	−0.0012	0.4862	177	0.5988	0.2742	0.6080	0.6356	0.5934	1.0687	0.5882	1.2339	0.5433	5.5784	0.9981
1971	0.5878	0.7195	0.0140	0.7223	209	0.7912	0.8402	0.7996	2.3310	0.8021	4.1132	0.7996	8.1890	0.7186	5.5545	0.9981
1972	1.1452	0.8695	0.0263	0.8691	239	0.9001	1.3098	0.9049	3.5791	0.9008	6.2449	0.8960	11.0783	0.7536	5.8334	0.9986
1973	2.1089	0.9784	0.0455	0.9771	272	0.9689	1.8790	0.9699	5.7492	0.9810	10.4193	0.9831	32.0709	0.9888	4.9017	0.9962
1974	2.4037	0.9888	0.0504	0.9880	298	0.9836	2.1856	0.9856	6.4980	0.9903	11.6430	0.9910	33.2250	0.9921	5.2552	0.9974
1975	1.8937	0.9662	0.0406	0.9656	301	0.9498	1.6631	0.9519	5.0655	0.9623	9.1416	0.9646	29.1029	0.9695	4.6489	0.9949
1976	1.7215	0.9524	0.0371	0.9522	316	0.9410	1.5850	0.9435	4.9120	0.9569	8.8980	0.9599	28.9102	0.9675	4.2245	0.9919
1977	1.6458	0.9450	0.0356	0.9453	333	0.9410	1.5850	0.9435	4.9120	0.9569	8.8980	0.9599	28.9102	0.9675	3.9484	0.9888
1978	1.8218	0.9609	0.0392	0.9615	356	0.9634	1.8078	0.9647	5.3853	0.9711	9.6601	0.9723	29.5528	0.9729	4.0106	0.9896
1979	2.0590	0.9760	0.0441	0.9768	380	0.9808	2.0763	0.9811	5.9445	0.9832	10.5658	0.9833	30.3393	0.9790	5.1674	0.9971
1980	1.4929	0.9272	0.0333	0.9286	382	0.9362	1.5474	0.9391	4.2491	0.9345	7.5213	0.9334	23.3841	0.9106	3.4344	0.9795
1981	1.1956	0.8795	0.0274	0.8806	390	0.8862	1.2350	0.8916	3.5003	0.8957	6.2975	0.8979	22.2300	0.8982	3.1825	0.9723
1982	0.8871	0.8091	0.0210	0.8086	397	0.8020	0.8784	0.8101	2.6479	0.8329	4.8922	0.8418	20.8257	0.8831	2.7466	0.9527
1983	0.4951	0.6880	0.0121	0.6841	401	0.6428	0.3917	0.6524	1.3999	0.6989	2.7605	0.7180	17.9364	0.8499	1.9946	0.8783
1984	0.6670	0.7452	0.0168	0.7420	423	0.7178	0.6063	0.7279	1.7906	0.7546	3.3863	0.7679	18.4520	0.8637	2.1574	0.9010
1985	0.3198	0.6244	0.0084	0.6171	428	0.5410	0.1260	0.5501	0.6921	0.6087	1.5404	0.6351	16.2623	0.8481	2.0168	0.8817
1986	−0.3731	0.3558	−0.0103	0.3483	429	0.2802	−0.5725	0.2833	−1.4780	0.2680	−2.6340	0.2673	−4.7303	0.3032	0.8122	0.4723
1987	−0.7050	0.2430	−0.0205	0.2393	441	0.1988	−0.8402	0.2004	−1.8687	0.2030	−3.2597	0.2070	−5.2459	0.2649	0.8779	0.5108
1988	−0.3059	0.3808	−0.0096	0.3701	468	0.3394	−0.4017	0.3440	−1.0163	0.3141	−1.8543	0.3084	−3.8416	0.2866	1.0166	0.5849
1989	0.4818	0.6833	0.0166	0.6698	499	0.6150	0.3310	0.6297	0.4162	0.5863	0.6471	0.5768	0.2805	0.5228		
1990	1.1739	0.8753	0.0451	0.8739	528	0.8361	1.1272	0.8702	1.5147	0.8167	2.4929	0.8000	2.4702	0.7214		
1991	0.7508	0.7708	0.0352	0.7405	534	0.6799	0.5281	0.7013	0.5486	0.6534	0.8835	0.6420	0.7381	0.6117		
1992	0.3469	0.6345	0.0228	0.5152	545	0.5152	0.1050	0.5418	0.0753	0.5152	0.1214	0.5152	0.0955	0.5152		

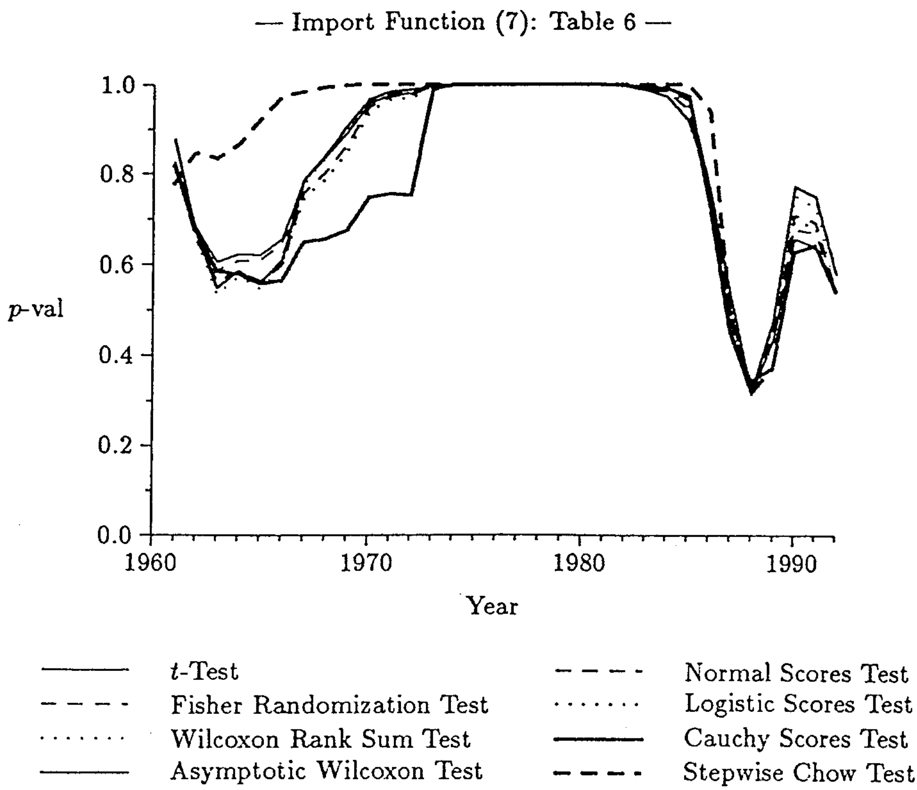

FIG. 1. p-values of non-parametric tests, using import function of equation (7) (Table 6).

logistic scores test and Cauchy scores test. Moreover, each test statistic is given by t_0, f_0, w_0, aw_0, ns_0, ls_0 and cs_0. The probability which is less than the test statistic represents p-val. The results of the stepwise Chow test, which is usually used in testing the structural change, are also shown in Tables 6 and 7. The stepwise Chow test statistic is denoted by F_0 and the p-value that corresponds to F_0 is given by p-val.

From the import function of equation (7), Table 6 and Fig. 1 indicate that the structural change occurs during the period 1973–1982 for the t-test, Fisher test, Wilcoxon test, asymptotic Wilcoxon test, normal scores test and logistic scores test, during the period 1974–1982 for the Cauchy scores test, and during the period 1968–1985 for the stepwise Chow test when the significance level is 1%. Also, when the significance level is 5%, we have the break point during the period 1971–1984 for the t-test and Fisher test, during the period 1971–1983 for the Wilcoxon test and asymptotic Wilcoxon test, during the period 1972–1984 for the normal scores test, during the period 1973–1984 for the logistic scores test and Cauchy scores test, and during the period 1967–1985 for the stepwise Chow test. It is concluded that the recursive residuals in the first period are larger than those in the last period.

From the import function of equation (8), Table 7 and Fig. 2 show that the structural change occurs during the periods 1973, 1974 and 1979 for the t-test, Fisher test, normal scores test, logistic scores test and Cauchy scores test, during the periods 1974 and 1979 for the Wilcoxon test and asymptotic Wilcoxon test, and during the period 1961–1982 for the stepwise Chow test, where the significance

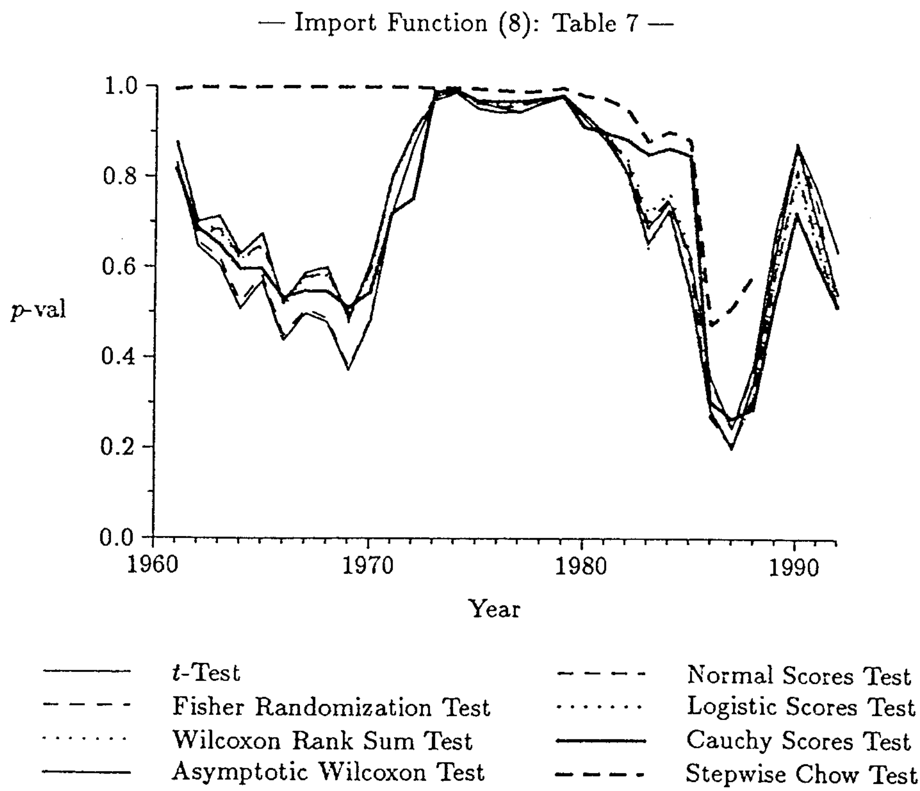

FIG. 2. p-values of non-parametric tests, using import function of equation (8) (Table 7).

level is 1%. However, except for the stepwise Chow test, we are unable to accept the fact that the structural change occurred during the period 1961–1992 when the significance level is 1%. For the stepwise Chow test the structural change is detected during the periods 1961–1976 and 1979.

Thus, according to the Chow test, which has often been used for the structural change in previous studies, it is difficult to know when the structural change takes place. However, the non-parametric tests shown in this paper gives us the break points more exactly.

5 Summarizing remarks

In previous studies, non-parametric test statistics have been approximated by a normal random variable from both computational and programming points of view. Recently, however, we have been able to perform the test exactly, thanks to progress in computers. In this paper, we have compared the sample powers in small samples, taking the scores tests (the Wilcoxon rank sum test, normal scores test, logistic scores test and Cauchy scores test) and the Fisher test.

In the case where we compare the t-test, Wilcoxon test and Fisher test, the following might be intuitively expected.

(1) When the underlying distribution is normal, the t-test gives us the most powerful test.

(2) In the situations where we cannot use the t-test, the Wilcoxon test and Fisher test are more powerful than the t-test. Moreover, the Fisher test performs better than the Wilcoxon test, because it utilizes more information than the Wilcoxon test. Therefore, in the case where the underlying distribution is non-normal or where two samples are heteroskedastic, we might have the following relationship among the sample powers: $\hat{p}_t \leqslant \hat{p}_w \leqslant \hat{p}_f$.

However, the results of the Monte Carlo simulations are as follows. The Wilcoxon test is as powerful as the t-test, even in the case where the two samples are identically and normally distributed. Moreover, when the underlying distribution is Cauchy, the Wilcoxon test performs much better than the t-test. In general, we have $\hat{p}_t \leqslant \hat{p}_f \leqslant \hat{p}_w$ in the situation where we cannot use the t-test. The fact proved by Chernoff and Savage (1958), which is the theorem that, under the alternative hypothesis of a shifting location parameter, the asymptotic relative efficiency of the normal scores test relative to the t-test is more than unity, holds even in the small-sample case. However, the normal scores test is less powerful than the Wilcoxon test. Therefore, in the small-sample case, it might be concluded from Tables 1–5 that we have $\hat{p}_t \leqslant \hat{p}_{ns} \leqslant \hat{p}_w$.

Finally, we take an example of testing structural change as an application to the non-parametric tests. According to the Chow test, it is difficult to know when the structural change takes place. However, the non-parametric tests shown in this paper gives us the break points more exactly.

Using the non-parametric tests, we can test the hypothesis, despite the functional form of the distribution of the disturbances.

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Notes

1. Note that Pitman's asymptotic relative efficiency relative to the t-test is defined as follows. Let $N_0 = n1_0 + n2_0$ be the sample size required to obtain the same power as the t-test for the sample size $N = n1 + n2$. Then, the limit of N/N_0 is called Pitman's asymptotic relative efficiency (see, for example, Kendall & Stuart, 1979), where $n1/N = n1_0/N_0$ and $n2/N = n2_0/N_0$. If we take $\mu = \mu'/N^{1/2}$, then Pitman's asymptotic relative efficiency does not depend on μ'.
2. Bradley (1968) showed that the t-test does not depend on the functional form of the underlying distribution for large $n1$, if there exists the fourth moment, which implies that the t-test is asymptotically a non-parametric test.
3. *StatXact* is a computer software package on non-parametric inference, which computes the exact probability using a non-parametric test. The network algorithm that Mehta and Patel (1983, 1986a, 1986b) and Mehta *et al.* (1984, 1985, 1988) developed is used for a permutation program.
4. For the non-parametric test in the case where we test if the functional form of the two distributions is different, we have the runs test (Kendal & Stuart, 1979).
5. Let z be a chi-square random variable with six degrees of freedom. Define $x = z/2$. Then x is distributed as $f(x)$.
6. We can interpret the Wilcoxon test as the scores test assumed to be a uniform distribution for the inverse function of $a(\cdot)$. In other words, the scores test defined as $a(x) = x/(n1 + n2 + 1)$ is equivalent to the Wilcoxon test.
7. Hoeffding (1952) showed that the Fisher test is asymptotically as powerful as t-test, even under normality assumption.

8. In Tables 1–5, note the following:
   (1) Perform m simulation runs, where $m = 10\,000$. Given the significance levels $\alpha = 0.10$, 0.05, 0.01, each value in Table 1 is a ratio of rejection numbers to m simulation runs, which represents the following probabilities: $\text{Prob}(t < -t_0) \leqslant \alpha$, $\text{Prob}(w < w_0) \leqslant \alpha$, $\text{Prob}(aw < -aw_0) \leqslant \alpha$, $\text{Prob}(ns < ns_0) \leqslant \alpha$, $\text{Prob}(ls < ls_0) \leqslant \alpha$, $\text{Prob}(cs < cs_0) \leqslant \alpha$ and $\text{Prob}(f < f_0) \leqslant \alpha$, where t_0, w_0, aw_0, ns_0, ls_0, cs_0 and f_0 are the statistics from the original data for each simulation run. For the t-distribution and the normal distribution, given the significance level, we have the following critical points (i.e. t_0 and aw_0):

α		0.10	0.05	0.01
t_0	$n1 = n2 = 5$	1.397	1.860	2.396
	$n1 = n2 = 7$	1.356	1.782	2.681
	$n1 = n2 = 9$	1.337	1.746	2.583
aw_0		1.282	1.645	2.326

   Let $\hat{p}_t$, $\hat{p}_w$, $\hat{p}_{aw}$, $\hat{p}_{ns}$, $\hat{p}_{ls}$, $\hat{p}_{cs}$ and $\hat{p}_f$ be the ratios for each test, indicating the probabilities that reject the null hypothesis under the alternative, i.e. the sample powers.
   (2) The estimated variance of each value in Table 1 is given by $\text{var}(\hat{p}_k) = [\hat{p}_k(1 - \hat{p}_k)]/m$ for $k = t$, w, aw, ns, ls, cs, f and $m = 10\,000$. Therefore, the standard errors of the estimated sample powers are at most 0.005 $(= [0.5(1 - 0.5)/10\,000]^{1/2})$, which is quite small.
   (3) $^{\circ}$, $^{\circ\circ}$, $^{\circ\circ\circ}$, *, ** and *** in Table 1 represent comparison with the t-test. We put the superscript $^{\circ}$ in the corresponding values when $(\hat{p}_k - \hat{p}_t)/[\text{Var}(\hat{p}_t)]^{1/2}$, $k = w$, aw, ns, ls, cs, f, is greater than 1; $^{\circ\circ}$ when it is greater than 2; and $^{\circ\circ\circ}$ when it is greater than 3. The superscript * is put in the corresponding values if $(\hat{p}_k - \hat{p}_t)/[\text{Var}(\hat{p}_t)]^{1/2}$ is less than -1; ** if it is less than -2; and *** if it is less than -3. Therefore, the values with $^{\circ}$ indicate a more powerful test than the t-test. The values with * represent a less powerful test than the t-test. Moreover, the number of $^{\circ}$ or * shows the level of the sample power.
9. The reason why a $\chi^2(6)/2$ random variable is subtracted by 3 is that the expectation of $\chi^2(6)/2$ is 3.
10. The reason why we use the recursive residual and not the conventional OLS residual is because the OLS residuals have the following problem. Let e_t be the residuals obtained from the regression equation $y_t = x_t\beta + u_t$, $t = 1, \ldots, T$, i.e. $e_t = y_t - x_t\hat{\beta}$. The residuals e_t, $t = 1, \ldots, T$, are not mutually independently distributed. Clearly, we have $E(e_s e_t) \neq 0$ for $s \neq t$. Therefore, we utilize the recursive residuals which are mutually independently distributed.
11. The estimation results by OLS are as follows. By equation (7), we have

$$\log M_t = \underset{(0.482\,85)}{-5.324\,81} + \underset{(0.040\,37)}{1.258\,13} \log \text{GDP}_t - \underset{(0.094\,36)}{0.110\,73} \log P_t$$

with $R^2 = 0.983\,93$, $\bar{R}^2 = 0.982\,96$, DW $= 0.4555$ and SE $= 0.105\,01$, for the estimation period 1958–1993. By equation (8), we have

$$\log M_t = \underset{(0.831\,59)}{-1.072\,50} + \underset{(0.156\,53)}{0.397\,26} \log \text{GDP}_t - \underset{(0.067\,56)}{0.157\,35} \log P_t + \underset{(0.112\,31)}{0.626\,66} \log M_{t-1}$$

with $R^2 = 0.992\,53$, $\bar{R}^2$-$0.991\,85$, DW $= 2.0564$ and SE $= 0.075\,89$, for the estimation period 1957–1993. There, the values in parentheses are the standard errors. R^2, $\bar{R}^2$, DW and SE are the coefficient of multiple determination, the adjusted R^2, the Durbin–Watson statistic, and the standard error of the disturbance respectively. For both equations (7) and (8), the recursive residuals are obtained from 1961–1993 (33 periods).

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