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Power Graphs of Finite Groups Determined by Hosoya Properties

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Abstract: Suppose $\mathcal{G}$ is a finite group. The power graph represented by $\mathcal{P}(\mathcal{G})$ of $\mathcal{G}$ is a graph, whose node set is $\mathcal{G}$, and two different elements are adjacent if and only if one is an integral power of the other. The Hosoya polynomial contains much information regarding graph invariants depending on the distance. In this article, we discuss the Hosoya characteristics (the Hosoya polynomial and its reciprocal) of the power graph related to an algebraic structure formed by the symmetries of regular molecular gones. As a consequence, we determined the Hosoya index of the power graphs of the dihedral and the generalized groups. This information is useful in determining the renowned chemical descriptors depending on the distance. The total number of matchings in a graph Γ is known as the Z-index or Hosoya index. The Z-index is a well-known type of topological index, which is popular in combinatorial chemistry and can be used to deal with a variety of chemical characteristics in molecular structures.

1. Introduction

A topological index is a numeric value that represents the symmetry of a molecular structure. Indeed, it is a mathematical classification of a chemical graph that offers a mathematical function in a quantitative structure–property relationship (QSPR). It links numerous physicochemical characteristics of molecular structured chemical substances, such as the strain energy, stability, and boiling point. Numerous characteristics of chemical compounds having a molecular structure can be examined using several kinds of topological indices. In 1947, H. Wiener presented the notion of the first topological index in researching the boiling point of paraffin, which he called the path number [1]. As a result, it was dubbed the Wiener index, and it was the moment that the idea of topological indices began.

Polya's [2] concept of computing the polynomials was used by many chemists to identify the molecular orbitals of the unsaturated hydrocarbons. In this context, the spectrum of a graph has been extensively examined. According to [3], Hosoya used this idea in 1988 to determine the polynomials of several chemical structures, which were labeled the Hosoya polynomials and attracted widespread attention. In 1996, Sagan et al. [4] then retitled the Hosoya polynomial as the Wiener polynomial, although most experts still refer to it as the Hosoya polynomial. The information regarding distance-based graph invariants may be obtained from the Hosoya polynomial. In [5], Cash noticed a connection between

the hyper Wiener index and the Hosoya polynomial. Estrada et al. [6] investigated several interesting applications of the expanded Wiener indices.

The graphs presented in this paper are all simple graphs, meaning they have no loops nor multiple edges. Suppose $\mathcal{G}$ is a finite group. The power graph represented by $\mathcal{P}(\mathcal{G})$ of $\mathcal{G}$ is a graph, in which $\mathcal{G}$ is its node set and two unlike elements are edge connected if and only if one of them is an integer power of the other. In [7], Kelarev and Quinn discussed the approach of directed power graphs related to groups and semigroups. Later, the authors of [8] illustrated the power graph $\mathcal{P}(\mathcal{S})$ of a semigroup $\mathcal{S}$ and identified the class of semigroups, whose power graphs are the complete graphs. Furthermore, they discussed that for any finite group $\mathcal{G}$, the associated power graph is the complete graph if and only if the group $\mathcal{G}$ is cyclic of order one or p^k, where p is any prime and $k \in \mathbb{N}$.

In the current literature of theory of graphs, the power graph is now an exciting topic in several branches of mathematics, that is group theory, ring theory, and Lie algebra. Cameron et al. [9] discussed the matching numbers and gave the upper, as well as the lower bounds for the perfect matching of power graphs of certain finite groups. They also derived a formula of matching numbers for any finite nilpotent groups. The authors of [10–13] presented an overview of finite groups with enhanced power graphs that enable the formation of a perfect code. They further established all possible perfect codes of the proper reduced power graphs and gave a necessary and sufficient condition for graphs having perfect codes. In [14], the authors concentrated on the power indices graph and classified all such graphs in some specified categories.

The authors of [15] examined the maximum clique and found the largest number of edges of power graphs for all the finite cyclic groups. Sriparna et al. [16] deliberated about the node connectivity of $\mathcal{P}(\mathbb{Z}_n)$, whenever n is the product of some prime numbers. Furthermore, several other researchers inquired about different concepts of algebraic graphs; for instance, see [17–20] and the references therein.

A matching or an independent edge set is the collection of edges that share no nodes. When a node is coincident with one of the matching edges, it is referred to as matched. Otherwise, there is an unmatched node. The maximum number of matchings in a graph is referred to as the Z-index or Hosoya index. Hosoya [21] first proposed the Z-index in 1971 and then extended the topological index as a common tool for quantum chemistry in [22]. It has since been shown to be useful in a variety of molecular chemistry problems, including the heat of vaporization, entropy, and the boiling point. The Z-index is a well-known topological index example that has significant relevance in combinatorial chemistry. Considering numerous graph structures, many researchers investigated extremal problems in regard to the Z-index. Extremal characteristics of different graphs, unicyclic graphs, and trees were extensively studied in [23–26].

In this article, we represent the cyclic group $\mathbb{Z}_n$ of order n, the generalized quaternion group Q_{4n}, and the dihedral group D_{2m} of order $4n$ and $2m$, respectively. It is very challenging to calculate the (reciprocal) Hosoya polynomial, as well as the Z-index of power graph $\mathcal{P}(\mathcal{G})$ of a group $\mathcal{G}$. In this regard, we provide both the Hosoya and the reciprocal Hosoya polynomials and also discuss the Z-index of the power graph $\mathcal{P}(\mathcal{G})$ of a group $\mathcal{G}$, when $\mathcal{G}$ is D_{2m} or Q_{4n}.

There are still several gaps in the current study about the determination of the Hosoya polynomials, the reciprocal Hosoya polynomials, and also the Z-index or Hosoya index of the power graphs of a finite cyclic group $\mathbb{Z}_n$, the dihedral group D_{2m}, and the generalized quaternion group Q_{4n}. We look at one of these problems in this article.

2. Basic Notions and Notations

This part reviews several fundamental graph-theoretic properties and well-known findings that will be important later in the article.

Suppose Γ is a simple finite undirected graph. The node and edge sets of Γ are represented by $V(\Gamma)$ and $E(\Gamma)$, respectively. The distance from u_1 to u_2 in a connected graph Γ denoted by $dis(u_1, u_2)$ is defined as the shortest distance between u_1 and u_2. The

total number of nodes, denoted by $|\Gamma|$, is said to be the order of Γ. Two nodes v_1 and v_2 are adjacent if there is an edge between them, and we denote them by $v_1 \sim v_2$, otherwise $v_1 \nsim v_2$. The valency or degree represented by $deg(u_1)$ of a node u_1 is the collection of nodes in Γ, which are adjacent to u_1. A $u_1 - u_2$ path having $dis(u_1, u_2)$ length is known as a $u_1 - u_2$ geodesic. The largest distance between a node u_1 and any other node of Γ is known as the eccentricity and is denoted by $ec(u_1)$. The diameter denoted by $diam(\Gamma)$ of Γ is the largest eccentricity among all the nodes of the graph Γ. Furthermore, the radius symbolized by $rad(\Gamma)$ of Γ is the lowest eccentricity among all the nodes of the graph Γ.

Suppose Γ is a graph of order n. According to Hosoya, the polynomial of Γ with a variable y is defined as follows:

$$\mathcal{H}(\Gamma, y) = \sum_{i \geq 0} dis(\Gamma, i) y^i.$$

The coefficient $dis(\Gamma, i)$ represents the number of pairs of nodes (v, w) so that $dis(v, w) = i$, where $i \leq diam(\Gamma)$. Ramane and Talwar [27] proposed the reciprocal status Hosoya polynomial of Γ, which is given as:

$$\mathcal{H}_{rs}(\Gamma, y) = \sum_{vw \in E(\Gamma)} y^{rs(v)+rs(w)},$$

where $rs(w) = \sum_{v \in V(\Gamma), w \neq v} \frac{1}{dis(w,v)}$ is referred to as the transmission or the reciprocal status of a node w.

Suppose Γ_1 and Γ_2 are two connected graphs, then $\Gamma_1 \vee \Gamma_2$ is the join of Γ_1 and Γ_2 whose node and edge sets are $V(\Gamma_1) \cup V(\Gamma_2)$ and $E(\Gamma_1) \cup E(\Gamma_2) \cup \{y \sim z : y \in V(\Gamma_1), z \in V(\Gamma_2)\}$, respectively. A complete graph is a graph that has an edge between any single node in the graph, and it is symbolized by K_n. Other unexplained terminologies and notations were taken from [28].

Definition 1. *Assume that $\mathcal{G}$ is a group. Then, the center of $\mathcal{G}$ is given as:*

$$Z(\mathcal{G}) = \{g_1 : g_1 \in \mathcal{G} \text{ and } g_1 g_2 = g_2 g_1, \text{ for all } g_2 \in \mathcal{G}\}.$$

The dihedral group D_{2m} is the group of symmetries, and its order is $2m$, where $m \geq 3$. The presentation of a dihedral group is given by:

$$D_{2m} = \langle a, b : a^m = b^2 = 1, bab = a^{m-1} \rangle.$$

Throughout this paper, we mean $m = p^\alpha$, where $\alpha \in \mathbb{N}$ and p is any odd prime number. We now split D_{2m} as follows:

$$H_1 = \langle a \rangle, H_2 = \bigcup_{\iota=0}^{\frac{m}{2}-1} H_2^\iota = \{a^\iota b, a^{\frac{m}{2}+\iota}b\} = \{b, ab, a^2b, \ldots, a^{m-1}b\},$$

where $0 \leq \iota \leq \frac{m}{2} - 1$ and $H_3 = \langle a \rangle \setminus \{e\}$. Since $|a| = m$ and $|a^i b| = 2$, for all $1 \leq i \leq m$, where the identity e is connected to every other node in its power graph, the subgraph induced by H_1 is a complete graph K_m.

Furthermore, the presentation of generalized quaternion group Q_{4n} of order $4n$ for $n = 2^k$, where $k \in \mathbb{N}$, is given as:

$$Q_{4n} = \langle y, z \mid y^{2n} = 1, y^n = z^2, zyz^{-1} = y^{-1} \rangle.$$

We now split Q_{4n} as follows:

$$\Omega = \{e, y^n\}, A_1 = \langle y \rangle, A_2 = \bigcup_{i=0}^{n-1} A_2^i, \text{ where } A_2^i = \left\{ y^i z, y^{n+i} z \right\}$$

and $A_3 = A_1 \setminus \Omega$. Since A_1 is cyclic, its induced subgraph is complete, and it is denoted by K_{2n}. A remarkable feature of Q_{4n} is that the involution y^n and the identity e are adjacent to every other node in their power graph. Several researchers [29–32] have analyzed the complete description of the above-mentioned groups and their power graphs. The first survey paper on power graphs was published in 2013 [33], and the most recent study was [34].

Proposition 1 ([31]). *The power graph $\mathcal{P}(D_{2m})$ of D_{2m} satisfies:*

$$\mathcal{P}(D_{2m}) = K_1 \vee \left(K_{m-1} \cup \overline{K}_m \right).$$

Proposition 2 ([31]). *The power graph $\mathcal{P}(Q_{4n})$ of Q_{4n} satisfies:*

$$\mathcal{P}(Q_{4n}) = K_2 \vee (K_{2n-2} \cup nK_2),$$

where nK_2 represents the n copies of K_2.

3. Hosoya Properties

The Hosoya polynomial and its reciprocal status of the power graphs of the dihedral and generalized quaternion groups are determined in this section.

3.1. Main Results

Theorem 1. *Let $\mathcal{P}(D_{2m})$ be the power graph of D_{2m}. Then:*

$$\mathcal{H}_{rs}(\mathcal{P}(D_{2m})) = (m-1)y^{\frac{7m-4}{2}} + my^{3m-1} + \binom{m-1}{2}y^{3m-2}.$$

Theorem 2. *Suppose $\mathcal{P}(Q_{4n})$ is a power graph of Q_{4n}. Then:*

$$\mathcal{H}_{rs}(\mathcal{P}(Q_{4n})) = \binom{2(n-1)}{2}y^{2(3n-1)} + 4(n-1)y^{7n-2} + y^{2(4n-1)} + 4ny^{6n} + ny^{2(2n+1)}.$$

Theorem 3. *The Hosoya index of $\mathcal{P}(D_{2m})$ is as follows:*

$$1 + \binom{m}{2} + m + \sum_{i=2}^{\frac{m}{2}}\left[\left(\frac{1}{i}\right)\prod_{k=0}^{i-1}\binom{m-2k}{2} + \left(\frac{m}{i-1}\right)\prod_{k=0}^{i-2}\binom{m-2k-1}{2} \right].$$

Theorem 4. *For $n \geq 2$, the Hosoya index of $\mathcal{P}(Q_{4n})$ is given as:*

$$1 + \sum_{i=1}^{n} d_i^1 + \sum_{i=1}^{2} d_i^2 + \sum_{i=1}^{n} d_i^3 + \sum_{i=2}^{n+1} d_i^4 + d_2^5 + \sum_{i=2}^{n} d_i^6 + \sum_{i=2}^{2n} d_i^7,$$

where:

$$d_i^1 = \frac{1}{i}\prod_{k=0}^{i-1}\binom{2(n-k)}{2}, \quad d_1^2 = 4n, \quad d_2^2 = 4n\left(n - \frac{1}{2}\right),$$

$$d_i^3 = \binom{n}{i}, \quad d_2^4 = 4n\binom{2(n-1)}{2},$$

$$d_i^4 = 2n\left\{ \frac{2}{i-1}\prod_{k=0}^{i-2}\binom{2(n-k-1)}{2} + \frac{2n-1}{i-2}\prod_{k=0}^{i-3}\binom{2(n-k-1)}{2} \right\},$$

where $3 \leq i \leq n$,

$$d_{n+1}^4 = \frac{4n(n-\frac{1}{2})}{i-2} \prod_{k=0}^{i-3} \binom{2(n-k-1)}{2}, d_2^5 = 8n(n-\frac{1}{2}), d_2^6 = 4n(n-1),$$

$$d_i^6 = 2n\left\{2\binom{n-1}{i-1} + \binom{n-1}{i-2} + 4(n-1)\binom{n-2}{i-2}\right\}, \text{ where } 3 \leq i \leq n,$$

$$d_i^7 = \sum_{j=1}^{i-1} \frac{1}{j} \prod_{k=0}^{j-1} \binom{2(n-k)}{2}\binom{n}{i-j}, \text{ where } 2 \leq i \leq 2n.$$

3.2. Hosoya Polynomial

To find the Hosoya polynomial, we first prove some important results.

Proposition 3. *Suppose* $\mathcal{P}(D_{2m})$ *is the power graph of* D_{2m}. *Then:*

$$dis(\mathcal{P}(D_{2m}), \ell) = \begin{cases} 2m, & \text{for } \ell = 0; \\ \frac{m(m+1)}{2}, & \text{for } \ell = 1; \\ \frac{3m(m-1)}{2}, & \text{for } \ell = 2. \end{cases}$$

Proof. As we know that $diam(\mathcal{P}(D_{2m})) = 2$, we need to determine $dis(\mathcal{P}(D_{2m}), 0)$, $dis(\mathcal{P}(D_{2m}), 1)$, and $dis(\mathcal{P}(D_{2m}), 2)$. Now, take a node set V_k for any pair of nodes of $\mathcal{P}(D_{2m})$, then:

$$|V_k| = \binom{|\mathcal{P}(D_{2m})|}{2} + |\mathcal{P}(D_{2m})| = m(2m+1).$$

Suppose:

$$C(\mathcal{P}(D_{2m}), \ell) = \{(i,j); i,j \in V(\mathcal{P}(D_{2m})) \mid dis(i,j) = \ell\},$$

and $dis(\mathcal{P}(D_{2m}), \ell) = |C(\mathcal{P}(D_{2m}), \ell)|$. Then:

$$V_k = C(\mathcal{P}(D_{2m}), 0) \cup C(\mathcal{P}(D_{2m}), 1) \cup C(\mathcal{P}(D_{2m}), 2). \tag{1}$$

As we know, $dis(i,i) = 0, \forall i \in V(\mathcal{P}(D_{2m}))$, so:

$$C(\mathcal{P}(D_{2m}), 0) = \{(i,i); i \in V(\mathcal{P}(D_{2m}))\},$$

and this is actually $V(\mathcal{P}(D_{2m}))$. Thus, $C(\mathcal{P}(D_{2m}), 0) = 2m$. Using Proposition 1, $\mathcal{P}(D_{2m}) = K_1 \vee (K_{m-2} \cup \overline{K_m})$ with $V(K_1), V(K_{m-1}) = H_3$, and $V(K_m) = H_2$. Therefore,

$$C(\mathcal{P}(D_{2m}), 1) = \{(i,j); i = e, j \in H_2\} \cup \{(i,j); i = e, j \in H_3\}$$
$$\cup \{(i,j); i,j \in H_3 \text{ and } i \neq j\}.$$

Consequently, $C(\mathcal{P}(D_{2m}), 1) = m + (m-1) + \binom{m-1}{2} = \frac{m(m+1)}{2}$. Using Equation (1), we obtain:

$$|V_k| = dis(\mathcal{P}(D_{2m}), 0) + dis(\mathcal{P}(D_{2m}), 1) + dis(\mathcal{P}(D_{2m}), 2).$$

Hence,

$$dis(\mathcal{P}(D_{2m}), 2) = |V_k| - dis(\mathcal{P}(D_{2m}), 0) - dis(\mathcal{P}(D_{2m}), 1)$$
$$= m(2m+1) - 2m - \frac{m(m+1)}{2}$$
$$= \frac{3m(m-1)}{2}.$$

□

Proposition 4. *Suppose $\mathcal{P}(Q_{4n})$ is the power graph of Q_{4n}. Then:*

$$dis(\mathcal{P}(Q_{4n}), \ell) = \begin{cases} 4n, & \text{for } \ell = 0; \\ 2n(n+2), & \text{for } \ell = 1; \\ 6n(n-1), & \text{for } \ell = 2. \end{cases}$$

Proof. As we know that $diam(\mathcal{P}(Q_{4n})) = 2$, we need to determine $dis(\mathcal{P}(Q_{4n}), 0)$, $dis(\mathcal{P}(Q_{4n}), 1)$, and $dis(\mathcal{P}(Q_{4n}), 2)$. Suppose V_k is the collection of all pairs of nodes of $\mathcal{P}(Q_{4n})$, then:

$$|V_k| = 2n(4n+1).$$

Let

$$C(\mathcal{P}(Q_{4n}), \ell) = \{(i, j); i, j \in V(\mathcal{P}(Q_{4n})) | dis(i, j) = \ell\},$$

then $dis(\mathcal{P}(Q_{4n}), \ell) = |C(\mathcal{P}(Q_{4n}), \ell)|$, and:

$$V_k = C(\mathcal{P}(Q_{4n}), 0) \cup C(\mathcal{P}(Q_{4n}), 1) \cup C(\mathcal{P}(Q_{4n}), 2). \tag{2}$$

Since, $dis(i, i) = 0$, for any $i \in V(\mathcal{P}(Q_{4n}))$, so:

$$C(\mathcal{P}(Q_{4n}), 0) = \{(i, i); i \in V(\mathcal{P}(Q_{4n}))\}$$

and this is equal to $V(\mathcal{P}(Q_{4n}))$. Thus, $C(\mathcal{P}(Q_{4n}), 0) = 4n$. Using Proposition 2, where $V(K_2) = \{e, y^n\}$, $V(K_{2n-2}) = A_3$, and $V(nK_n) = A_2 = \bigcup_{j=0}^{n-1} A_2^j$, therefore, we have:

$$C(\mathcal{P}(Q_{4n}), 1) = \{(i, j); i \in \Omega, j \in A_2\} \cup \bigcup_{j=0}^{n-1} \left\{(i, j); (i, j) \in A_2^j \text{ and } j \neq i\right\}$$

$$\cup \{(i, j); i \in \Omega, j \in A_3\} \cup \{(i, j); (i, j) \in A_3 \text{ and } j \neq i\}$$

$$\cup \{(i, j); (i, j) \in \Omega \text{ and } j \neq i\}.$$

Consequently, $dis(\mathcal{P}(Q_{4n}), 1) = 4n + n(1) + 2(2n-2) + \binom{2n-2}{2} + 1 = 2n(n+2)$. Using Equation (2), we obtain:

$$|V_k| = dis(\mathcal{P}(Q_{4n}), 0) + dis(\mathcal{P}(Q_{4n}), 1) + dis(\mathcal{P}(Q_{4n}), 2).$$

Hence,

$$\begin{aligned} dis(\mathcal{P}(Q_{4n}), 2) &= |V_k| - dis(\mathcal{P}(Q_{4n}), 0) - dis(\mathcal{P}(Q_{4n}), 1) \\ &= 2n(4n+1) - 4n - 2n(n+2) \\ &= 6n(n-1). \end{aligned}$$

$\square$

The following results yield the Hosoya polynomials of the power graphs of the dihedral and the generalized quaternion groups.

Theorem 5. *Consider the power graph $\mathcal{P}(D_{2m})$ of D_{2m}. Then:*

$$\mathcal{H}(\mathcal{P}(D_{2m}), y) = \frac{m}{2}\left(3(m-1)y^2 + (m+1)y + 4\right).$$

Proof. By substituting the coefficients $dis(\mathcal{P}(D_{2m}), \ell)$ derived in Propositions 3 and 4 into the formula for the Hosoya polynomial, we obtain:

$$\mathcal{H}(\mathcal{P}(D_{2m}), y) = dis(\mathcal{P}(D_{2m}), 2)y^2 + dis(\mathcal{P}(D_{2m}), 1)y^1 + dis(\mathcal{P}(D_{2m}), 0)y^0$$
$$= \left(\frac{3m(m-1)}{2}\right)y^2 + \left(\frac{m(m+1)}{2}\right)y + 2my^0$$
$$= \frac{m}{2}\left(3(m-1)y^2 + (m+1)y + 4\right).$$

We obtain the desired polynomial. $\square$

Theorem 6. *Consider the power graph $\mathcal{P}(Q_{4n})$ of Q_{4n}. Then:*

$$H(\mathcal{P}(Q_{4n}), y) = n\left(6(n-1)y^2 + 2(n+2)y + 4\right).$$

Proof. By substituting the coefficients $dis(\mathcal{P}(Q_{4n}), \ell)$ derived in Propositions 3 and 4 into the formula for the Hosoya polynomial, we obtain:

$$H(\mathcal{P}(Q_{4n}), y) = dis(\mathcal{P}(Q_{4n}), 2)y^2 + dis(\mathcal{P}(Q_{4n}), 1)y^1 + dis(\mathcal{P}(Q_{4n}), 0)y^0$$
$$= (6n(n-1))y^2 + (2n(n+2))y + (4n)y^0$$
$$= n\left(6(n-1)y^2 + 2(n+2)y + 4\right).$$

This proves the statement. $\square$

4. Reciprocal Status Hosoya Polynomial

This part determines the reciprocal status of every node in the power graphs.

Proposition 5. *If v is a node of $\mathcal{P}(D_{2m})$, then:*

$$rs(v) = \begin{cases} 2m-1, & \text{when } v = e; \\ m, & \text{when } v \in H_2; \\ \frac{3m-2}{2}, & \text{when } v \in H_3. \end{cases}$$

Proof. Using Proposition 1, $\mathcal{P}(D_{2m}) = K_1 \vee \left(K_{m-1} \cup \overline{K}_m\right)$ with node set $\{e\} \cup H_2 \cup H_3$. Therefore, if $v = e$, then $ec(v) = 1$, and following the concept of the reciprocal status, we have:

$$rs(v) = \left(\frac{1}{1}\right)(m + m - 1) = 2m - 1.$$

When $v \in H_2$, implying $ec(v) = 2$, also, we apply the concept of the reciprocal status, so we have:

$$rs(v) = \left(\frac{1}{1}\right)(1) + \frac{1}{2}(2m - 2) = m.$$

When $v \in H_3$, implying $ec(v) = 2$, also, we apply the concept of the reciprocal status, so we have:

$$rs(v) = \left(\frac{1}{1}\right)(m-1) + \frac{m}{2} = \frac{3m-2}{2}.$$

Combining them, we obtain the required quantity. $\square$

Proposition 6. *If v is a node of $\mathcal{P}(Q_{4n})$, then:*

$$rs(v) = \begin{cases} 4n-1, & \text{when } v \in \Omega; \\ 2n+1, & \text{when } v \in A_2; \\ 3n-1, & \text{when } v \in A_3. \end{cases}$$

Proof. Using Proposition 2, the node set of $\mathcal{P}(Q_{4n})$ is $\Omega \cup A_2 \cup A_3$. Therefore, when $v \in \Omega$, implying $ec(v) = 1$, also, we apply the concept of the reciprocal status, so we have:

$$rs(v) = \left(\frac{1}{1}\right)\{1 + 2n - 2 + 2n\} = 4n - 1.$$

When $v \in A_2$, implying $ec(v) = 2$, also, we apply the concept of the reciprocal status, so we have:

$$rs(v) = 3\left(\frac{1}{1}\right) + \left(\frac{1}{2}\right)\left\{2(n-1) + \left(\frac{4n}{2} - 2\right)\right\} = 2n + 1.$$

When $v \in A_3$, implying $ec(v) = 2$, and by the definition of the reciprocal status, we have:

$$rs(v) = \left(\frac{1}{1}\right)(2n - 3 + 2) + \left(\frac{1}{2}\right)(2n) = 3n - 1.$$

Combining them, we obtain the required quantity. $\square$

Proof of Theorems 1 and 2

The following results compute the reciprocal status Hosoya polynomial of the power graph of a group $\mathcal{G}$.

Proof of Theorem 1. Using Proposition 5, this means that there are three types of edges $(u \sim v, u \sim w, v \sim v)$ in $\mathcal{P}(D_{2m})$, according to their end nodes' reciprocal statuses, where $u = 2m - 1, v = \frac{3m-2}{2}$, and $c = m$. The edge partition is shown in the following Table 1.

By incorporating the edge set's partition specified in Table 1 into the formula for the reciprocal status Hosoya polynomial, we obtain the following.

$$\mathcal{H}_{rs}(\mathcal{P}(D_{2m})) = \sum_{E_{u \sim v}} y^{u+v} + \sum_{E_{u \sim w}} y^{u+w} + \sum_{E_{v \sim v}} y^{v+v}$$

$$= (m-1)y^{2m-1+\frac{3m-2}{2}} + my^{2m-1+m} + \binom{m-1}{2}y^{\frac{3m-2}{2}+\frac{3m-2}{2}}$$

$$= (m-1)y^{\frac{7m-4}{2}} + my^{3m-1} + \binom{m-1}{2}y^{3m-2}.$$

$\square$

Table 1. $\mathcal{P}(D_{2m})$ is partitioned into edges based on their reciprocal statuses.

Type of Edge	Edge Set's Partition	Edges Count
$u \sim v$	$E_{u \sim v} = \{ab \in E(\mathcal{P}(D_{2m})) : rs(a) = u, rs(b) = v\}$	$\|E_{u \sim v}\| = m - 1$
$u \sim w$	$E_{u \sim w} = \{ab \in E(\mathcal{P}(D_{2m})) : rs(a) = u, rs(b) = w\}$	$\|E_{u \sim w}\| = m$
$v \sim v$	$E_{v \sim v} = \{ab \in E(\mathcal{P}(D_{2m})) : rs(a) = v, rs(b) = v\}$	$\|E_{v \sim v}\| = \binom{m-1}{2}$

Proof of Theorem 2. From Proposition 6, there are five types of edges $(u \sim u, u \sim v, v \sim v, v \sim w, w \sim w)$ in $\mathcal{P}(Q_{4n})$. Consequently, the edge partitioning is shown in Table 2 along with the reciprocal status of their end nodes, when $u = \frac{6n}{2} - 1, v = 4n - 1, w = 2n + 1$.

We obtain the reciprocal status Hosoya polynomial formula by substituting the edge partition of $\mathcal{P}(Q_{4n})$, which is provided in Table 2.

$$\mathcal{H}_{rs}(\mathcal{P}(Q_{4n})) = \sum_{E_{u \sim u}} y^{u+u} + \sum_{E_{u \sim v}} y^{u+v} + \sum_{E_{v \sim v}} y^{v+v} + \sum_{E_{v \sim w}} y^{v+w} + \sum_{E_{w \sim w}} y^{w+w}$$

$$= \binom{2(n-1)}{2}y^{2(3n-1)} + 4(n-1)y^{(3n-1)+(4n-1)} + (1)y^{2(4n-1)}$$

$$+ (4n)y^{(4n-1)+(2n+1)} + (n)y^{2(2n+1)}$$

$$= \binom{2(n-1)}{2}y^{2(3n-1)} + 4(n-1)y^{7n-2} + y^{2(4n-1)} + (4n)y^{6n} + (n)y^{2(2n+1)}.$$

$\square$

Table 2. $\mathcal{P}(Q_{4n})$ is partitioned into edges based on their reciprocal statuses.

Type of Edge	Edge Set's Partition	Edges Count
$u \sim u$	$E_{u \sim u} = \{ab \in E(\mathcal{P}(D_{2m})) : rs(a) = u, rs(b) = u\}$	$|E_{u \sim u}| = \binom{2(n-1)}{2}$
$u \sim v$	$E_{u \sim v} = \{ab \in E(\mathcal{P}(D_{2m})) : rs(a) = u, rs(b) = v\}$	$|E_{u \sim v}| = 4(n-1)$
$v \sim v$	$E_{v \sim v} = \{ab \in E(\mathcal{P}(D_{2m})) : rs(a) = v, rs(b) = v\}$	$|E_{v \sim v}| = 1$
$v \sim w$	$E_{v \sim w} = \{ab \in E(\mathcal{P}(D_{2m})) : rs(a) = v, rs(b) = w\}$	$|E_{v \sim w}| = 4n$
$w \sim w$	$E_{w \sim w} = \{ab \in E(\mathcal{P}(D_{2m})) : rs(a) = w, rs(b) = w\}$	$|E_{w \sim w}| = n$

5. Hosoya Index

The Hosoya index of the power graphs of finite groups is examined in this section. On a graph with n nodes, the greatest feasible value of the Hosoya index is provided by the complete graph K_n [35]. In general, the Hosoya index of K_n, where $n \geq 2$, is as follows:

$$1 + \sum_{i=1}^{\frac{n}{2}} \left(\frac{1}{i}\right) \prod_{k=0}^{i-1} \binom{n - 2k}{2},$$

this may be seen concerning the entire non-void matchings specified in Table 3, whereas d_i represents the cardinality i matchings, where $1 \leq i \leq \frac{n}{2}$.

Table 3. The total non-void matchings in K_n.

K_m	d_1	d_2	d_3	d_4	$\cdots$	d_i
K_2	$\binom{2}{2}$					
K_3	$\binom{3}{2}$					
K_4	$\binom{4}{2}$	$\frac{1}{2}\binom{4}{2}\binom{2}{2}$				
K_5	$\binom{5}{2}$	$\frac{1}{2}\binom{5}{2}\binom{3}{2}$				
K_6	$\binom{6}{2}$	$\frac{1}{2}\binom{6}{2}\binom{4}{2}$	$\frac{1}{3}\binom{6}{2}\binom{4}{2}\binom{2}{2}$			
K_7	$\binom{7}{2}$	$\frac{1}{2}\binom{7}{2}\binom{5}{2}$	$\frac{1}{3}\binom{7}{2}\binom{5}{2}\binom{3}{2}$			
K_8	$\binom{8}{2}$	$\frac{1}{2}\binom{8}{2}\binom{6}{2}$	$\frac{1}{3}\binom{8}{2}\binom{6}{2}\binom{4}{2}$	$\frac{1}{4}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
K_m	$\binom{m}{2}$	$\frac{1}{2}\binom{m}{2}\binom{m-2}{2}$	$\frac{1}{3}\binom{m}{2}\binom{m-2}{2}\binom{m-4}{2}$	$\frac{1}{4}\binom{m}{2}\binom{m-2}{2}\binom{m-4}{2}\binom{m-6}{2}$	$\cdots$	$\frac{1}{i}\prod_{k=0}^{i-1}\binom{m-2k}{2}$

Proof of Theorem 3. From the structure of $\mathcal{P}(D_{2m})$, the identity e is the only node that is connected to every other node in $V(\mathcal{P}(D_{2m}))$. Therefore, there are three types of edges in $\mathcal{P}(D_{2m})$, i.e.,:

Type 1: $u_1 \sim u_2$, for $u_1, u_2 \in H_3$;

Type 2: $u_1 \sim u_2$, for $u_1 \in H_3, u_2 = e$;

Type 3: $u_1 \sim u_2$, for $u_1 \in H_2, u_2 = e$.

Since we know that the subgraph induced by H_1 is complete, i.e., K_m, thus, $\mathcal{P}(D_{2m})$ has two distinct types of matchings:

$\quad T_1 \quad$ Matchings of Type 1 and Type 2 edges;
$\quad T_2 \quad$ Matchings of Type 1 and Type 3 edges:

T_1: For each type, the number of matchings may be calculated as follows: Due to the fact that the edges of Type 1 and Type 2 are the edges of a complete graph K_m, which is induced by the nodes in H_1, so the number of matchings in this type can be obtained by counting the matchings in K_m, which are given in Table 4, where d_i denotes the number of matchings of order i, for $1 \leq i \leq \frac{m}{2}$;

T_2: Every matching of this kind may be generated by substituting one edge of Type 3 for any edge of Type 1. Given that every Type-1 edge is also an edge of K_{m-1}, which is induced by the nodes of H_3, so every matching of Type-1 edges is also a matching of K_{m-1}. The total number of certain matchings is described in Table 5, where for any $1 \leq i \leq \frac{m}{2}$, d_i signifies the total number of matchings of order i.

Given that Type 3 has m edges, the essential matchings may be produced as:

- Matchings having one order: These are the m such matches that correspond to the m Type-3 edges;
- Matchings having two orders: All of these matchings may be achieved by inserting a Type-3 edge through every other matching having one order in K_{m-1}. Using Table 5, there are m edges of Type 3, as well as $\binom{m-1}{2}$ matchings having one order in K_{m-1}. As a result of the product rule, the number of matchings having two order is: $m\binom{m-1}{2}$;
- Matchings having three orders: Every one of these matchings may be achieved by inserting a Type-3 edge into every other matching of order two in K_{m-1}. Thus, there are m Type-3 edges, as well as $\frac{1}{2}\binom{m-1}{2}\binom{m-3}{2}$ matchings having two order in K_{m-1}, by Table 5. As a result of the product rule, the number of order three matchings is:

$$\frac{m}{2}\binom{m-1}{2}\binom{m-3}{2};$$

- Order four matchings: Any of these matchings may be generated by inserting one edge of Type 3 through every order three matching in K_{m-1}. Thus, there are m Type-3 edges, as well as $\frac{1}{3}\binom{m-1}{2}\binom{m-3}{2}\binom{m-5}{2}$ matchings having three orders in K_{m-1}, by Table 5. As a consequence of the product rule, the number of matchings of order four is given by:

$$\frac{m}{3}\binom{m-1}{2}\binom{m-3}{2}\binom{m-5}{2};$$

- Order i matchings: In general, every order i matching may be generated by adding one edge of Type 3 to every order $i-1$ matching in K_{m-1}. According to Table 5, there are m possible edges of Type 3 and $\frac{1}{i-1}\prod_{k=0}^{i-2}\binom{m-2k-1}{2}$ matchings of order $i-1$ in K_{m-1}. Therefore, by the product rule, the number of matchings of order i is:

$$\frac{m}{i-1}\prod_{k=0}^{i-2}\binom{m-2k-1}{2}.$$

Next, using the sum rule, the entire matchings in each order (matchings (T_1) + matchings (T_2)) may be calculated as: The number of matchings of order one is as follows:

$$m+\binom{m}{2}.$$

Order two has the sequel number of matchings:

$$m\binom{m-1}{2}+\frac{1}{2}\binom{m}{2}\binom{m-2}{2}.$$

Order three has the following number of matchings:

$$\frac{m}{2}\binom{m-1}{2}\binom{m-3}{2}+\frac{1}{3}\binom{m}{2}\binom{m-2}{2}\binom{m-4}{2}.$$

Order four has the following number of matchings:

$$\frac{m}{3}\binom{m-1}{2}\binom{m-3}{2}\binom{m-5}{2}+\frac{1}{4}\binom{m}{2}\binom{m-2}{2}\binom{m-4}{2}\binom{m-6}{2}.$$

Generally, order i has the following number of matchings:

$$\frac{1}{i} \prod_{k=0}^{i-1} \binom{m-2k}{2} + \frac{m}{i-1} \prod_{k=0}^{i-2} \binom{m-2k-1}{2},$$

where $2 \leq i \leq \frac{m}{2}$.

Thus, the Hosoya index of $\mathcal{P}(D_{2m})$ is given by:

$$1 + m + \binom{m}{2} + \sum_{i=2}^{\frac{m}{2}} \left\{ \frac{1}{i} \prod_{k=0}^{i-1} \binom{m-2k}{2} + \frac{m}{i-1} \prod_{k=0}^{i-2} \binom{m-2k-1}{2} \right\}.$$

$\square$

Table 4. The total non-void matchings in K_m.

K_m	d_1	d_2	d_3	d_4	$\cdots$	d_i
K_3	$\binom{3}{2}$					
K_5	$\binom{5}{2}$	$\frac{1}{2}\binom{5}{2}\binom{3}{2}$				
K_7	$\binom{7}{2}$	$\frac{1}{2}\binom{7}{2}\binom{5}{2}$	$\frac{1}{3}\binom{7}{2}\binom{5}{2}\binom{3}{2}$			
K_9	$\binom{9}{2}$	$\frac{1}{2}\binom{9}{2}\binom{7}{2}$	$\frac{1}{3}\binom{9}{2}\binom{7}{2}\binom{5}{2}$	$\frac{1}{4}\binom{9}{2}\binom{7}{2}\binom{5}{2}\binom{3}{2}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
K_m	$\binom{m}{2}$	$\frac{1}{2}\binom{m}{2}\binom{m-2}{2}$	$\frac{1}{3}\binom{m}{2}\binom{m-2}{2}\binom{m-4}{2}$	$\frac{1}{4}\binom{m}{2}\binom{m-2}{2}\binom{m-4}{2}\binom{m-6}{2}$	$\cdots$	$\frac{1}{i}\prod_{k=0}^{i-1}\binom{m-2k}{2}$

Table 5. The total non-void matchings in K_m.

K_m	d_1	d_2	d_3	d_4	$\cdots$	d_i
K_2	$\binom{2}{2}$					
K_4	$\binom{4}{2}$	$\frac{1}{2}\binom{4}{2}\binom{2}{2}$				
K_6	$\binom{6}{2}$	$\frac{1}{2}\binom{6}{2}\binom{4}{2}$	$\frac{1}{3}\binom{6}{2}\binom{4}{2}\binom{2}{2}$			
K_8	$\binom{8}{2}$	$\frac{1}{2}\binom{8}{2}\binom{6}{2}$	$\frac{1}{3}\binom{8}{2}\binom{6}{2}\binom{4}{2}$	$\frac{1}{4}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
K_{m-1}	$\binom{m-1}{2}$	$\frac{1}{2}\binom{m-1}{2}\binom{m-3}{2}$	$\frac{1}{3}\binom{m-1}{2}\binom{m-3}{2}\binom{m-5}{2}$	$\frac{1}{4}\binom{m-1}{2}\binom{m-3}{2}\binom{m-5}{2}\binom{m-7}{2}$	$\cdots$	$\frac{1}{i-1}\prod_{k=0}^{i-2}\binom{m-2k-1}{2}$

Proof of Theorem 4. Using Proposition 2, the node set is $V(\mathcal{P}(Q_{4n})) = \Omega \cup A_2 \cup A_3$, where $A_2 = \bigcup_{j=0}^{n-1} A_2^j$. Therefore, we have the sequel types of edges in $\mathcal{P}(Q_{4n})$:

Type 1: $u_1 \sim u_2$, for any $u_1, u_2 \in A_3$;

Type 2: $u_1 \sim u_2$, for any $u_1, u_2 \in \Omega$;

Type 3: $u_1 \sim u_2$, for any $u_1 \in A_3, u_2 \in \Omega$;

Type 4: $u_1 \sim u_2$, for any $u_1 \in A_2, u_2 \in \Omega$;

Type 5: $u_1 \sim u_2$, for any $u_1, u_2 \in A_2^j \subseteq A_2$, where $0 \leq j \leq n-1$.

Seven cases of matchings occur amongst the edges of $\mathcal{P}(Q_{4n})$ according to the following categories:

(d^1) Matchings amongst the Type-1, Type-2, as well as Type-3 edges;

(d^2) Matchings amongst the Type-4 edges;

(d^3) Matchings amongst the Type-5 edges;

(d^4) Matchings amongst the Type-1 and Type-4 edges;

(d^5) Matchings amongst the Type-3 and Type-4 edges;

(d^6) Matchings amongst the Type-4 and Type-5 edges;

(d^7) Matchings amongst the Type-1, Type-2, Type-3, and Type-5 edges.

The following method computes all the types of matchings as mentioned above:

(d^1) As we know that the subgraph induced by A_1 is complete, that is, K_{2n}, so all the Type-1, Type-2, and Type-3 edges are exactly the edges of K_{2n}, and all the matchings between these edges are counted in Table 6, where d_i^1 denotes the number of matchings of order i, where $1 \leq i \leq n$;

(d^2) For $i = 1, 2$, let d_i^2 indicate the number of order i matchings:

> **For** (d_1^2): The number of Type-4 edges that are $4n$, which is equal to the number of order one matchings. Consequently, $(d_1^2) = 4n$;
>
> **For** (d_2^2): Suppose $u_1 \sim u_2 = e$ is a Type-4 edge with $u_2 \in \Omega$ and $u_1 \in A_2^j$ for a fixed $0 \leq j \leq n - 1$. Then, in addition to the edge e, every edge of Type 4 with one end in $A_2 \setminus \{u_1\}$ and another in $\Omega \setminus \{u_2\}$ forms a matching of order two. As a result,
>
> $$(d_2^2) = 4(n - \frac{1}{2})(n) = 4n(n - \frac{1}{2}).$$
>
> Hence, there is no order larger than two matching in this situation;

(d^3) Type 5 has n edges, none of which have a similar node. Thus, for any order i, there exists a matching such that $1 \leq i \leq n$. Suppose (d_i^3) represents the number of order i matchings. Then, $(d_i^3) = \binom{n}{i}$;

(d^4) Assume that (d_i^4) represents the number of order i matchings, where $1 \leq i \leq n + 1$. Thus, in this context, $(d_1^4) = 0$. There are no Type-1 edges that connect a node to any Type-4 edge in $\mathcal{P}(Q_{4n})$. Hence, we may obtain a matching in this situation by joining every matching of Type-1 edges to any matching of the Type-4 edges. The edges of Type 1 are also the edges of K_{2n-2}, and there are (d_ℓ^1) matchings of order ℓ between them, where $1 \leq \ell \leq n - 1$. Every (d_ℓ^1) can be found in Table 6. In between the edges of Type 4, there are $(d_1^2) = 4n$ and $(d_2^2) = 4n(n - \frac{1}{2})$ matchings having one and two orders, respectively.

As a result of the product rule, we obtain:

$$d_2^4 = d_1^2 \times d_1^1 = 4n d_1^1.$$

When $3 \leq i \leq n$, then:

$$
\begin{aligned}
d_i^4 &= d_1^2 \times d_{i-1}^1 + d_2^2 \times d_{i-2}^1 \\
&= 4n d_{i-1}^1 + 4n\left(n - \frac{1}{2}\right) d_{i-2}^1 \\
&= 2n\left(2 d_{i-1}^1 + 2\left(n - \frac{1}{2}\right) d_{i-2}^1\right).
\end{aligned}
$$

Furthermore, when $i = n + 1$, then:

$$d_i^4 = d_2^2 \times d_{i-2}^1 = 4n\left(n - \frac{1}{2}\right) d_{n-1}^1;$$

(d^5) For $i = 1, 2$, (d_i^5) represents the total matchings of order i. Then, $(d_1^5) = 0$. We can only utilize matchings of order one between the edges of Type 4 in this case. Otherwise, we are unable to employ any Type-3 edge, since both kinds of edges often share the nodes in Ω. As a result, we can only obtain matchings of order two in this case. Assume that $N = \{e = u_1 \sim u_2\}$ is the order one matching amongst the Type-4 edges with $u_1 \in A_2^j$, for $0 \leq j \leq n - 1$. Then, any Type-3 edge that is non-adjacent with u_2 can lead to construct an order two matchings. Given that there are $2(n - 1)$ such Type-3 edges, every of which may be utilized in every one of $4n$ matchings of order one amongst Type-4 edges, so we obtain:

$$(d_2^5) = 8n(n - 1);$$

(d^6) For $1 \le i \le n$, (d_i^6), denote the number of order i matchings, then (d_1^6) = 0. To find matchings, both matchings of order one and two between the edges of Type 4 will be considered and any matching of order ℓ between the edges of Type 5, where $1 \le \ell \le n-1$. Thus, by counting these matchings using the product rule, we obtain:

$$d_2^6 = 1 \times 4 \times n \times \binom{n-1}{1} = 4n(n-1),$$

and for $3 \le i \le n$:

$$d_i^6 = 2n\left\{2\binom{n-1}{i-1} + \binom{n-1}{i-2} + 4(n-1)\binom{n-2}{i-2}\right\};$$

(d^7) Given that the edges of Type 1, Type 2, as well as Type 3 are also the edges of K_{2n}, which is induced by A_1, so we may utilize them to identify matchings between the edges of Type 5 and the edges of K_{2n}. Let d_i^7 be the number of order i matchings. Then, $d_i^7 = 0$. Because no edge of Type 5 shares a node with an edge of K_{2n}, this corresponds to each matching of the edges of Type 5. Therefore, each matching of the edges of K_{2n} can be used to determine a match in this situation. Since there exist d_κ^1 matchings of the cardinality $1 \le \kappa \le n$ among the edges of K_{2n}, as shown in Table 6, as well as $d_j^3 = \binom{n}{j}$ matchings of the order $1 \le j \le n$ among the Type 5 edges, thus, in this example, the greatest order of a matching is $2n$. As a result, we may determine d_i^7, for $2 \le i \le 2n$, as follows:

$$d_2^7 = d_1^1 d_1^3,$$
$$d_3^7 = d_1^1 d_2^3 + d_2^1 d_1^3,$$
$$d_4^7 = d_1^1 d_3^3 + d_2^1 d_2^3 + d_3^1 d_1^3,$$
$$\vdots$$
$$d_i^7 = \sum_{j=1}^{i-1} d_j^1 d_{i-j}^3.$$

As a consequence, by the sum rule, the Hosoya index of $\mathcal{P}(Q_{4n})$ is as follows:

$$1 + \sum_{i=1}^{7}(d^i) = 1 + \sum_{i=1}^{n} d_i^1 + \sum_{i=1}^{2} d_i^2 + \sum_{i=1}^{n} d_i^3 + \sum_{i=2}^{n+1} d_i^4 + d_2^5 + \sum_{i=2}^{n} d_i^6 + \sum_{i=2}^{2n} d_i^7,$$

where:

$$d_i^1 = \frac{1}{i}\prod_{k=0}^{i-1}\binom{2(n-k)}{2}, \quad d_1^2 = 4n, \quad d_2^2 = 4n\left(n - \frac{1}{2}\right), \quad d_i^3 = \binom{n}{i},$$

$$d_2^4 = 4n\binom{2(n-1)}{2},$$

$$d_i^4 = 2n\left\{\frac{2}{i-1}\prod_{k=0}^{i-2}\binom{2(n-k-1)}{2} + \frac{2n-1}{i-2}\prod_{k=0}^{i-3}\binom{2(n-k-1)}{2}\right\},$$

for $3 \le i \le n$,

$$d_{n+1}^4 = \frac{2n(2n-1)}{i-2}\prod_{k=0}^{i-3}\binom{2(n-k-1)}{2},$$

$$d_2^5 = 8n\left(n - \frac{1}{2}\right), \quad d_2^6 = 4n(n-1),$$

$$d_i^6 = 2n\left\{2\binom{n-1}{i-1} + \binom{n-1}{i-2} + 4(n-1)\binom{n-2}{i-2}\right\}, \text{ for } 3 \le i \le n,$$

$$d_i^7 = \sum_{j=1}^{i-1}\frac{1}{j}\prod_{k=0}^{j-1}\binom{2(n-k)}{2}\binom{n}{i-j}, \text{ for } 2 \le i \le 2n.$$

□

Table 6. The total non-void matchings in K_{2n}.

K_{2n}	d_1^1	d_2^1	d_3^1	d_4^1	$\cdots$	d_i^1
K_2	$\binom{2}{2}$					
K_4	$\binom{4}{2}$	$\frac{1}{2}\binom{4}{2}\binom{2}{2}$				
K_6	$\binom{6}{2}$	$\frac{1}{2}\binom{6}{2}\binom{4}{2}$	$\frac{1}{3}\binom{6}{2}\binom{4}{2}\binom{2}{2}$			
K_8	$\binom{8}{2}$	$\frac{1}{2}\binom{8}{2}\binom{6}{2}$	$\frac{1}{3}\binom{8}{2}\binom{6}{2}\binom{4}{2}$	$\frac{1}{4}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
K_{2n-1}	$\binom{2n}{2}$	$\frac{1}{2}\binom{2n}{2}\binom{2n-2}{2}$	$\frac{1}{3}\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}$	$\frac{1}{4}\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}\binom{2n-6}{2}$	$\cdots$	$\frac{1}{i}\prod_{k=0}^{i-1}\binom{2(n-k)}{2}$

6. Concluding Remarks

This work aimed to discuss the structural properties of the power graphs of finite non-abelian groups. In this work, we found the Hosoya properties, that is the Hosoya polynomials, the reciprocal status Hosoya polynomials, and the Z-index of the power graphs of certain finite groups. The reciprocal status Hosoya polynomials described in Theorems 1 and 2 are the most notable results of this context. We further illustrated the Z-index in Theorems 3 and 4 of the power graphs of the dihedral and the generalized quaternion groups, respectively.

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