Power Series with the Riemann Zeta-function in the Coefficients

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1. Introduction. Let $\zeta(s)$ be the Riemann zeta-function, and $\zeta(s, \alpha)$ with a parameter $\alpha > 0$ the Hurwitz zeta-function defined by

$$\zeta(s, \alpha) = \sum_{n=0}^{\infty} (n + \alpha)^{-s} \quad (\operatorname{Re} s > 1),$$

and its meromorphic continuation over the whole s-plane. Let $\Gamma(s)$ be the gamma-function, and $(s)_n = \Gamma(s + n)/\Gamma(s)$ for any integer n Pochhammer's symbol.

The main aim of this note is to investigate two types of power series whose coefficients involve the Riemann zeta-function (see Sections 2 and 3) based on Mellin-Barnes' type integral formulae. Further, as for generalizations of these power series, we shall introduce hypergeometric type generating functions of $\zeta(s)$ and derive their basic properties in the final section. Proofs of the results in the following sections are only sketched. Detailed version of the proofs will appear in a forthcoming paper.

2. Binomial type series. A simple relation

$$\sum_{n=2}^{\infty} \{\zeta(n) - 1\} = 1,$$

which was firstly mentioned by Goldbach in 1729 (see [10, Section 1]), follows immediately from the inversion of the order of the double sum $\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} m^{-n}$. This is in fact derived as a special case of Ramanujan's formula

$$(2.1) \quad \zeta(\nu, 1 + x) = \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} \zeta(\nu + n)(-x)^n$$

for $|x| < 1$ and any complex $\nu \notin \{-1,0,1,2,\ldots\}$, which gives a base of his various evaluations of sums involving $\zeta(s)$ (see [7, Sections 5 and 6]). Noting the relations $\zeta(s, 1) = \zeta(s)$ and $(\partial/\partial\alpha)^n \zeta(s, \alpha) = (-1)^n (s)_n \zeta(s + n, \alpha)$, we see that (2.1) is actually the Taylor series expansion of $\zeta(\nu, 1 + x)$ as a function of x near $x = 0$. Srivastava [9][10] proved various summation formulae related to (2.1), while Klusch [6] considered a generalization of (2.1) to the Lerch zeta-function. This direction has recently been pursued by Yoshimoto, Kanemitsu, and the author [15]. Rane [8] applied (2.1) to study the mean square of Dirichlet L-functions.

For our later purpose we shall prove (2.1) as an application of Mellin-Barnes' type integrals. Suppose first that $\operatorname{Re} \nu > 1$, and set

$$(2.2) \quad F_\nu(x) = \frac{1}{2\pi i} \int_{(b)} \frac{\Gamma(\nu + s)\Gamma(-s)}{\Gamma(\nu)} \zeta(\nu + s) x^s ds$$

for $x > 0$, where b is fixed with $1 - \operatorname{Re} \nu < b < 0$, and (b) denotes the vertical straight line from $b - i\infty$ to $b + i\infty$. We can shift the path of integration in (2.2) to the right, provided $0 < x < 1$. Collecting the residues at the poles $s = 0,1,2,\ldots$ of the integrand, we see that $F_\nu(x)$ is equal to the right-hand infinite series in (2.1). On the other hand, since $\zeta(\nu + s) = \sum_{n=1}^{\infty} n^{-\nu-s}$ converges absolutely on the path $\operatorname{Re} s = b$, the term-by-term integration is permissible, and this gives

$$F_\nu(x) = \sum_{n=1}^{\infty} (n + x)^{-\nu} = \sum_{n=0}^{\infty} (n + 1 + x)^{-\nu},$$

where each term in the resulting expression could be evaluated by taking $-z = x/n$ in

$$\Gamma(a)(1 - z)^{-a} = \frac{1}{2\pi i} \int_{(\sigma)} \Gamma(a + s)\Gamma(-s)(-z)^s ds$$

for $|\arg(-z)| < \pi$ and $-\operatorname{Re} a < \sigma < 0$ (cf. [14], p. 289, 14.51, Corollary]). We therefore obtain (2.1) by analytic continuation.

3. Exponential type series. Chowla and Hawkins [2] found that the sum

$$G_0(x) = \sum_{n=2}^{\infty} \zeta(n) \frac{(-x)^n}{n!}$$

has the asymptotic formula

$$(3.1) \quad G_0(x) = x \log x + (2\gamma - 1)x + \frac{1}{2} + O(e^{-A\sqrt{x}})$$

as $x \to +\infty$, where γ is Euler's constant and A is a certain positive constant. They conjectured that the error term in (3.1) cannot be essentially sharpened. Let a be an arbitrary fixed real number. Buschman and Srivastava [1] introduced a

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more general formulation

$$G_a(x) = \sum_{n>a+1} \zeta(n-a) \frac{(-x)^n}{n!},$$

where n runs through all nonnegative integers with $n > a+1$, and studied its asymptotic behaviour as $x \to +\infty$. The special cases $a = -2, -1$ and 1 have been investigated by Tennenbaum [11], Verma [12], and Verma and Prasad [13], respectively.

Let ν be an arbitrary fixed complex number. It is in fact possible to treat a slightly general sum

$$G_\nu(x) = \sum_{n>\mathrm{Re}\,\nu+1} \zeta(n-\nu) \frac{(-x)^n}{n!},$$

based on the formula

$$G_\nu(x) = \frac{1}{2\pi i} \int_{(c)} \Gamma(-s)\zeta(s-\nu)x^s ds \tag{3.2}$$

for $x > 0$, where c is fixed with $\mathrm{Re}\,\nu + 1 < c < [\mathrm{Re}\,\nu] + 2$. Here $[\mathrm{Re}\,\nu]$ denotes the greatest integer not exceeding $\mathrm{Re}\,\nu$. (3.2) can be proved by shifting the path (c) to the right, and collecting the residues at the poles $s = [\mathrm{Re}\,\nu] + 2, [\mathrm{Re}\,\nu] + 3, \ldots$ of the integrand. While the main method of [1] is Euler-Maclaurin's summation device, our treatment of $G_\nu(x)$ is due to a refinement of original [2].

Shifting the path of integration in (3.2) to the left in an appropriate manner, we can show

Theorem 1. *The following formulae hold for all $x \geq 1$.*

(i) *If* $\nu \notin \{-1,0,1,2,\ldots\}$,

$$G_\nu(x) = \Gamma(-\nu-1)x^{\nu+1} - \sum_{n=0}^{[\mathrm{Re}\,\nu]+1} \zeta(n-\nu)\frac{(-x)^n}{n!} + \mathscr{G}_\nu(x);$$

(ii) *If* $\nu \in \{-1,0,1,2,\ldots\}$,

$$G_\nu(x) = -\frac{(-x)^{\nu+1}}{(\nu+1)!}\left(\log x + 2\gamma - \sum_{n=1}^{\nu+1}\frac{1}{n}\right) - \sum_{n=0}^{\nu}\zeta(n-\nu)\frac{(-x)^n}{n!} + \mathscr{G}_\nu(x),$$

where empty sums are considered to be zero, and $\mathscr{G}_\nu(x)$ is the error term satisfying the estimate

$$\mathscr{G}_\nu(x) = O(x^{-C}) \tag{3.3}$$

for any $C > 0$. Here the implied constant depends only on C and ν.

Remark. This theorem gives a refinement of the results in [1].

Chowla and Hawkins suggested in [2] that the error term in (3.1) is expressible in terms of 'almost' Bessel functions, however, it seems that the functions have not been precisely determined.

Let $K_\nu(z)$ be the modified Bessel function of the third kind defined by

$$K_\nu(z) = \frac{\pi}{2\sin\pi\nu}\{I_{-\nu}(z) - I_\nu(z)\},$$

where $I_\nu(z)$ is the Bessel function with purely imaginary argument (see [4, p. 5, 7.2.2, (12) and (13)]). We can indeed show that $\mathscr{G}_\nu(x)$ has the Voronoï type summation formula involving $K_{\nu+1}(z)$.

Theorem 2. *For any $x \geq 1$, we have*

$$\mathscr{G}_\nu(x) = 2\left(\frac{x}{2\pi}\right)^{\frac{1}{2}(\nu+1)} \times \sum_{n=1}^{\infty} n^{-\frac{1}{2}(\nu+1)}\{e^{-\frac{\pi i}{4}(\nu+1)}K_{\nu+1}(2e^{\frac{\pi i}{4}}\sqrt{2n\pi x}) + e^{\frac{\pi i}{4}(\nu+1)}K_{\nu+1}(2e^{-\frac{\pi i}{4}}\sqrt{2n\pi x})\}.$$

Let $(\nu, m) = \Gamma\left(\frac{1}{2}+\nu+m\right)/m!\Gamma\left(\frac{1}{2}+\nu-m\right)$ for any integer $m \geq 0$ be Hankel's symbol. Applying an asymptotic expansion for $K_{\nu+1}(z)$ (cf. [4, p. 24, 7.4.1, (4)]) to Theorem 2, we can further prove

Corollary. *The asymptotic formula*

$$\mathscr{G}_\nu(x) = \sqrt{2}\left(\frac{x}{2\pi}\right)^{\frac{1}{2}\nu+\frac{1}{4}} e^{-2\sqrt{\pi x}} \times \left\{\sum_{m=0}^{M-1}(\nu+1, m)\cos\left(2\sqrt{\pi x}+\frac{\pi}{4}\times\left(\nu+\frac{3}{2}+m\right)\right)(32\pi x)^{-\frac{m}{2}} + O(x^{-\frac{M}{2}})\right\}$$

holds for all $x \geq 1$ and all integers $M \geq 0$.

Remark. This gives an affirmative answer to the conjecture of Chowla and Hawkins mentioned above.

4. Generating functions of $\zeta(s)$. Let α and ν be arbitary complex numbers with $\nu \notin \{1,0,-1,\ldots\}$. We define

$$f_\nu(\alpha; z) = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\zeta(\nu+n)z^n \quad (|z| < 1),$$

$$e_\nu(z) = \sum_{n=0}^{\infty}\frac{1}{n!}\zeta(\nu+n)z^n \quad (|z| < +\infty).$$

Since $\zeta(\nu+n) \to 1$ uniformly for $n = 0,1,2,\ldots$, as $\mathrm{Re}\,\nu \to +\infty$, we see that $f_\nu(\alpha; z) \to (1-z)^{-\alpha}$ and $e_\nu(z) \to e^z$, as $\mathrm{Re}\,\nu \to +\infty$. This suggests us to define the hypergeometric type generating functions of $\zeta(s)$ as

$$\mathscr{F}_\nu(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}\zeta(\nu+n)z^n \quad (|z| < 1), \tag{4.1}$$

$$(4.2)\quad \mathscr{F}_\nu(\alpha\,;\gamma\,;z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\gamma)_n n!}\zeta(\nu+n)z^n \quad (|z|<+\infty),$$

where α, β and γ are arbitrary fixed complex numbers with $\gamma \notin \{0, -1, -2, \ldots\}$. Then we can observe, when $\mathrm{Re}\,\nu \to +\infty$, that

$$\mathscr{F}_\nu(\alpha, \beta\,;\gamma\,;z)\to F(\alpha,\beta\,;\gamma\,;z),$$
$$\mathscr{F}_\nu(\alpha\,;\gamma\,;z)\to F(\alpha\,;\gamma\,;z),$$

where $F(\alpha,\beta\,;\gamma\,;z)$ and $F(\alpha\,;\gamma\,;z)$ denote hypergeometric functions of Gauss and Kummer, respectively.

Corresponding to Euler's integral formulae for $F(\alpha,\beta\,;\gamma\,;z)$ and $F(\alpha\,;\gamma\,;z)$ (cf. [3, p. 59, 2.1.3, (10), and p. 255, 6.5, (1)]), we can deduce from term-by-term integration that

$$(4.3)\quad \frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\gamma)}\mathscr{F}_\nu(\alpha,\beta\,;\gamma\,;z) = \int_0^1 \tau^{\beta-1}(1-\tau)^{\gamma-\beta-1}f_\nu(\alpha\,;\tau z)\,d\tau$$

for $0<\mathrm{Re}\,\beta<\mathrm{Re}\,\gamma$ and $|z|<1$, and

$$(4.4)\quad \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}\mathscr{F}_\nu(\alpha\,;\gamma\,;z) = \int_0^1 \tau^{\alpha-1}(1-\tau)^{\gamma-\alpha-1}e_\nu(\tau z)\,d\tau$$

for $0<\mathrm{Re}\,\alpha<\mathrm{Re}\,\gamma$ and $|z|<+\infty$. Moreover, corresponding to Mellin-Barnes' integral formulae for $F(\alpha,\beta\,;\gamma\,;z)$ and $F(\alpha\,;\gamma\,;z)$ (cf. [3, p. 62, 2.1.3, (15), and p. 256, 6.5, (4)]), we can show by the same path shifting argument as in Section 2 that

$$(4.5)\quad \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)}\mathscr{F}_\nu(\alpha,\beta\,;\gamma\,;z)=\frac{1}{2\pi i} \times\int_{(b)}\frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(-s)}{\Gamma(\gamma+s)}\zeta(\nu+s)(-z)^s ds$$

for $\mathrm{Re}\,\nu>1$, $\max(-\mathrm{Re}\,\alpha, -\mathrm{Re}\,\beta, 1-\mathrm{Re}\,\nu)<b<0$ and $|\arg(-z)|<\pi$, and

$$(4.6)\quad \frac{\Gamma(\alpha)}{\Gamma(\gamma)}\mathscr{F}_\nu(\alpha\,;\gamma\,;z)=\frac{1}{2\pi i} \times\int_{(c)}\frac{\Gamma(\alpha+s)\Gamma(-s)}{\Gamma(\gamma+s)}\zeta(\nu+s)(-z)^s ds$$

for $\mathrm{Re}\,\nu>1$, $\max(-\mathrm{Re}\,\alpha, 1-\mathrm{Re}\,\nu)<c<0$ and $|\arg(-z)|<\pi/2$.

Formulae (4.1)-(4.6) are fundamental in deriving various properties of $\mathscr{F}_\nu(\alpha,\beta\,;\gamma\,;z)$ and $\mathscr{F}_\nu(\alpha\,;\gamma\,;z)$. Further investigations and detailed proofs will be given in forthcoming papers.

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