Research Article

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Power moments of automorphic *L*-functions related to Maass forms for $SL_3(\mathbb{Z})$

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Abstract: Let f be a self-dual Hecke-Maass eigenform for the group $SL_3(\mathbb{Z})$. For $\frac{1}{2} < \sigma < 1$ fixed we define $m(\sigma)$ (≥ 2) as the supremum of all numbers m such that

$$\int_1^T |L(s, f)|^m \mathrm{d}t \ll_{f,\varepsilon} T^{1+\varepsilon},$$

where $L(s, f)$ is the Godement-Jacquet L-function related to f. In this paper, we first show the lower bound of $m(\sigma)$ for $\frac{2}{3} < \sigma < 1$. Then we establish asymptotic formulas for the second, fourth and sixth powers of $L(s, f)$ as applications.

Keywords: power moments, L-function, automorphic form

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1 Introduction

Let f be a self-dual Hecke-Maass eigenform for the group $SL_3(\mathbb{Z})$ of type $\nu = (\alpha, \beta)$. Then the Langlands' parameters for f are

$$\mu_f(1) = \alpha + 2\beta - 1, \quad \mu_f(2) = \alpha - \beta, \quad \mu_f(3) = 1 - 2\alpha - \beta.$$

It is known that f has the following Fourier-Whittaker expansion:

$$f(z) = \sum_{\gamma \in U_2(\mathbb{Z}) \setminus SL_2(\mathbb{Z})} \sum_{m \geq 1} \sum_{n \neq 0} \frac{A_f(m, n)}{m|n|} W_J\left(M\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} z, \nu, \psi_{1,1}\right),$$

where $U_2 = \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}$, $W_J(z, \nu, \psi_{1,1})$ is the Jacquet-Whittaker function, $\psi_{1,1}$ is a character of $U_3(\mathbb{R})$, $M = \operatorname{diag}(m|n|, m, 1)$ and $A_f(m, n)$ are the Fourier coefficients of f. The function $W_J(z, \nu, \psi_{1,1})$ represents an exponential decay in y_1 and y_2 for

$$z = \begin{pmatrix} 1 & x_{12} & x_{13} \\ & 1 & x_{23} \\ & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}.$$

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From Kim and Sarnak [1] and Sarnak [2] we know that

$$A_f(m,n) \ll |mn|^{\frac{5}{14}+\varepsilon}.$$

From [3], the Rankin-Selberg theory shows that

$$\sum_{mn^2\le N} |A_f(m,n)|^2 \ll_f N.$$

Due to $A_f(m,n) = A_{\tilde{f}}(n,m)$, then

$$\sum_{m^2n\le N} |A_f(m,n)|^2 \ll_f N \tag{1.1}$$

also holds, where $\tilde{f}$ is the contragredient form of f. According to these estimates, we have

$$\sum_{m\le N} \frac{|A_f(m,1)|^2}{m} \ll \log N, \quad \sum_{n\le N} \frac{|A_f(1,n)|^2}{n} \ll \log N. \tag{1.2}$$

As in [4] and [5], the Godement-Jacquet L-function associated with f is defined as

$$L(s,f) = \sum_{n=1}^{\infty} \frac{A_f(1,n)}{n^s}, \quad \text{for } \Re s > 1.$$

This L-function has a standard functional equation and analytic continuation to an entire function on complex plane $\mathbb{C}$. Due to the fact that f is a Hecke eigenform, the Fourier coefficients are multiplicative and the L-function has an Euler product (see [5, pp. 173–174]), for $\Re s > 1$,

$$L(s,f) = \prod_p (1 - A_f(1,p)p^{-s} + A_f(p,1)p^{-2s} - p^{-3s})^{-1}.$$

Then the L-function associated with the dual Maass form $\tilde{f}$ takes the form

$$L(s,\tilde{f}) = \sum_{n=1}^{\infty} \frac{A_f(n,1)}{n^s} = \prod_p (1 - A_f(p,1)p^{-s} + A_f(1,p)p^{-2s} - p^{-3s})^{-1}.$$

We write $s = \sigma + it$ and suppose that $\frac{1}{2} < \sigma < 1$ is fixed. Let $m(\sigma)$ (≥ 2) be the supremum of all numbers m (≥ 2) such that

$$\int_1^T |L(s,f)|^m \mathrm{d}t \ll_{f,\varepsilon} T^{1+\varepsilon}, \tag{1.3}$$

where the $\ll$-constant may depend on $L(s,f)$ and ε. Naturally, we want to seek lower bounds for $m(\sigma)$, which occurs frequently in applications. In the cases of full modular group $SL_2(\mathbb{Z})$ and the congruence group, many scholars have obtained lot of results (e.g., see [6–25], etc.).

In this paper, we focus our attention on the Hecke-Maass eigenforms for the group $SL_3(\mathbb{Z})$. In this situation, for one thing, we do not know whether the Ramanujan conjecture is true; for another, the square and fourth mean value estimates of $L(s,f)$ are weaker than ones over $SL_2(\mathbb{Z})$. Our results are as follows.

Theorem 1. *Let $m(\sigma)$ for each $\frac{2}{3} < \sigma < 1$ be defined by* (1.3). *Then we have*

$$m(\sigma) \ge \frac{4(3-2\sigma)}{5(4-3\sigma)(1-\sigma)}. \tag{1.4}$$

From Theorem 1 we can get the following corollary immediately.

Corollary. *We have*

$$m\left(\frac{2}{3}\right) \geq 2,\ m\left(\frac{97-\sqrt{769}}{90}\right) \geq 3, \ldots,\ m\left(\frac{103-\sqrt{349}}{90}\right) \geq 12, \ldots.$$

Remark. Due to the fact that $L(s,f)$ is an L-function of degree 3, then Perelli's mean value theorem [26] shows that, for $\frac{1}{2} \leq \sigma \leq 1$ and $T \geq 1$ uniformly,

$$\int_1^T |L(\sigma+it,f)|^2 \mathrm{d}t \ll T^{\max(3(1-\sigma),1)+\varepsilon},$$

which implies

$$\int_1^T |L(\sigma+it,f)|^2 \mathrm{d}t \ll T^{1+\varepsilon} \qquad \left(\frac{2}{3} \leq \sigma \leq 1\right).$$

Thus, we restrict the range of σ in Theorem 1 into $\frac{2}{3} < \sigma < 1$.

As applications of Theorem 1, we can establish the asymptotic formulas for the second, fourth and sixth powers of $L(s,f)$.

Theorem 2. *For any $\varepsilon > 0$ and σ fixed, we have*

$$\int_1^T |L(\sigma+it,f)|^2 \mathrm{d}t = T\sum_{n=1}^{\infty} |A_f(1,n)|^2 n^{-2\sigma} + O\left(T^{\frac{4-3\sigma}{2}+\varepsilon}\right), \tag{1.5}$$

$$\int_1^T |L(\sigma+it,f)|^4 \mathrm{d}t = T\sum_{n=1}^{\infty} |A_f * A_f(1,n)|^2 n^{-2\sigma} + O\left(T^{\frac{27+\sqrt{69}-30\sigma}{2\sqrt{69}-6}+\varepsilon}\right), \tag{1.6}$$

$$\int_1^T |L(\sigma+it,f)|^6 \mathrm{d}t = T\sum_{n=1}^{\infty} |A_f * A_f * A_f(1,n)|^2 n^{-2\sigma} + O\left(T^{\frac{79+\sqrt{481}-90\sigma}{2\sqrt{481}-22}+\varepsilon}\right), \tag{1.7}$$

where $A_f * A_f(1,n) = \sum_{n=ml} A_f(1,m)A_f(1,l)$ is the Dirichlet convolution of $A_f(1,n)$ with itself. The asymptotic formulas (1.5), (1.6) *and* (1.7) *follow for $\frac{2}{3} < \sigma < 1$, $\frac{33-\sqrt{69}}{30} < \sigma < 1$ and $\frac{101-\sqrt{481}}{90} < \sigma < 1$, respectively.*

Notation. Throughout this paper, the letter ε stands for a sufficiently small positive number, and the value of ε may change from statement to statement.

2 Some lemmas

In order to prove Theorems 1 and 2, we first introduce some lemmas.

Lemma 2.1. *Let $T \leq t \leq 2T$ and $k \geq 1$ be a fixed integer. Then for $\frac{1}{2} < \sigma < 1$, we have*

$$|L(\sigma+it,f)|^k \ll 1 + \log T \int_{-\log^2 T}^{\log^2 T} \left|L(\sigma - \frac{1}{\log T} + it + iv, f)\right|^k e^{-|v|} \mathrm{d}v.$$

Proof. The proof of this lemma is similar to [27, Lemma 7.1], and we just need to use the following functional equation:

$$G_v(s)L(s,f) = \widetilde{G}_v(1-s)L(1-s,\widetilde{f}),$$

where

$$G_v(s) = \pi^{-\frac{3s}{2}}\Gamma\left(\frac{s+1-2\alpha-\beta}{2}\right)\Gamma\left(\frac{s+\alpha-\beta}{2}\right)\Gamma\left(\frac{s-1+\alpha+2\beta}{2}\right),$$
$$\widetilde{G}_v(s) = \pi^{-\frac{3s}{2}}\Gamma\left(\frac{s+1-\alpha-2\beta}{2}\right)\Gamma\left(\frac{s-\alpha+\beta}{2}\right)\Gamma\left(\frac{s-1+2\alpha+\beta}{2}\right),$$

in place of the functional equation of $\zeta(s)$. □

Lemma 2.2. *For $m = m(\sigma)$,*

$$\int_1^T |L(\sigma+it,f)|^{m(\sigma)}\mathrm{d}t \ll T^{1+\varepsilon} \tag{2.1}$$

is equivalent to

$$\sum_{r\le R} |L(\sigma+it_r,f)|^{m(\sigma)} \ll T^{1+\varepsilon}, \tag{2.2}$$

where

$$t_r \in [T,2T] \text{ for } r = 1,\dots,R; \quad |t_r - t_s| \ge \log^4 T \text{ for } 1 \le r \ne s \le R. \tag{2.3}$$

Proof. Let

$$L(\sigma+it_m,f) = \max_{t\in I_m}|L(\sigma+it,f)|, \quad I_m = [2T - m\log^4 T, 2T-(m-1)\log^4 T],$$

where $m = 1,2,\dots,[T\log^{-4} T]$. Denote by $\{t_r\}$ either of the sets $\{t_{2m}\}$ or $\{t_{2m-1}\}$. Then the t_r's satisfy (2.3) and

$$\begin{aligned}\int_T^{2T} |L(\sigma+it,f)|^{m(\sigma)}\mathrm{d}t &\ll \sum_{m=1}^{[t\log^{-4}T]} \int_{2T-m\log^4 T}^{2T-(m-1)\log^4 T} |L(\sigma+it_m,f)|^{m(\sigma)}\mathrm{d}t \\ &\ll \sum_{m=1}^{[t\log^{-4}T]} |L(\sigma+it_m,f)|^{m(\sigma)}\log^4 T \\ &\ll T^{1+\varepsilon}.\end{aligned}$$

And then replacing T by $\frac{T}{2}, \frac{T}{2^2}, \dots$ and adding we can get (2.1). On the other hand, by Lemma 2.1, we have

$$\begin{aligned}\sum_{r\le R} L(\sigma+it_r,f)|^{m(\sigma)}\mathrm{d}t &\ll R + \log T\sum_{r\le R}\int_{-\log^2 T}^{\log^2 T}\left|L\left(\sigma-\frac{1}{\log T}+it_r+iv,f\right)\right|^{m(\sigma)}\mathrm{d}v \\ &\ll R+\log T\sum_{r\le R}\int_{t_r-\log^2 T}^{t_r+\log^2 T}\left|L\left(\sigma-\frac{1}{\log T}+it,f\right)\right|^{m(\sigma)}\mathrm{d}t \\ &\ll T\log^{-4}T+\log T\int_1^{2T+\log^2 T}\left|L\left(\sigma-\frac{1}{\log T}+it,f\right)\right|^{m(\sigma)}\mathrm{d}t \\ &\ll T^{1+\varepsilon},\end{aligned}$$

which implies (2.1). □

Lemma 2.3. *We suppose that* $\frac{1}{2} < \sigma < 1$ *is fixed and*

$$R \ll T^{1+\varepsilon} V^{-m(\sigma)}, \tag{2.4}$$

where for t_r *defined by* (2.3) *we have*

$$|L(\sigma + it_r, f)| \geq V \geq T^{\varepsilon} \ (r = 1, 2, \dots, R), \tag{2.5}$$

which is equivalent to

$$\sum_{r \leq R} |L(\sigma + it_r, f)|^{m(\sigma)} \ll T^{1+\varepsilon}. \tag{2.6}$$

Proof. We suppose that (2.6) is true and let $\{t_{V,1}, \dots, t_{V,R_1}\}$ be the subset of $\{t_r\}$ such that

$$|L(\sigma + it_{V,j}, f)| \geq V \ \ (j = 1, \dots, R_1).$$

Then from (2.6) we have

$$R_1 V^{m(\sigma)} \leq \sum_{r \leq R} |L(\sigma + it_r, f)|^{m(\sigma)} \ll T^{1+\varepsilon},$$

thus for $R_1 = R$, (2.4) holds.

Inversely, we let (2.4) hold and denote by $t_{V,1}, \dots, t_{V,R(V)}$ those of the points $t_1, \dots, t_R$ for which

$$V \leq |L(\sigma + it_{V,j}, f)| \leq 2V \ \ (j = 1, \dots, R(V)).$$

For each V, we have $O(\log T)$ choices. And from the following Lemma 2.6, we take $V = T^{\frac{5(1-\sigma)}{4}}$, $V = 2^{-1} T^{\frac{5(1-\sigma)}{4}}$, $V = 2^{-2} T^{\frac{5(1-\sigma)}{4}}, \dots$. Then we can obtain

$$\sum_{r \leq R} L(\sigma + it_r, f)|^{m(\sigma)} \mathrm{d}t \ll RT^{\varepsilon} + \sum_{V} \sum_{j \leq R(V)} (2V)^{m(\sigma)} \ll RT^{\varepsilon} + \sum_{V} T^{1+\varepsilon} \ll T^{1+\varepsilon}.$$

□

Lemma 2.4. *Let* $t_1 < \cdots < t_R$ *be real numbers such that* $t_r \in [T, 2T]$ *for* $r = 1, \dots, R$; $|t_r - t_s| \geq \log^4 T$ *for* $1 \leq r \neq s \leq R$. *If*

$$T^{\varepsilon} \ll V \leq \left| \sum_{M < n \leq 2M} a(n) n^{-\sigma - it_r} \right|, \tag{2.7}$$

where $\sum_{n \leq M} |a(n)|^2 \ll M^{1+\varepsilon}$ *for* $1 \ll M \ll T^C$ $(C > 0)$, *then we have*

$$R \ll T^{\varepsilon} (M^{2-2\sigma} V^{-2} + TV^{-f(\sigma)}), \tag{2.8}$$

where

$$f(\sigma) = \begin{cases} \dfrac{2}{3 - 4\sigma}, & \text{if } \dfrac{1}{2} < \sigma \leq \dfrac{2}{3}, \\[2mm] \dfrac{10}{7 - 8\sigma}, & \text{if } \dfrac{2}{3} \leq \sigma \leq \dfrac{11}{14}, \\[2mm] \dfrac{34}{15 - 16\sigma}, & \text{if } \dfrac{11}{14} \leq \sigma \leq \dfrac{13}{15}, \\[2mm] \dfrac{98}{31 - 32\sigma}, & \text{if } \dfrac{13}{15} \leq \sigma \leq \dfrac{57}{62}, \\[2mm] \dfrac{5}{1 - \sigma}, & \text{if } \dfrac{57}{62} \leq \sigma \leq 1 - \varepsilon. \end{cases} \tag{2.9}$$

Proof. We can get this lemma by following a similar argument to [6, Lemma 8.2] replacing $a(n) \ll M^{\varepsilon}$ by $\sum_{n \leq M} |a(n)|^2 \ll M^{1+\varepsilon}$. □

Lemma 2.5. [27, Theorem 5.2] *Let $a_1, \dots, a_N$ be arbitrary complex numbers. Then*

$$\int_0^T \left| \sum_{n\le N} a_n n^{it} \right|^2 \mathrm{d}t = T \sum_{n\le N} |a_n|^2 + O\left(\sum_{n\le N} n|a_n|^2 \right),$$

and the above formula remains also valid if $N = \infty$, provided that the series on the right hand side of the aforementioned formula converge.

Lemma 2.6. [28, Corollary 1.2] *Let $\frac{1}{2} \le \sigma \le 1$ be fixed, we have*

$$|L(\sigma + it, f)| \ll |t|^{\frac{5}{4}(1-\sigma)+\varepsilon}.$$

Lemma 2.7. *For any $\varepsilon > 0$, we have*

$$\int_0^T \left| L\left(\frac{2}{3} + it, f\right) \right|^2 \mathrm{d}t \ll T^{1+\varepsilon},$$
$$\int_0^T \left| L\left(\frac{2}{3} + it, f\right) \right|^4 \mathrm{d}t \ll T^{\frac{17}{12}+\varepsilon}.$$

Proof. The first result is a general result of Perelli [26], which we can also get from Lemma 2.5 with $m = 3$ and $\sigma = \frac{2}{3}$ in Liu and Zhang [29]. From Lemma 2.6 and the first result, we can easily get the second result. □

Lemma 2.8. *For t_r satisfying* (2.3), *we have*

$$\sum_{r\le R} \left| L\left(\frac{2}{3} + it_r, f\right) \right|^2 \mathrm{d}t \ll T^{1+\varepsilon},$$
$$\sum_{r\le R} \left| L\left(\frac{2}{3} + it_r, f\right) \right|^4 \mathrm{d}t \ll T^{\frac{17}{12}+\varepsilon}.$$

Proof. Following a similar argument of Lemma 2.2, with the help of Lemma 2.7 we can obtain this lemma. □

Lemma 2.9. [27, Lemma 8.3] *Let $F(s)$ be regular in the region $\mathfrak{D} : \alpha \le \sigma \le \beta, t \ge 1$ and let $F(s) \ll e^{Ct^2}$ for $s \in \mathfrak{D}$. Then for any fixed $q > 0$ and $\alpha < \gamma < \beta$, we have*

$$\int_2^T |F(\gamma + it)|^q \mathrm{d}t \ll \left(\int_1^{2T} |F(\alpha + it)|^q \mathrm{d}t + 1 \right)^{\frac{\beta-\gamma}{\beta-\alpha}} \left(\int_1^{2T} |F(\beta + it)|^q \mathrm{d}t + 1 \right)^{\frac{\gamma-\alpha}{\beta-\alpha}}.$$

In the following two lemmas, though the definitions of $\varphi_k(m)$ and $\psi_k(n)$ are different from ones in Lemmas 2.11 and 2.12 of [18], we still can get these two lemmas by following similar arguments, respectively.

Lemma 2.10. *Let $\varphi_k(n)$ be the arithmetic function generated by $L(s, f)^k$, that is*

$$\varphi_k(n) = \underbrace{A_f * \cdots * A_f(1, n)}_{k \text{ times}}. \tag{2.10}$$

Then we have

$$\sum_{n\le x} \varphi_k(n) \ll x^{1+\varepsilon}, \quad \sum_{n\le x} \varphi_k^2(n) \ll x^{1+\varepsilon}.$$

Lemma 2.11. *Let* $0 < \delta < \frac{1}{2}$ *be a fixed constant and*

$$\psi_k(n) = \begin{cases} \varphi_{2k}(n) - \sum\limits_{\substack{n=ml \\ m \le T, l \le T}} \varphi_k(m)\varphi_k(l), & T < n \le T^2, \\ \varphi_{2k}(n), & n > T^2. \end{cases}$$

Then we have

$$\sum_{n \ge T} \psi_k^2(n) n^{-2-2\delta} = O(1).$$

3 Proofs of Theorems 1 and 2

3.1 Proof of Theorem 1

In this section, we restrict the range of σ into $\frac{2}{3} < \sigma < 1$ and shall give lower bounds for $m(\sigma)$ by establishing formulas of type

$$R \ll T^{1+\varepsilon} V^{-m(\sigma)}.$$

Recalling Mellin's formula

$$e^{-x} = (2\pi i)^{-1} \int_{2-i\infty}^{2+i\infty} \Gamma(\omega) x^{-\omega} \mathrm{d}\omega (x > 0). \tag{3.1}$$

Taking $x = \frac{n}{Y}$ and multiplying (3.1) by $A_f(1, n_1)A_f(1, n_2)n_1^{-s}n_2^{-s}$, where $n = n_1 n_2$ and summing over n, we can obtain

$$\sum_{n=1}^{\infty} \left(\sum_{n=n_1n_2} A_f(1, n_1)A_f(1, n_2) \right) e^{-\frac{n}{Y}} n^{-s} = (2\pi i)^{-1} \int_{2-i\infty}^{2+i\infty} Y^{\omega} \Gamma(\omega) L(s + \omega, f)^2 \mathrm{d}\omega. \tag{3.2}$$

Shifting the line of integration in (3.2) to $\Re\omega = \frac{2}{3} - \sigma$, we encounter a simple pole at $\omega = 0$ with residue $L(s, f)^2$ and get, as $Y \to \infty$,

$$\sum_{n \le Y \log^2 Y} \left(\sum_{n=n_1n_2} A_f(1, n_1)A_f(1, n_2) \right) e^{-\frac{n}{Y}} n^{-s} + o(1) = L(s, f)^2 + (2\pi i)^{-1} \int_{\Re\omega = \frac{2}{3} - \sigma} Y^{\omega} \Gamma(\omega) L(s + \omega, f)^2 \mathrm{d}\omega. \tag{3.3}$$

The integral part of (3.3) for which $\Im\omega \ge \log^2 T$ is $o(1)$ as $T \to \infty$ by Stirling's formula. Then let $s = \sigma + it_r$ and thus for each t_r we have

$$L(\sigma + it_r, f)^2 \ll 1 + \left| \sum_{n \le Y \log^2 Y} \left(\sum_{n=n_1n_2} A_f(1, n_1)A_f(1, n_2) \right) e^{-\frac{n}{Y}} n^{-\sigma - it_r} \right| + \int_{-\log^2 T}^{\log^2 T} Y^{\frac{2}{3} - \sigma} \left| L\left(\frac{2}{3} + it_r + iv, f \right) \right|^2 e^{-|v|} \mathrm{d}v. \tag{3.4}$$

Combining (2.5) with (3.4), we can obtain

$$\begin{aligned} V^2 &\ll \left| \sum_{n \le Y \log^2 Y} \left(\sum_{n=n_1n_2} A_f(1, n_1)A_f(1, n_2) \right) e^{-\frac{n}{Y}} n^{-\sigma - it_r} \right| \\ &\ll \log T \max_{M \le \frac{1}{2} Y \log^2 Y} \left| \sum_{M < n \le 2M} \left(\sum_{n=n_1n_2} A_f(1, n_1)A_f(1, n_2) \right) e^{-\frac{n}{Y}} n^{-\sigma - it_r} \right| \end{aligned} \tag{3.5}$$

or

$$V^2 \ll Y^{\frac{2}{3}-\sigma} \left| L\left(\frac{2}{3} + it_r', f\right) \right|^2, \tag{3.6}$$

where $V \gg T^{\varepsilon}$ and t_r' is defined as

$$\left| L\left(\frac{2}{3} + it_r', f\right) \right| = \max_{-\log^2 T \le v \le \log^2 T} \left| L\left(\frac{2}{3} + it_r + iv, f\right) \right|.$$

For convenience, denote by R_1' and R_2' those points which satisfy (3.5) and (3.6), respectively.

Recalling (1.1), we know that Lemma 2.4 is valid. We first consider R_1'. By Lemma 2.4, we have

$$R_1' \ll \log Y \times T^{\varepsilon}(M^{2-2\sigma}V^{-4} + TV^{-2f(\sigma)}) \ll T^{\varepsilon}(Y^{2-2\sigma}V^{-4} + TV^{-2f(\sigma)}). \tag{3.7}$$

While for R_2', by Lemma 2.8, Hölder's inequality and (3.6), we can obtain

$$R_2' \ll Y^{\frac{2}{3}-\sigma}V^{-2} \sum_{r \le R_2'} \left| L\left(\frac{2}{3} + it_r', f\right) \right|^2 \ll Y^{\frac{2}{3}-\sigma}V^{-2}T^{1+\varepsilon} \tag{3.8}$$

and

$$R_2' \ll Y^{\frac{2}{3}-\sigma}V^{-2} \sum_{r \le R_2'} \left| L\left(\frac{2}{3} + it_r', f\right) \right|^2 \ll Y^{\frac{2}{3}-\sigma}V^{-2}R_2'^{\frac{1}{2}}T^{\frac{17}{24}+\varepsilon}. \tag{3.9}$$

For (3.8), if we take $Y = V^{\frac{6}{4-3\sigma}}T^{\frac{3}{4-3\sigma}}$, then we have

$$R \ll R_1' + R_2' \ll T^{\varepsilon}\left(Y^{2-2\sigma}V^{-4} + TV^{-2f(\sigma)} + Y^{\frac{2}{3}-\sigma}V^{-2}T^{1+\varepsilon}\right) \ll T^{\varepsilon}\left(V^{\frac{-4}{4-3\sigma}}T^{\frac{6-6\sigma}{4-3\sigma}} + TV^{-2f(\sigma)}\right). \tag{3.10}$$

For (3.9), if we take $Y = T^{\frac{17}{8}}$, then we have

$$R \ll R_1' + R_2' \ll T^{\varepsilon}\left(Y^{2-2\sigma}V^{-4} + TV^{-2f(\sigma)} + Y^{\frac{4}{3}-2\sigma}V^{-4}T^{\frac{17}{12}}\right) \ll T^{\varepsilon}\left(V^{-4}T^{\frac{17}{4}-\frac{17}{4}\sigma} + TV^{-2f(\sigma)}\right). \tag{3.11}$$

Therefore, combining (3.10) with (3.11) we have

$$R \ll T^{\varepsilon}\left(TV^{-2f(\sigma)} + V^{\frac{-4}{4-3\sigma}}T^{\frac{6-6\sigma}{4-3\sigma}} + V^{-4}T^{\frac{17}{4}-\frac{17}{4}\sigma}\right). \tag{3.12}$$

We assume that the second and the third terms in (3.12) do not exceed TV^{-x} and TV^{-y}, for values x and y which can be determined by Lemma 2.6, then we can obtain

$$x \le \frac{4(3-2\sigma)}{5(1-\sigma)(4-3\sigma)}, \quad y \le \frac{7-3\sigma}{5(1-\sigma)}.$$

Thus, we have

$$R \ll T^{1+\varepsilon}V^{-z}$$

with

$$z = \min\left(2f(\sigma), \frac{4(3-2\sigma)}{5(1-\sigma)(4-3\sigma)}, \frac{7-3\sigma}{5(1-\sigma)}\right).$$

For $\frac{2}{3} < \sigma \le 1 - \varepsilon$, we always have

$$\frac{4(3-2\sigma)}{5(1-\sigma)(4-3\sigma)} < \frac{7-3\sigma}{5(1-\sigma)}.$$

Recalling the value of $f(\sigma)$ in Lemma 2.4, we can take

$$z = \frac{4(3-2\sigma)}{5(1-\sigma)(4-3\sigma)}, \quad \frac{2}{3} < \sigma \le 1 - \varepsilon.$$

Thus, we complete the proof of Theorem 1.

3.2 Proof of Theorem 2

In this section, we give the proof of Theorem 2 by following a similar argument to [6, Theorem 2]. Let σ_k^* denote the infimum of all numbers σ for which

$$\int_1^T |L(\sigma+it,f)|^{2k}\mathrm{d}t \ll T^{1+\varepsilon}$$

holds for any $\varepsilon > 0$, where $k \geq 1$ is a fixed integer, $\frac{1}{2} \leq \sigma_k^* < 1$.

Writing $s = \sigma + it$, we have

$$\int_1^T |L(\sigma+it,f)|^{2k}\mathrm{d}t = \int_1^T \left|\sum_{n\leq T}\varphi_k(n)n^{-\sigma-it}\right|^2 \mathrm{d}t + O\left(\int_1^T \left|L(\sigma+it,f) - \left(\sum_{n\leq T}\varphi_k(n)n^{-\sigma-it}\right)^2\right|\mathrm{d}t\right), \tag{3.13}$$

where $\varphi_k(n)$ is given by Lemma 2.10.

Combining Abel's summation formula with Lemmas 2.5 and 2.10, we can obtain

$$\int_1^T \left|\sum_{n\leq T}\varphi_k(n)n^{-\sigma-it}\right|^2 \mathrm{d}t = T\sum_{n\leq T}\varphi_k^2(n)n^{-2\sigma} + O\left(\sum_{n\leq T}\varphi_k^2(n)n^{1-2\sigma}\right) = T\sum_{n=1}^{\infty}\varphi_k^2(n)n^{-2\sigma} + O(T^{2-2\sigma+\varepsilon}). \tag{3.14}$$

Let

$$F(\sigma+it,f) = L^{2k}(\sigma+it,f) - \left(\sum_{n\leq T}\varphi_k(n)n^{-\sigma-it}\right)^2.$$

And applying Lemma 2.9 with $q = 1$, $\alpha = \sigma_k^* + \delta$, $\beta = 1+\delta$, $\gamma = \sigma$, where $0 < \delta < \frac{1}{2}$ is a fixed constant which may be chosen arbitrarily small, for fixed k we have

$$\frac{\beta-\sigma}{\beta-\alpha} = \frac{1+\delta-\sigma}{1-\sigma_k^*} \leq \frac{1-\sigma}{1-\sigma_k^*} + \delta^{\frac{1}{2}}$$

and

$$\frac{\sigma-\alpha}{\beta-\alpha} = \frac{\sigma-\sigma_k^*-\delta}{1-\sigma_k^*} \leq \frac{\sigma-\sigma_k^*}{1-\sigma_k^*}.$$

Recalling the definition of σ_k^*, by Lemma 2.5 we have

$$\int_1^{2T} |F(\alpha+it,f)|\mathrm{d}t \leq \int_1^{2T} |L(\sigma_k^*+\delta+it,f)|^{2k}\mathrm{d}t + \int_1^{2T}\left|\sum_{n\leq T}\varphi_k(n)n^{-\sigma_k^*-\delta-it}\right|^2 \mathrm{d}t \ll T^{1+\delta} + T^{2-2\sigma_k^*+\varepsilon} \ll T^{1+\delta}.$$

Moreover,

$$F(\beta+it,f) = \sum_{n=1}^{\infty}\varphi_{2k}(n)n^{-1-\delta-it} - \left(\sum_{n\leq T}\varphi_k(n)n^{-1-\delta-it}\right)^2 = \sum_{n>T}\psi_k(n)n^{-1-\delta-it},$$

where $\psi_k(n)$ is given by Lemma 2.11.

By Lemma 2.5, Lemma 2.10 and Hölder's inequality, we can obtain

$$\int_1^{2T} |F(\beta+it,f)|\mathrm{d}t \ll T^{\frac{1}{2}}\left(\int_1^{2T}\left|\sum_{n\geq T}\psi_k(n)n^{-1-\delta-it}\right|^2 \mathrm{d}t\right)^{\frac{1}{2}} \ll T^{\frac{1}{2}}.$$

Thus, Lemma 2.9 shows

$$\int_1^{2T} |F(\sigma + it, f)| \mathrm{d}t \ll T^{(1+\delta)\left(\frac{1-\sigma}{1-\sigma_k^*} + \delta^{\frac{1}{2}}\right) + \frac{\sigma-\sigma_k^*}{2-2\sigma_k^*}}.$$

Note that

$$(1 + \delta)\left(\frac{1 - \sigma}{1 - \sigma_k^*} + \delta^{\frac{1}{2}}\right) + \frac{\sigma - \sigma_k^*}{2 - 2\sigma_k^*} \leq \frac{2 - \sigma - \sigma_k^*}{2 - 2\sigma_k^*} + \varepsilon$$

holds for any $\varepsilon > 0$ if $\delta = \delta(\varepsilon)$ is sufficiently small. Noting that for the exponent of the O-term in (3.14), we have

$$2 - 2\sigma < \frac{2 - \sigma - \sigma_k^*}{2 - 2\sigma_k^*} < 1.$$

Thus,

$$\int_1^T |L(\sigma + it, f)|^{2k} \mathrm{d}t = T \sum_{n=1}^{\infty} \varphi_k^2(n) n^{-2\sigma} + R(k, \sigma; T),$$

and for fixed σ satisfying $\sigma_k^* < \sigma < 1$, we have

$$R(k, \sigma; T) \ll T^{\frac{2-\sigma-\sigma_k^*}{2-2\sigma_k^*} + \varepsilon}.$$

From Theorem 1 we have

$$\int_1^T \left| L\left(\frac{2}{3} + it, f\right)\right|^2 \mathrm{d}t \ll T^{1+\varepsilon},$$

$$\int_1^T \left| L\left(\frac{33 - \sqrt{69}}{30} + it, f\right)\right|^4 \mathrm{d}t \ll T^{1+\varepsilon},$$

$$\int_1^T \left| L\left(\frac{101 - \sqrt{481}}{90} + it, f\right)\right|^6 \mathrm{d}t \ll T^{1+\varepsilon}.$$

Recalling the definition of σ_k^*, we can take $\sigma_1^* = \frac{2}{3}$, $\sigma_2^* = \frac{33-\sqrt{69}}{30}$ and $\sigma_3^* = \frac{101-\sqrt{481}}{90}$, from which we can obtain Theorem 2 immediately.

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