POWER SERIES SOLUTIONS FOR THE *m*th-ORDER-MATRIX ORDINARY DIFFERENTIAL EQUATION*

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Abstract. A description of the fundamental solution of the mth-order linear ordinary differential equation with matrix coefficients is given in terms of power series and the Green function. The second-order equation is discussed.

1. Introduction. For describing a basis of fundamental solutions for the *m*th-order linear differential equation with matrix coefficients, the standard procedure would be to transform it into an equivalent first-order equation of higher dimension. This approach does not exploit the fact that it is an *m*th-order equation and therefore does not give the detailed information desired.

We discuss this problem by applying the classical method of power series to the *m*th-order equation itself. By appropriately choosing the initial conditions, a set of m linearly independent solutions is easily obtained. It is then found that their matrix series coefficients are to be determined by certain matrix recurrence relations which depend only on the initial values and the coefficients of a fixed fundamental solution. This solution turns out to be the Green function of the given equation. A particular description is given for the second-order equation.

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2. Fundamental solutions. We consider the *m*th-order differential equation

$$x^{(m)}(t) = \sum_{j=0}^{m-1} A_j x^{(j)}(t) \tag{1}$$

with the initial values

$$x^{(j)}(t_0) = x_0^{(j)}; \qquad j = 0, m-1 \tag{2}$$

for arbitrary $n \times n$ complex matrices A_j. Existence, uniqueness and the analyticity of the above initial value problem is assumed. The variable x will denote an n-dimensional complex vector or an $n \times n$ complex matrix. The solutions $C_j(t)$ having the initial values

$$x^{(j)}(0) = I, x^{(k)}(0) = 0 \qquad k \neq j; \qquad k, j = 0, m-1,$$

where I denotes the identity matrix, will be referred as the fundamental solutions of Eq. (1).

We can write

$$C_j(t) = \sum_{k=0}^{\infty} C_{j,k}\, t^k/k!$$

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By substitution in (1), the following relations are obtained:

$$C_{j,k+m} = \sum_{i=0}^{m-1} A_i\, C_{j,i+k}; \qquad k \geq 0 \text{ integer,}$$
$$C_{j,j} = I,\; C_{j,s} = 0 \qquad s \neq j; \qquad s, j = 0, m-1. \tag{3}$$

It appears that in order to describe the fundamental solutions $C_j(t)$, we are required to solve (3) for each j. This is not the case. We claim that

$$C_{0,k+m} = C_{m-1,k+m-1}\, A_0\,, \tag{4}$$

$$C_{j,k+m} = \sum_{i=0}^{j} C_{m-1,k+m-j+i}\, A_i\,, \qquad 0 < j < m-1, \tag{5}$$

$$C_{m-1,k+m} = \sum_{i=0}^{m-1} A_i\, C_{m-1,i+k} \tag{6}$$

for each integer $k \geq 0$.

The last relation follows from (3) with $j = m - 1$. The other two will be established by induction. Due to the initial values of the matrix functions $C_j(t)$, we can assume that $C_{0,s} = C_{m-1,s-1}\, A_0$, $s = 1, k + m$, once (4) is established. Thus

$$C_{0,k+1+m} = \sum_{i=0}^{m-1} A_i\, C_{0,i+k+1} = \sum_{i=0}^{m-1} A_i\, C_{m-1,i+k} = C_{m-1,k+m}\, A_0\,.$$

Similarly,

$$C_{j,k+1+m} = \sum_{i=0}^{m-1} A_i\, C_{j,i+k+1} = \sum_{i=0}^{m-1} A_i \sum_{s=0}^{j} C_{m-1,i+k+s-j+1}\, A_s$$
$$= \sum_{s=0}^{j} \left[\sum_{i=0}^{m-1} A_i\, C_{m-1,i+k+s-j+1} \right] A_s = \sum_{i=0}^{j} C_{m-1,k+m+i-j+1}\, A_i$$

for each $j = 1, m - 2$.

By applying the Leibniz differentiation rule for an integral, it follows easily that

$$U(t) = \int_{t_0}^{t} C_{m-1}(t-s)\, f(s)\, ds \tag{7}$$

is the free response solution for the nonhomogeneous equation

$$x^{(m)}(t) = \sum_{j=0}^{m-1} A_j\, x^{(j)}(t) + f(t), \tag{8}$$

where $f(t)$ is continuous on a neighborhood of t_0.

Therefore, the solution to the initial value problem (8), (2) is given by

$$x(t) = \sum_{j=0}^{m-1} C_j(t-t_0)x^{(j)}(t_0) + \int_{t_0}^{t} C_{m-1}(t-s)\, f(s)\, ds,$$

in clear analogy to the scalar case.

The Green's function $G(t) = C_{m-1}(t)$ is given by

$$G(t) = (t^{m-1}/(m-1)!)\, I + \sum_{k=0}^{\infty} (t^{k+m}/(k+m)!)\, G_{k+m} \tag{9}$$

where the matrix coefficients G_k are obtained by solving the difference equation

$$G_{k+m} = \sum_{i=0}^{m-1} A_i\, G_{i+k}\,,\ G_s = 0;\ s = 0, m-2,\ G_{m-1} = I \tag{10}$$

for each integer $k \geq 0$. The other fundamental solutions are given by

$$C_j(t) = (t^{j-1}/(j-1)!)\, I + \sum_{k=0}^{\infty} (t^{k+m}/(k+m)!)\, C_{j,k+m} \tag{11}$$

where the matrix coefficients $C_{j,k+m}$ are to be determined recursively from relations (4), (5).

3. The second-order equation. In the theory of linear vibrations, the second-order matrix equation

$$x''(t) = Ax'(t) + Bx(t) \tag{12}$$

is often encountered. It is usually assumed that the matrices A, B are symmetric and the solution techniques in such situations involve certain diagonalization procedures. From our discussion it follows that its general solution is given by $x(t) = C_0(t - t_o)x(t_o) + C_1(t - t_o)x'(t_o)$ with the fundamental solutions

$$C_0(t) = I + \sum_{k=0}^{\infty} (t^{k+2}/(k+2)!)\, F_{k+2}\,,\ C_1(t) = tI + \sum_{k=0}^{\infty} (t^{k+2}/(k+2)!)\, G_{k+2} \tag{13}$$

where the matrices F_{k+2}, G_{k+2} are obtained by solving the difference equation

$$\begin{aligned} G_{k+2} &= AG_{k+1} + BG_k\,;\ G_0 = 0,\ G_1 = I; \\ F_{k+2} &= G_{k+1}B;\ F_0 = I,\ F_1 = 0. \end{aligned} \tag{14}$$

For the undamped equation $x''(t) = Bx(t)$, the above system can easily be solved to obtain: $F_{2k} = B^k$, $F_{2k+1} = 0$; $G_{2k} = 0$, $G_{2k+1} = B^k$. Thus

$$x(t) = \text{Cos}\,(t - t_o)\,\sqrt{-B}\,x(t_o) + (\text{Sen}\,(t - t_o)\,\sqrt{-B}/\sqrt{-B})\,x'(t_o) \tag{15}$$

where the notation $F(A)$ is employed for a matrix function arising from a complex power series $F(z)$ by replacing z for the matrix A (see [8]).

Since the sum of a convergent matrix power series equals (for each matrix A) a matrix polynomial in A whose coefficients are obtained by Hermite interpolation over spectral values [5], the above fundamental solutions can be computed in terms of the eigenvalues of B and the scalar functions Cos $t\sqrt{-z}$ and (Sen $t\sqrt{-z}/\sqrt{-z}$). Using the matrix reduction theorem mentioned, it can be further shown that the fundamental solutions (13) are given by

$$C_0(t) = b_0(t)\, I + \sum_{k=0}^{2n-1} b_k(t) G_{k-1} B,\ C_1(t) = \sum_{k=0}^{2n-1} b_k(t) G_k \tag{16}$$

where the scalar functions $b_k(t)$ are obtained by solving the interpolation scheme

$$F^{(j)}(z_i) = \sum_{k=0}^{2n-1} j! \binom{k}{j} b_k\, z_i^{k-j}, \qquad j = 0, n_i - 1$$

with z_i the eigenvalue of multiplicity n_i of the matrix

$$D = t \begin{bmatrix} 0 & I \\ B & A \end{bmatrix},$$

and $F(z) = e^z$.

Finally, solutions of exponential type $x(t) = \exp(\lambda t)\, v$ would exist for the Eq. (12) iff $F_k v + \lambda G_k v = \lambda^k v$ for every integer $k \geq 0$. This yields to the well-known condition det $(\lambda^2 I - \lambda A - B) = 0$.

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