Power of Randomization in Automata on Infinite Strings

A. Prasad Sistla

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Joint work with Rohit Chadha and Mahesh Viswanathan

OutLine

Automata on Infinite Strings

Probabilistic Automata on Finite Strings

Probabilistic Buchi Automata

Finite State Probabilistic Monitors

Expressiveness results for FPMS

Decidability and Complexity results for FPMs

Decidability and Expressiveness for PBAs

Hierarchical PBAs

Automata on Infinite Strings (Rabin, Buchi, Muller, McNaughton)

- Finite State Automata on Finite Strings:
  DFA $\equiv$ *NFA* $\equiv$ *RExprs*

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- Finite State Automata on Finite Strings:
  $DFA \equiv NFA \equiv RExprs$
- Buchi Automata (BA) on Infinite Strings:
  $\mathcal{A} = (\Sigma, Q, \delta, q_0, F)$
  - Σ — input alphabet
  - Q — automaton states
  - $\delta \subseteq Q \times \Sigma \times q$ — transition relation
  - q_0 — start state
  - $F \subseteq Q$ — accepting set

Buchi Automata

- A run r on an infinite input $a = a_0, ...a_i, ...$ is a sequence $(r_0, ..., r_i, ...)$ such that $r_0 = q_0$ and $(r_i, a_i, r_{i+1}) \in \delta$ for $i \geq 0$.

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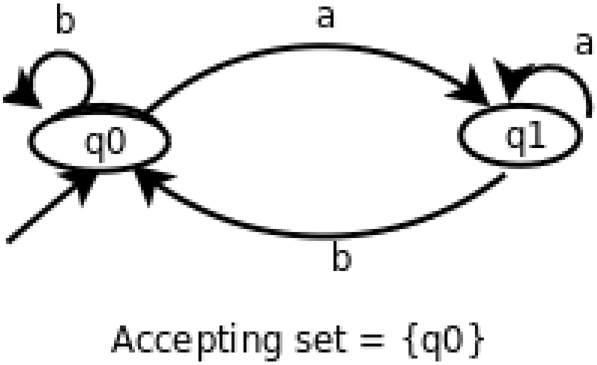
- **An example BA:** Accepts strings in which every a is eventually followed by b.



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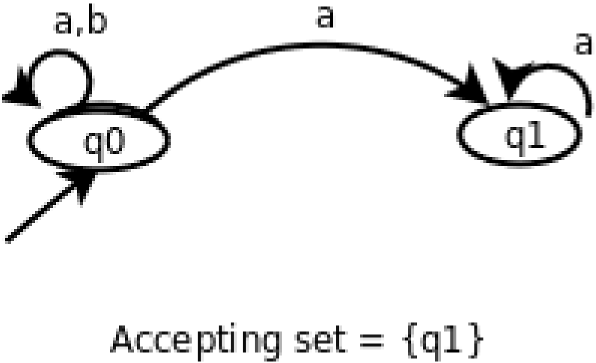

Accepts strings in which eventually only *a* appears.

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- **Applications:** Verification of concurrent programs, decision procedures for logics: Linear Temporal Logic, S1S, Theory of Linear order, etc.

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- Universality; Given a BA $\mathcal{A}$, check if $L(\mathcal{A}) = \Sigma^\omega$. It is **PSPACE**-complete.
- Language Containment; Given a BAs $\mathcal{A}, \mathcal{B}$, check if $L(\mathcal{A}) \subseteq L(\mathcal{B})$. **PSPACE**-complete.

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Probabilistic Finite string Automata (PFA) (Rabin)

- $\mathcal{A} = (\Sigma, Q, \delta, q_0, F)$.
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- $L_{>\frac{1}{2}}(\mathcal{A})$ can be a non-regular set (Rabin 1960s).
- Determining non-emptiness of $L_{>\frac{1}{2}}(\mathcal{A})$ and $L_{\geq\frac{1}{2}}(\mathcal{A})$ are undecidable. Both are **R.E.**-complete. (Paz 1971, Soloma 1973).

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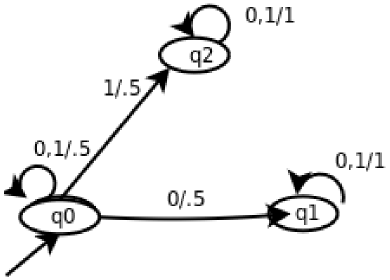
- Example. $\Sigma = \{0, 1\}$.



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  If $r_0 \neq q_0$ then $\phi(S_u) = 0$.
- $PrOfAcc_{\mathcal{A}}(a) = \phi(Inf(F))$. (Note $Inf(F) \in \Delta$).

OutLine

Finite State Probabilistic Monitors (FPM)

- A FPM is a PBA with a designated **reject** state, an **absorbing** state. All other states are accepting states. Let $PrOfRej_{\mathcal{A}}(a) = 1 - PrOfAcc_{\mathcal{A}}(a)$.

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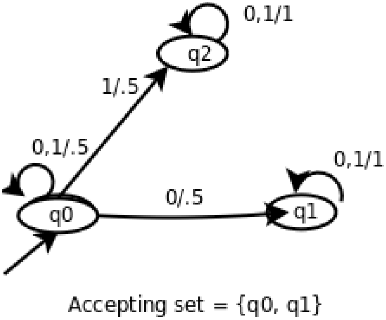

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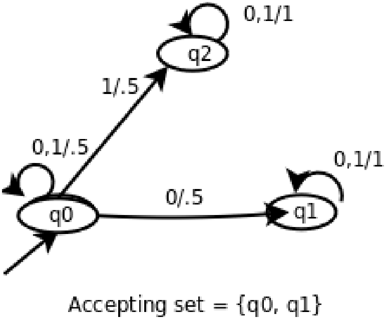

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- Observe $PrOfRej_B(a) = val(a)$.

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- Model checking safety properties of open finite state probabilistic programs.

Properties of Infinite Executions

- Language $C \subseteq \Sigma^\omega$ is a safety property, if it is limit closed. That is, for any $a \in \Sigma^\omega$, if $prefixes(a) \subseteq (Prefixes(C))$ then $a \in C$.
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- Only Safety properties can be monitored using deterministic monitors.
- $C \subseteq \Sigma^\omega$ is an almost safety property if it is a countable union of safety properties.
  Example: Set of sequences in which 1 appears at least 3 times.

Monitoring Non-safety Properties

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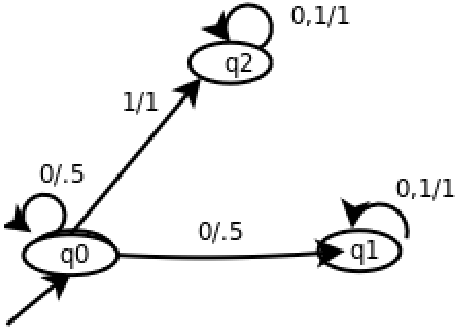
The following FPM monitors (i.e., accepts) the set of sequences in which 1 appears eventually. Not a safety property!



q1--- rejecting state

Classes of Monitorable Languages

Consider an alphabet Σ and a language $\mathcal{L} \subseteq \Sigma^\omega$.

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- $\mathcal{L}$ is Monitorable with Non-strict Cut-off if there is a FPM $\mathcal{A}$ such that $\mathcal{L}$ is the set of strings rejected by $\mathcal{A}$ with probability $\leq \frac{1}{2}$. MNC is the class of all such languages.

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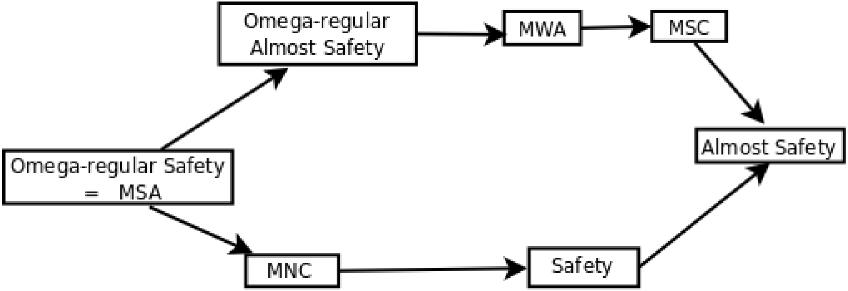
Expressiveness results



Containment Lattice among Language Classes

Monitoring Non-ω-regular Languages

- For any $\mathcal{A}, \mathcal{B}$, can construct $\mathcal{C}$ such that for any input *a*, $PrOfRej_{\mathcal{C}}(a) = PrOfRej_{\mathcal{A}}(a) \times PrOfRej_{\mathcal{B}}(a)$.

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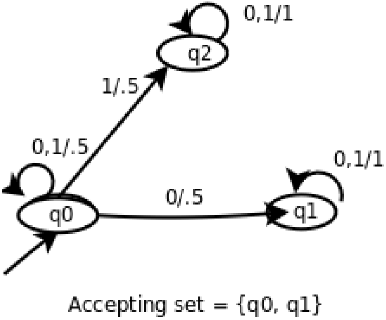

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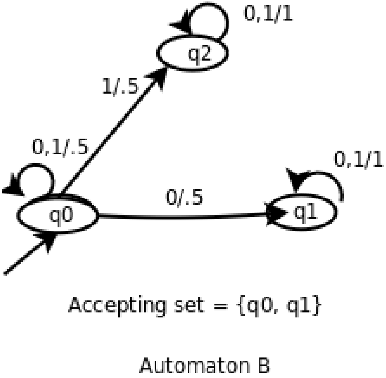

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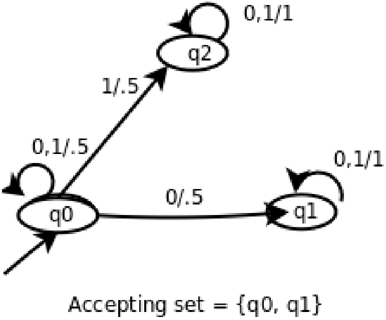

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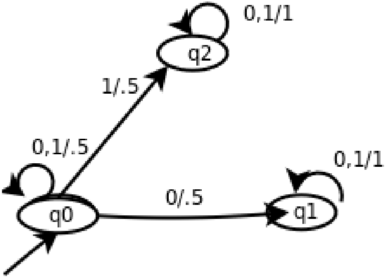

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- Let L be the set of inputs rejected by $\mathcal{C}$ with prob $\leq \frac{1}{2}$.
- $L = \{a : val(a) \leq \frac{1}{\sqrt{2}}\}$. $L \in MNC$ and not ω-regular.

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Decidability and Complexity results

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Decidability and Complexity results

- Given a FPM $\mathcal{A}$, determining if there is at least one input *a* such that $PrOfRej_{\mathcal{A}}(a) < 1$ is **PSPACE**-complete.
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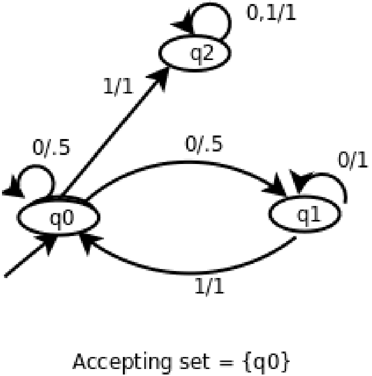
- Given a FPM $\mathcal{A}$, determining if there is at least one input *a* such that $PrOfRej_{\mathcal{A}}(a) \leq \frac{1}{2}$ is **co-R.E.**-complete.

Summary of Complexity and Decidability results

	EMPTINESS	UNIVERSALITY
Msa	**PSPACE**-complete	**NL**-complete
Mwa	**PSPACE**-complete	**PSPACE**-complete
Msc	**co-R.E.**-complete	Π^1_1-complete
Mnc	**R.E.**-complete	**co-R.E.**-complete

OutLine

Decidability and Expressiveness for PBAs



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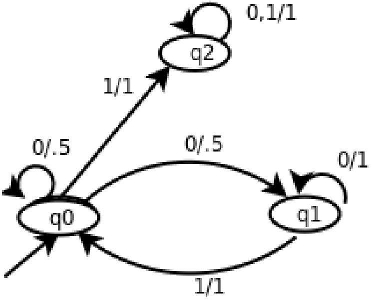

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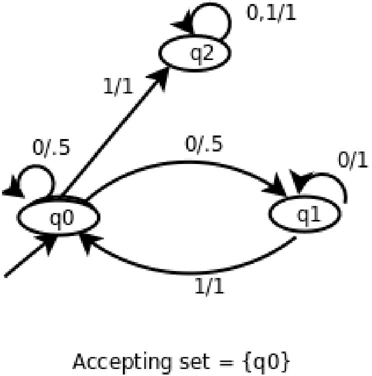

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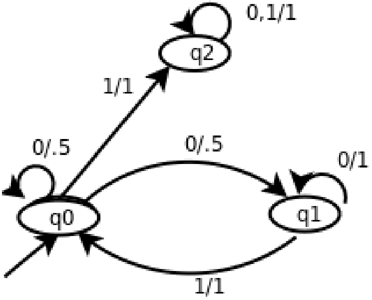

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OutLine

Automata on Infinite Strings

Probabilistic Automata on Finite Strings

Probabilistic Buchi Automata

Finite State Probabilistic Monitors

Expressiveness results for FPMS

Decidability and Complexity results for FPMs

Decidability and Expressiveness for PBAs

Hierarchical PBAs

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