SIAM J. APPL. MATH.
Vol. 30, No. 2, March 1976

POWER REFLECTION FROM A LOSSY ONE-DIMENSIONAL RANDOM MEDIUM*

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Abstract. We consider the problem of an electromagnetic plane wave normally incident upon a slab of material whose constitutive parameters are subjected to lossy random perturbations. A transmission line model is adopted, wherein the four distributed parameters are assumed to be strongly mixing random functions of distance along the line. We study the reflection of energy at the input in the diffusion limit, an asymptotic limit involving weak random perturbations and long transmission lines. In the presence of dissipation, the probability density function for the modulus of the reflection coefficient approaches a nontrivial limit as the line length approaches infinity. We compute the mean and fluctuations of the voltage and power reflection coefficients with respect to this limiting density as a function of the dissipation.

1. Introduction. Consider the problem of an electromagnetic plane wave normally incident upon a slab of material whose permittivity, conductivity and permeability have been randomly perturbed. Assume that the slab is homogeneous in the plane transverse to the propagation direction and that in the absence of perturbations the constitutive parameters reduce to those of free space. This one-dimensional problem is mathematically equivalent to the analysis of a transmission line whose corresponding four distributed parameters are randomly perturbed [2], [5]. We shall adopt this transmission line model.

The nondissipative or lossless case has been studied by a number of authors ([2], [3], [7] and the references therein). For a general class of random perturbations known as strongly-mixing processes [1], [8] it is known that the modulus of the input reflection coefficient converges in probability to unity as the slab thickness becomes infinite; in another case [9] this convergence to unity has been shown to hold almost surely. This behavior has also been observed in numerical simulation studies [3], [4], [6], [9]. Tutubalin [10], however, has shown that if the random perturbations are weakly dissipative, the marginal density function for the modulus of the reflection coefficient approaches a nontrivial limit as the slab thickness becomes infinite. Our goal in this paper is to extend [8] to consider this dissipative case within the context of strongly mixing processes and the diffusion limit.

We adopt a transmission line model wherein all four distributed parameters are subjected to random perturbations. As in [8], we obtain the Riccati equation for the input reflection coefficient as a function of slab thickness. This equation is studied in the diffusion approximation, an asymptotic limit involving weak random fluctuations and long transmission lines. Using a theorem of Khas'minskii [1], we derive the forward equation for the marginal density function of the reflection coefficient modulus and explicitly determine the limiting equilibrium

* Received by the editors November 6, 1974. This work was supported by the Office of Naval Research under Contract N00014-67-A-0467-0015.

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solution. With this density function, we compute the mean and fluctuations of the input voltage and power reflection coefficients as functions of a characteristic loss parameter.

2. Formulation of the problem. Let $(\Omega, \mathscr{F}, P)$ denote an underlying probability space with element $\omega \in \Omega$. $E\{\cdot\}$ will be used to indicate expectation, i.e., integration with respect to the probability measure P. Frequency will be denoted by f. We consider the following transmission line equations:

$$dV/dx = -ZI, \qquad dI/dx = -YV, \tag{1}$$

where an $e^{-i2\pi ft}$ time-dependence has been omitted and where we assume that

$$\begin{aligned} &Z = R - i2\pi fL, \quad Y = G - i2\pi fC, \\ &R = \varepsilon^2 R_2(x, f, \omega), \\ &G = \varepsilon^2 G_2(x, f, \omega), \quad R_2 \text{ and } G_2 \text{ almost surely nonnegative}, \\ &L = L_0 + \varepsilon L_1(x, f, \omega) + \varepsilon^2 L_2(x, f, \omega), \\ &C = C_0 + \varepsilon C_1(x, f, \omega) + \varepsilon^2 C_2(x, f, \omega), \\ &E\{L_1(x, f, \omega)\} = E\{C_1(x, f, \omega)\} = 0, \end{aligned} \tag{2}$$

with ε a small real parameter. The distributed parameters of the line, therefore, are subjected to small random perturbations. We shall assume that the stochastic processes in (2) are almost surely bounded, wide-sense stationary and strongly mixing. A precise statement of the strong mixing property is given in [1]. Loosely speaking, a strongly mixing process is one where random variables formed by sampling the process at two values of x tend to become independent as the separation between the sampling points becomes infinite. Note, however, that we make no other assumptions regarding the details of the joint statistics of the six processes since they do not affect the diffusion limit.

We define the characteristic impedance, characteristic admittance, wave-number and scattering parameters as follows:

$$\begin{aligned} &Z_0 = \sqrt{L_0/C_0}, \quad Y_0 = Z_0^{-1}, \quad k_0 = 2\pi f\sqrt{L_0 C_0} \\ &a = \tfrac{1}{2}(Y_0^{1/2}V + Z_0^{1/2}I), \qquad b = \tfrac{1}{2}(Y_0^{1/2}V - Z_0^{1/2}I), \end{aligned} \tag{3}$$

where a and b are functions of x, f, ω and ε. We shall use a prime to denote normalized parameters as follows:

$$L_j' \equiv L_j Y_0, \quad C_j' \equiv C_j Z_0, \quad j = 1, 2; \quad R_2' \equiv R_2 Y_0, \quad G_2' \equiv G_2 Z_0. \tag{4}$$

Define

$$\begin{aligned} &\alpha_1 \equiv (L_0 C_0)^{-1/2}(L_1' + C_1'), &&\beta_1 \equiv (L_0 C_0)^{-1/2}(L_1' - C_1'), \\ &\alpha_2 \equiv (L_0 C_0)^{-1/2}(L_2' + C_2'), &&\beta_2 \equiv (L_0 C_0)^{-1/2}(L_2' - C_2'), \\ &\alpha_3 \equiv k_0^{-1}(R_2' + G_2'), &&\beta_3 \equiv k_0^{-1}(R_2' - G_2'), \\ &\alpha \equiv \varepsilon\alpha_1 + \varepsilon^2(\alpha_2 + i\alpha_3), &&\beta \equiv \varepsilon\beta_1 + \varepsilon^2(\beta_2 + i\beta_3). \end{aligned} \tag{5}$$

Note that $E\{\alpha_1\} = E\{\beta_1\} = 0$ and that $\alpha_3 \geqq |\beta_3|$. The parameter α_3 is a measure of the total dissipation in each realization of the process. We assume that $E\{\alpha_3\} > 0$, i.e., that this dissipation is present in a nontrivial way. In terms of this notation, the scattering parameters a and b satisfy the following stochastic differential equations:

$$\begin{aligned} \frac{da}{dx} &= ik_0a + \frac{i}{2}k_0\alpha a - \frac{i}{2}k_0\beta b, \\ \frac{db}{dx} &= -ik_0b - \frac{i}{2}k_0\alpha b + \frac{i}{2}k_0\beta a. \end{aligned} \tag{6}$$

We shall consider the case where the randomly perturbed transmission line occupies the interval $0 \leqq x \leqq l$. We define

$$\Gamma(l, f, \omega, \varepsilon) \equiv e^{-i2k_0l}\frac{a(l, f, \omega, \varepsilon)}{b(l, f, \omega, \varepsilon)}, \tag{7}$$

which represents the right input reflection coefficient of the line, with the rapid phase variation removed. From (6) it follows that Γ satisfies the first order Riccati equation:

$$\frac{d\Gamma}{dl} = ik_0\alpha\Gamma - \frac{i}{2}k_0\beta\, e^{i2k_0l}\Gamma^2 - \frac{i}{2}k_0\beta\, e^{-i2k_0l}, \qquad l \geqq 0. \tag{8}$$

We shall assume that the line is matched at $x = 0$ (i.e., $a(0) = 0$) and that it is excited by a wave incident from the right at $x = l$ (i.e., $b(l) \neq 0$). Consequently, we consider the initial value problem consisting of (8) and the initial condition $\Gamma(0) = 0$.

Let

$$\Gamma \equiv r\, e^{i\phi}, \qquad \theta \equiv \phi + 2k_0l. \tag{9}$$

Then (8) transforms into the following pair of equations:

$$\begin{aligned} \frac{dr}{dl} &= \frac{\varepsilon}{2}k_0\beta_1(r^2 - 1)\sin\theta \\ &\quad + \frac{\varepsilon^2}{2}k_0[-2\alpha_3r + \beta_3(r^2 + 1)\cos\theta + \beta_2(r^2 - 1)\sin\theta] \\ \frac{d\phi}{dl} &= \frac{\varepsilon}{2}k_0[2\alpha_1 - \beta_1(r + r^{-1})\cos\theta] \\ &\quad + \frac{\varepsilon^2}{2}k_0[2\alpha_2 - \beta_2(r + r^{-1})\cos\theta + \beta_3(r - r^{-1})\sin\theta]. \end{aligned} \tag{10}$$

3. Application of the limit theorem. Observe that for fixed coordinates r and θ, the coefficients of ε on the right side of (10) have a zero expectation. Let $\tau \equiv \varepsilon^2 l$. We shall study (10) in the diffusion approximation, an asymptotic limit in which $\varepsilon \to 0$ and $l \to \infty$ in such a way that τ remains fixed. This limit has been amply discussed in the literature ([1], [2], [3] and references therein); reference [8] in particular, deals with an application quite similar to the present case. Therefore, we shall only state the result here.

Recall that the processes in (2) were assumed to be wide sense stationary. Define

$$\begin{aligned} R_1(u) &\equiv E\{\beta_1(u+s)\beta_1(s)\} \\ &\equiv k_0^{-2}\omega_0^2 E\{[L_1'(u+s) - C_1'(u+s)][L_1'(s) - C_1'(s)]\}, \\ D &\equiv \frac{k_0^2}{8}\int_0^\infty R_1(u)\cos 2k_0 u\, du, \\ \gamma &\equiv k_0 D^{-1} E\{\alpha_3\}. \end{aligned} \tag{11}$$

An application of the diffusion limit leads to the conclusion that the marginal density function associated with the reflection coefficient $r(\tau/\varepsilon^2, f, \omega)$ behaves asymptotically as the solution of the forward equation:

$$\begin{aligned} (4D)^{-1}\frac{\partial p}{\partial \tau} &= \frac{\partial}{\partial z}\left[(z^2-1)\frac{\partial p}{\partial z}\right] + \frac{\gamma}{4}\frac{\partial}{\partial z}[(z^2-1)p], \\ z &\equiv (1+r^2)/(1-r^2). \end{aligned} \tag{12}$$

4. Limiting density function and moments. To obtain the equilibrium or limiting density function as $\tau \to \infty$, call it $p_\infty(z)$, we look for a solution of

$$\begin{aligned} &\frac{\partial}{\partial z}\left[(z^2-1)\frac{\partial p_\infty}{\partial z}\right] + \frac{\gamma}{4}\frac{\partial}{\partial z}[(z^2-1)p_\infty] = 0, \\ &p_\infty(z) \geqq 0 \quad \text{on } [1,\infty) \quad \text{and} \quad \int_1^\infty p_\infty(z)\,dz = 1. \end{aligned} \tag{13}$$

Problem (13) can be solved by repeated quadratures; the solution is

$$p_\infty(z) = \frac{\gamma}{4} e^{-\gamma(z-1)/4}. \tag{14}$$

This density function agrees with that obtained by Tutubalin [10]. Let $E_\infty\{\cdot\}$ denote expectation with respect to this limiting density function. Then

$$E_\infty\{r^n\} = \int_1^\infty \left(\frac{z-1}{z+1}\right)^{n/2} p_\infty(z)\,dz, \qquad n = 1, 2, \cdots, \tag{15}$$

represent the equilibrium moments. In particular, the mean and fluctuations of the voltage reflection coefficient modulus are given by $E_\infty\{r\}$ and $[E_\infty\{r^2\} - E_\infty^2\{r\}]^{1/2}$ while the mean and fluctuations of the power reflection coefficient are given by $E_\infty\{r^2\}$ and $[E_\infty\{r^4\} - E_\infty^2\{r^2\}]^{1/2}$. These quantities are plotted in Fig. 1 as a function of the loss parameter γ.

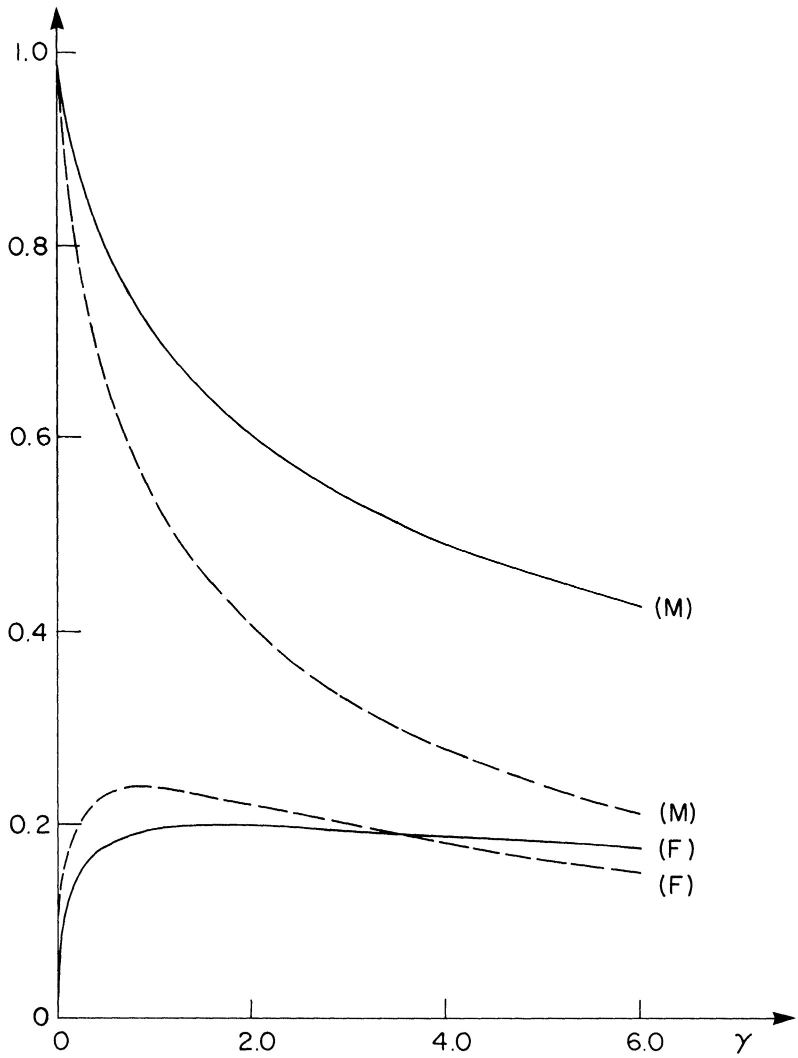

FIG. 1. *Equilibrium values of the reflection coefficient vs. loss parameter* γ.
Voltage reflection coefficient: ———
Power reflection coefficient: – – – – –
The graphs labeled (M) *and* (F) *are mean values and fluctuations, respectively.*

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