Power Shaping

Ortega, Romeo; Jeltsema, Dimitri; Scherpen, Jacquelien M.A.

Published in:
Proceedings of the Latinamerican Congress on Automatic Control, CLCA 2002

Please refer to this paper as:
R. Ortega, D. Jeltsema, J.M.A. Scherpen, "Power-shaping: A new paradigm for stabilization of nonlinear RLC circuits", Proc. Latimamerican Congress on Automatic Control, CLCA2002, Oaxaca, Mexico, December 2002, nr. 72.

POWER SHAPING: A NEW PARADIGM FOR STABILIZATION OF NONLINEAR RLC CIRCUITS *

Romeo Ortega$^{\flat}$, Dimitri Jeltsema‡ and Jacquelien M.A. Scherpen‡

$^{\flat}$*Laboratoire des Signaux et Systémes*
Supelec, Plateau du Moulon
91192 Gif-sur-Yvette, France

‡*Systems and Control Engineering Group*
Delft University of Technology
P.O. Box 5031, 2600 GA Delft, The Netherlands

Abstract: In this paper we prove that for a class of RLC circuits with convex energy function and weak electromagnetic coupling it is possible to "add a differentiation" to the port terminals preserving passivity—with a new storage function that is directly related to the circuit power. The result is of interest in circuits theory, but also has applications in control problems as it suggests the paradigm of power shaping stabilization as an alternative to the well–known method of energy shaping. We show in the paper that, in contrast with energy shaping designs, power shaping is not restricted to systems without pervasive dissipation and naturally allows to add "derivative" actions in the control. These important features, that stymie the applicability of energy shaping control, make power shaping very practically appealing, as illustrated with examples in the paper.

1. INTRODUCTION

Passivity is a fundamental property of dynamical systems that constitutes a cornerstone for many major developments in systems and control theory, including optimal ($\mathcal{H}_2$ and $\mathcal{H}_\infty$) control, realization theory and adaptive control. Passivity has also been instrumental to reformulate, in an elegant and unifying manner, the central problem of feedback stabilization—either in its form of feedback passivation for general nonlinear systems (Byrnes *et al.*, 1991; Jankovic *et al.*, 1996) or as energy–shaping control for systems with physical structures (Ortega *et al.*, 1998).

In this paper we are interested in (possibly nonlinear) RLC circuits consisting of arbitrary interconnections of resistors, inductors, capacitors and voltage and current sources. It is well–known that, if the resistors, inductors and capacitors are passive, *i.e.*, if their energy functions are positive, then the overall interconnected circuit is also passive with port variables the external sources voltages and currents, and storage function the total stored energy (Desoer and Kuh, 1969). This property was exploited by Youla in 1959 (Youla *et al.*, 1959) who proved that terminating the port variables of a passive RLC circuit with a passive resistor would ensure that "finite energy inputs will be mapped into finite energy outputs," what in modern parlance says that adding damping injection to a passive system ensures $\mathcal{L}_2$–stability. Passivity can also be used to stabilize a non–zero equilibrium point, but in this case we must modify the storage function to assign a minimum at this point. If the storage function is the total energy we refer to this step as energy shaping, which combined with damping injection constitute the two main stages of passivity–based control (PBC) (Ortega and Spong, 1989). As explained in (Ortega *et al.*, 1998) there are several ways to achieve energy shaping, the most physically appealing being the so–called energy balancing PBC (or control by interconnection) method. With this procedure the storage function assigned to the closed–loop passive map is the difference between the total energy of the system and the energy supplied by the controller, hence the name energy balancing. Unfortunately, energy balancing PBC is stymied by the presence of pervasive dissipation, that is, the existence of resistive elements whose power does not vanish at the desired equilibrium point. Another practical drawback of energy–shaping control is the limited ability to "speed up" the transient response (preserving, of course, a provable stable behavior.) Indeed, as tuning in this kind of controllers is essentially restricted to the damping injection gain, the transients may turn out to be somehow sluggish, and the overall performance level below par.

Our main contribution in this paper is the establishment of a new passivity property for a class of RLC circuits that provides the basis for a novel PBC design methodology that does not suffer from the two aforementioned drawbacks. To define the class, we assume that the energy of the inductors and capacitors are not just positive but actually *convex* functions, and assume that the electromagnetic coupling between the dynamic elements is weak. Indeed, for the case of RC or RL circuits this condition is conspicuous

* This work has been done in the context of the European sponsored project GeoPlex with reference code IST-2001-34166. Further information is available at http://www.geoplex.cc.

by its absence—as already reported in (Ortega and Shi, 2002).

The new passivity property, which is by itself of interest in circuits theory, has two key features that makes it attractive for control design as well. First, that the storage function is not the total energy, but a function directly related with the *power* in the circuit. Second, that the port variables of the new passive system include *derivatives* of the sources voltages and/or currents. The utilization of power (instead of energy) storage functions immediately suggests the paradigm of power shaping stabilization as an alternative to the well–known method of energy shaping. We show in the paper that, in contrast with energy shaping designs, power shaping is applicable also to systems with pervasive dissipation, the only restriction for stabilization being the degree of underactuation of the circuit. Further, establishing passivity with respect to "differentiated" port variables allows the direct incorporation of (approximate) derivative actions, whose predictive nature can speed–up the transient response.

The remaining of the paper is organized as follows. In Section 2 we briefly review the method of energy balancing passivity-based control (EB-PBC). Next, in Section 3, a simple RL-circuit example is presented to motivate the concept of stabilization via power shaping. To generalize the ideas to a broad class of RLC we need some preliminary material from the ground breaking paper (Brayton and Moser, 1964), that is introduced in Section 4. Finally, we present the main result in Section 5.

2. ENERGY BALANCING PASSIVITY–BASED CONTROL

In (Ortega *et al.*, 2001) we presented a new method to stabilize the following class of nonlinear systems—that includes passive systems.

Definition 1. We say that the m–port system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} \\ \mathbf{y} &= \hat{\mathbf{y}}(\mathbf{x}), \end{aligned} \tag{1}$$

with state $\mathbf{x} = \mathrm{col}(x_1, \dots, x_n) \in \mathbb{R}^n$, and power port variables $\mathbf{u}, \mathbf{y} \in \mathbb{R}^m$, satisfies the energy balance inequality if, along all trajectories compatible with $\mathbf{u}(\cdot) : [0, t] \to \mathbb{R}^m$, we have

$$\underbrace{\mathcal{E}[\mathbf{x}(t)] - \mathcal{E}[\mathbf{x}(0)]}_{\text{stored energy}} \leq \underbrace{\int_0^t \mathbf{u}^\top(t')\hat{\mathbf{y}}[\mathbf{x}(t')]dt'}_{\text{supplied energy}} \tag{2}$$

where $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}$ is the stored energy function. If $\mathcal{E}(\mathbf{x})$ is positive semidefinite then we say that the system is *passive* with port variables $(\mathbf{u}, \mathbf{y})$.

The proposition below, established in (Ortega *et al.*, 2001), constitutes the basis for energy–balancing PBC. (For simplicity, we present only the case of static state feedback, the case of dynamic controllers may be found in (Ortega *et al.*, 2001).)

Proposition 1. Consider m–port systems that satisfy the energy balance equation (2). If we can find a vector function $\hat{\mathbf{u}} : \mathbb{R}^n \to \mathbb{R}^m$ such that the partial differential equation

$$\frac{\partial^\top \mathcal{E}_a}{\partial \mathbf{x}}(\mathbf{x})[\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\hat{\mathbf{u}}(\mathbf{x})] = -\hat{\mathbf{y}}^\top(\mathbf{x})\hat{\mathbf{u}}(\mathbf{x}), \tag{3}$$

can be solved for the scalar function $\mathcal{E}_a : \mathbb{R}^n \to \mathbb{R}$, and the function $\mathcal{E}_d(\mathbf{x}) := \mathcal{E}(\mathbf{x}) + \mathcal{E}_a(\mathbf{x})$ has an isolated minimum at $\mathbf{x}^\star$, then the state–feedback $\mathbf{u} = \hat{\mathbf{u}}(\mathbf{x})$ is an energy balancing PBC, i.e., $\mathbf{x}^\star$ is a stable equilibrium with the difference between the stored and the supplied energies constituting a Lyapunov function.

This result, although quite general, is of limited interest. First of all, these kind of state models do not reveal the role played by the energy function in the system dynamics. Hence it is difficult to incorporate prior information to select a $\hat{\mathbf{u}}(x)$ to solve the PDE (3). In (Ortega *et al.*, 2002) energy balancing PBC is developed for a more suitable class of models, the so–called port–controlled Hamiltonian systems, that explicitly exhibit the existence of dynamic invariants. Second, and perhaps more importantly, it is shown in (Ortega *et al.*, 1998) that, beyond the realm of mechanical systems, the applicability of energy balancing control is severely stymied by the system's natural dissipation. Indeed, it is easy to see that a necessary condition for the *global* solvability of the PDE (3) is that $\hat{\mathbf{y}}^\top(\mathbf{x})\hat{\mathbf{u}}(\mathbf{x})$ vanishes at all the zeros of $\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\hat{\mathbf{u}}(\mathbf{x})$, that is, the implication

$$\mathbf{f}(\bar{\mathbf{x}}) + \mathbf{g}(\bar{\mathbf{x}})\hat{\mathbf{u}}(\bar{\mathbf{x}}) = 0 \Rightarrow \hat{\mathbf{y}}^\top(\bar{\mathbf{x}})\hat{\mathbf{u}}(\bar{\mathbf{x}}) = 0 \tag{4}$$

should hold (We will denote with $\mathbf{x}^\star$ the desired equilibrium). Now, $\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\hat{\mathbf{u}}(\mathbf{x})$ is obviously zero at the equilibrium $\mathbf{x}^\star$, hence the power extracted from the controller should also be zero at the equilibrium. This means that energy balancing PBC is applicable only if the system does not have pervasive damping, *i.e.*, if it can be stabilized extracting a finite amount of energy from the controller. This is the case in regulation of mechanical systems where the extracted power is the product of force and velocity and we want to drive the velocity to zero. Unfortunately, it is no longer the case for most electrical or electromechanical systems where power involves the product of voltages and currents and the latter may be nonzero for nonzero equilibria. For instance, a series RC circuit is energy–balancing stabilizable (because in steady state there is no current drained from the source), but not an RL circuit—see the following section.

Remark 1. For linear systems it is, of course, possible to overcome the dissipation obstacle by shifting the equilibrium of the systems equation to zero. As the terms dependent on $\mathbf{x}^\star, \mathbf{u}^\star$ cancel in the incremental model, the original (quadratic) storage function—but expressed now in terms of the incremental variables—qualifies as a storage function for the shifted model. Unfortunately, this simple solution is not applicable for the nonlinear case, as there is no systematic procedure to generate, from the knowledge of $\mathcal{E}(\mathbf{x})$, a storage function for the "input–shifted" system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}^\star + \mathbf{g}(\mathbf{x})\mathbf{w}, \ \mathbf{y} = \hat{\mathbf{y}}(\mathbf{x}),$$

with $\mathbf{w} := \mathbf{u} - \mathbf{u}^\star$, and $(\mathbf{w}, \mathbf{y})$ the new port variables. As shown in (Maschke *et al.*, 2000) the natural solution of adding to $\mathcal{E}(\mathbf{x})$ a term $-\int_0^t \mathbf{w}^\top(t')\hat{\mathbf{y}}[\mathbf{x}(t')]dt'$ is also restricted to systems without pervasive damping.

3. TOWARDS POWER SHAPING CONTROL

Let us illustrate with an example how the limitations of energy balancing PBC can be overcome via power balancing. Consider a voltage–controlled nonlinear series RL circuit as shown in Figure 1.

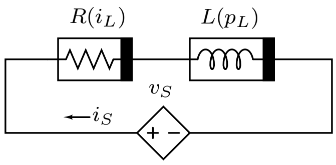

Fig. 1. Nonlinear RL circuit.

The behavior of the inductor is characterized by a function, $p_L = \hat{p}_L(i_L)$, relating the flux linkages p_L and the current i_L, and Faraday's law: $\dot{p}_L = v_L$, where v_L is the inductor voltage. The resistor is a static element described by its characteristic function $v_R = \hat{v}_R(i_R)$, where v_R, i_R are the resistors voltage and current, respectively. The dynamics of the circuit is obtained from Kirchhoff's voltage law as

$$v_L = L(i_L)\frac{di_L}{dt} = -\hat{v}_R(i_L) + v_S, \tag{5}$$

where v_S is the voltage at the port terminal, which is our control action, we used $i_R = i_L$, and defined $L(i_L) := \frac{\partial}{\partial i_L}\hat{p}_L(i_L)$.

The energy stored in an inductor, $\mathcal{E}_L(p_L)$, is related with the current via the relation $i_L = \frac{\partial}{\partial p_L}\mathcal{E}_L(p_L)$. Differentiating the inductor's energy we obtain

$$\begin{aligned}\dot{\mathcal{E}}_L(p_L) &= \frac{\partial \mathcal{E}_L}{\partial p_L}(p_L)\dot{p}_L \; (= i_L v_L) \\ &= i_S v_S - i_R \hat{v}_R(i_R),\end{aligned}$$

where, to obtain the last equation, we used the fact that i_S, the port current, is equal to i_L. If we assume that the resistor is passive, that is, that the energy that it dissipates is nonnegative, i.e., $\int_0^t i_R(t')\hat{v}_R[i_R(t')]dt' \geq 0$, and integrate from 0 to t, we recover the energy balance inequality (2). If we further assume that the inductor is also passive—that is, its stored energy is nonnegative—we verify that the circuit defines a passive system with port variables (v_S, i_S) and storage function $\mathcal{E}_L(p_L)$.

We define as control objective the stabilization of an equilibrium $i_L^\star$ of (5), whose corresponding equilibrium supply voltage is given by $v_S^\star = \hat{v}_R(i_L^\star)$. If we further assume that the function $\hat{v}_R(i_R)$ is zero only at zero, it is clear that, at any equilibrium $i_L^\star \neq 0$, the extracted power $i_L^\star \hat{v}_R(i_L^\star)$ is nonzero, hence the circuit is not energy–balancing stabilizable—not even in the linear case!

To overcome this problem let us define the function

$$F(i_R) := \int_0^{i_R} \hat{v}_R(i'_R)di'_R, \tag{6}$$

known in the circuits literature (Millar, 1951) as the resistors *content*, which has units of power—in particular, for linear resistors it is half the dissipated power. Furthermore, notice that for passive resistors the function is nonnegative. Summarizing, we have the following result.

Proposition 2. Consider the RL circuit of Figure 1. If the inductor is passive and has a twice differentiable *convex* energy function, that is,

$$\frac{\partial^2 \mathcal{E}_L}{\partial p_L^2}(p_L) \geq 0,$$

then, along the trajectories of the system, we have the power balance inequality

$$F[i_L(t)] - F[i_L(0)] \leq \int_0^t v_S^\top(t')\frac{di_S}{dt'}(t')dt'. \tag{7}$$

Furthermore, if the resistor is passive, then the circuit defines a passive system with port variables $(v_S, \frac{di_S}{dt})$ and storage function the resistor content.

The properties of Proposition 2 differ from the classical energy balancing and passivity properties in two important respects: the presence of the derivative of i_S and the use of a new power–like storage function. These two properties suggest, similarly to energy balancing PBC, to shape the resistors content. That is, to look for functions $\hat{v}_S(i_L), F_a(i_L)$ such that

$$\dot{F}_a(i_L) \equiv -\hat{v}_S(i_L)\frac{di_L}{dt}. \tag{8}$$

If we furthermore ensure that

$$i_L^\star = \arg\min\{F(i_L) + F_a(i_L)\},$$

then i_{L*} will be a stable equilibrium with Lyapunov function $F_d(i_L) := F(i_L)+F_a(i_L)$, that is, the system is stabilized via power shaping!

Clearly, for any choice of $F_a(i_L)$, (8) is trivially solved with the control $v_S = \hat{v}_S(i_L)$, where

$$\hat{v}_S(i_L) = -\frac{\partial F_a}{\partial i_L}(i_L).$$

If the resistance characteristic is exactly known we can take $F_a(i_L) = -F(i_L) + \frac{R_a}{2}(i_L - i_L^\star)^2$, with $R_a > 0$ some tuning parameter. But clearly, we only need to "dominate" $F(i_L)$ to assign the desired minimum, which (together with the fact that $L(i_L)$ is completely unknown) exhibits the robustness of the design procedure.

Detailed proofs for general RL and RC circuits can be found in (Ortega and Shi, 2002). An important observation, that will be proved for more general nonlinear RLC circuits in the following section, is that we can express the circuit dynamics (5) in terms of the resistor content as

$$L(i_L)\frac{di_L}{dt} = -\frac{\partial F}{\partial i_L}(i_L) + v_S.$$

The identification of a gradient–like description of RLC circuits is the main contribution of the seminal paper (Brayton and Moser, 1964).

4. PASSIVITY OF BRAYTON-MOSER CIRCUITS

The previous developments show that, using the content (resp. co-content in the RC case (Ortega and Shi, 2002)) as storage functions, we can identify new passivity properties of RL and RC circuits. In this section we will establish similar properties for RLC circuits. Towards this end, we strongly rely on some fundamental results reported in (Brayton and Moser, 1964). Furthermore, we assume that the current-controlled resistors are contained in Σ_L and the voltage-controlled resistors are contained in Σ_C. The class of RLC considered here is then composed by an interconnection of Σ_L and Σ_C.

4.1 Brayton and Moser's Equations

In the early sixties Brayton and Moser (Brayton and Moser, 1964) have shown that the dynamic behavior of a topologically complete circuit (where we restrict, for simplicity, to circuits having only voltage sources in series with the inductors) is governed by the following differential equation

$$\mathbf{Q}(\mathbf{x})\dot{\mathbf{x}} = \frac{\partial P}{\partial \mathbf{x}}(\mathbf{x}) - \mathbf{B}\mathbf{v}_S \tag{9}$$

where $\mathbf{x} = \mathrm{col}(\mathbf{i}_L, \mathbf{v}_C)$, $\mathbf{B} = \mathrm{col}(\mathbf{B}_S, 0)$ with $\mathbf{B}_S \in \mathbb{R}^{n_L \times n_S}$, $\mathbf{Q}(x) = \mathrm{diag}(-\mathbf{L}(i_L), \mathbf{C}(v_C)) \in \mathbb{R}^{n\times n}$, $n = n_L + n_C$, and $P : \mathbb{R}^n \to \mathbb{R}$ is called the mixed-potential and is given by

$$P(\mathbf{x}) = \mathbf{i}_L^\top \mathbf{\Gamma}\mathbf{v}_C + F(\mathbf{i}_L) - G(\mathbf{v}_C), \tag{10}$$

where $\mathbf{\Gamma} \in \mathbb{R}^{n_L \times n_C}$ is a (full rank) matrix that captures the interconnection structure between the inductors and capacitors. The functions $F(\mathbf{i}_L)$ and $G(\mathbf{v}_C)$ are the resistors content (like in (6)) and co-content, respectively.

4.2 Generation of New Storage Function Candidates

Let us next see how the Brayton-Moser equations (9) can be used to generate storage functions for RLC circuits. From (9) we have that

$$\dot{P}(\mathbf{x}) = \dot{\mathbf{x}}^\top \mathbf{Q}(\mathbf{x})\dot{\mathbf{x}} + \dot{\mathbf{x}}^\top \mathbf{B}\mathbf{v}_S. \tag{11}$$

Compare the latter with the right-hand side of (7) of Proposition 2 (notice that $\dot{\mathbf{x}}^\top \mathbf{B}\mathbf{v}_S = \mathbf{i}_S^\top \mathbf{v}_S$). Unfortunately, even under the reasonable assumption that the inductor and capacitor have convex energy functions, the presence of the negative sign in the first main diagonal block of $\mathbf{Q}(\mathbf{x})$ makes the quadratic form sign–indefinite, and not negative (semi–)definite as desired. Hence, we cannot establish a power-balance inequality from (11). Moreover, to obtain the passivity property an additional difficulty stems from the fact that $P(\mathbf{x})$ is also not sign-definite. To overcome these difficulties we borrow inspiration from (Brayton and Moser, 1964) and look for other suitable pairs, say $\mathbf{Q}_A(\mathbf{x})$ and $P_A(\mathbf{x})$, which we call *admissible*, that preserve the form of (9). More precisely, we want to find matrix functions $\mathbf{Q}_A(\mathbf{x})$ verifying

$$\mathbf{Q}_A^\top(\mathbf{x}) + \mathbf{Q}_A(\mathbf{x}) \leq 0, \tag{12}$$

and scalar functions $P_A : \mathbb{R}^n \to \mathbb{R}$ (if possible, positive semi-definite). If (12) holds, it is clear that $\dot{P}_A(\mathbf{x}) \leq \dot{\mathbf{x}}^\top \mathbf{B}\mathbf{v}_S$, from which we obtain a power balance equation with the desired port variables. Furthermore, if $P_A(\mathbf{x})$ is positive semi-definite we are able to establish the required passivity property.

In the proposition below we will provide a complete characterization of the admissible pairs $\mathbf{Q}_A(\mathbf{x})$ and $P_A(\mathbf{x})$. For, we find it convenient to use the general form, i.e., $\mathbf{Q}(\mathbf{x})\dot{\mathbf{x}} = \frac{\partial}{\partial \mathbf{x}}\tilde{P}(\mathbf{x})$, where for the case considered here $\tilde{P}(\mathbf{x}) = P(\mathbf{x}) - \mathbf{x}^\top \mathbf{B}\mathbf{v}_S$.

Proposition 3. For any $\lambda \in \mathbb{R}$ and any constant symmetric matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$

$$\tilde{P}_A = \frac{1}{2}\frac{\partial^\top \tilde{P}}{\partial \mathbf{x}}\mathbf{M}\frac{\partial \tilde{P}}{\partial \mathbf{x}}(\mathbf{x}) + \lambda \tilde{P}.$$

$$\mathbf{Q}_A = \left[\lambda \mathbf{I} + \frac{1}{2}\frac{\partial^2 \tilde{P}}{\partial \mathbf{x}^2}\mathbf{M} + \frac{1}{2}\frac{\partial}{\partial \mathbf{x}}\left(\mathbf{M}\frac{\partial \tilde{P}}{\partial \mathbf{x}}\right)\right]\mathbf{Q}.$$

An important observation regarding Proposition 3 is that, for suitable choices of λ and $\mathbf{M}$, we can now try to generate a matrix $\mathbf{Q}_A(\mathbf{x})$ with the required negativity property (12).

4.3 Power-Balance Inequality

Before we present our main result we first remark that in order to preserve the port variables $(\mathbf{v}_S, \frac{d\mathbf{i}_s}{dt})$, we must ensure that the transformed dynamics can be expressed in the form (9), which is equivalent to requiring that $\tilde{P}_A(\mathbf{x}) = P_A(\mathbf{x}) - \mathbf{x}^\top \mathbf{B}\mathbf{v}_S$. This naturally restricts the freedom in the choices for λ and $\mathbf{M}$ in Proposition 3.

Theorem 1. Consider a (possibly nonlinear) RLC circuit satisfying (9). Assume:

- A.1 The inductors and capacitors are passive and have strictly convex energy functions.
- A.2 The voltage-controlled resistors in Σ_C are passive, linear and time-invariant. Also, $\det(\mathbf{R}_C) \neq 0$, and thus $G(\mathbf{v}_C) = \frac{1}{2}\mathbf{v}_C^\top \mathbf{R}_C^{-1}\mathbf{v}_C \geq 0$.
- A.3 Uniformly in $\mathbf{x}$ we have
$$\left\|\mathbf{C}^{\frac{1}{2}}(\mathbf{v}_C)\mathbf{R}_C\mathbf{\Gamma}^\top \mathbf{L}^{-\frac{1}{2}}(\mathbf{i}_L)\right\| < 1,$$
where $||\cdot||$ denotes the spectral norm of a matrix.

Under these conditions, we have the following power balance inequality

$$P_A[\mathbf{x}(t)] - P_A[\mathbf{x}(0)] \leq \int_0^t \mathbf{v}_S^\top(t')\frac{d\mathbf{i}_S}{dt'}(t')dt', \tag{13}$$

where the transformed mixed-potential function is defined as

$$P_A(\mathbf{x}) = F(\mathbf{i}_L) + \tfrac{1}{2}\mathbf{i}_L\Gamma\mathbf{R}_C\Gamma^\top \mathbf{i}_L + \tfrac{1}{2}(\mathbf{\Gamma}^\top \mathbf{i}_L - \mathbf{R}_C^{-1}\mathbf{v}_C)^\top \mathbf{R}_C(\Gamma^\top \mathbf{i}_L - R_C^{-1}\mathbf{v}_C).$$

If, furthermore

A.4 The current-controlled resistors are passive,

then, the circuit defines a passive system with port variables $(\mathbf{v}_S, \frac{d\mathbf{i}_S}{dt})$ and storage function the transformed mixed-potential $P_A(\mathbf{x})$.

Proof. The proof consists in first defining the parameters λ and $\mathbf{M}$ of Proposition 3 so that, under the conditions A.1–A.4 of the theorem, the resulting $\mathbf{Q}_A(\mathbf{x})$ satisfies (12) and $P_A(\mathbf{x})$ is a positive semi-definite function. First, notice that under assumption A.2 the co-content is linear and quadratic. To ensure that $P(\mathbf{x})$ is linear in v_S, as is required to preserve the desired port variables, we may select $\lambda = 1$ and $\mathbf{M} = \mathrm{diag}(0, 2\mathbf{R}_C)$. Now, using (13) we obtain after some straight forward calculations

$$\mathbf{Q}_A(\mathbf{x}) = \begin{bmatrix} -\mathbf{L}(\mathbf{i}_L) & 2\mathbf{R}_C\mathbf{\Gamma}\mathbf{C}(\mathbf{v}_C) \\ 0 & -\mathbf{C}(\mathbf{v}_C) \end{bmatrix}.$$

Assumption A.1 ensures that $\mathbf{L}(\mathbf{i}_L)$ and $\mathbf{C}(\mathbf{v}_C)$ are positive definite. Hence, a Schur complement analysis proves that, under Assumption A.3, (13) holds. This proves the power balance inequality. Passivity follows from the fact that, under Assumption A.2 and A.4, the mixed-potential function $P_A(\mathbf{x})$ is positive semi-definite for all $\mathbf{x}$. This completes the proof. $\triangleleft$

Remark 2. Assumption A.3 is satisfied if the voltage-controlled resistances in $\mathbf{R}_C$ are 'small'. Recalling that these resistors are contained in Σ_C, this means that the coupling between Σ_L and Σ_C, that is, the coupling between the inductors and capacitors, is weak.

5. STABILIZATION VIA POWER SHAPING

The theorem below proves that complete RLC circuits with strictly convex energy function and linear voltage controlled resistors are stabilizable via power–shaping provided the number of control signals is 'sufficiently large' to shape the mixed potential function and add the damping.

Theorem 2. Consider a complete RLC circuit satisfying Assumptions A.1 and A.2 of Theorem 1, and a desired (admissible) equilibrium $\mathbf{x}^\star \in \mathbb{R}^n$. Assume we can find a function $P_a : \mathbb{R}^{n_L} \to \mathbb{R}$, verifying:

A.5 *(Realizability)*

$$\mathbf{B}_S^\perp \frac{\partial P_a}{\partial \mathbf{i}_L}(\mathbf{i}_L) = 0, \tag{14}$$

where $\mathbf{B}_S^\perp \mathbf{B}_S = 0$.

A.6 *(Equilibrium assignment)*

$$\frac{\partial}{\partial \mathbf{i}_L}[P_a(\mathbf{i}_L) + F(\mathbf{i}_L)] + \mathbf{\Gamma}\tilde{\mathbf{R}}_C\mathbf{\Gamma}^\top \mathbf{i}_L = 0 \tag{15}$$

verifies that $\mathbf{i}_L = \mathbf{i}_L^\star$.

A.7 *(Damping injection)*

$$\mathbf{\Psi}(\mathbf{i}_L) := \frac{\partial^2}{\partial \mathbf{i}_L^2}[P_a(\mathbf{i}_L) + F(\mathbf{i}_L)] \geq R_a\mathbf{I}, \tag{16}$$

for some sufficiently large $R_a > 0$, so that

$$\left(\tfrac{2}{R_a}\mathbf{\Psi} - \mathbf{I}\right)\mathbf{L} + \mathbf{L}\left(\tfrac{2}{R_a}\mathbf{\Psi} - \mathbf{I}\right) > \tfrac{2}{R_a^2}\mathbf{L}\mathbf{\Gamma}\mathbf{C}^{-1}\mathbf{\Gamma}^\top\mathbf{L}. \tag{17}$$

If, the current controlled resistors are linear and we take P_a quadratic we can simplify the condition above to

$$R_a > \left\|\mathbf{L}^{\frac{1}{2}}(\mathbf{i}_L)\mathbf{\Gamma}\mathbf{C}^{-\frac{1}{2}}(\mathbf{v}_C)\right\|. \tag{18}$$

Under these conditions, the circuit is stabilizable via *power shaping* with the control law

$$\mathbf{v}_S = -\left(\mathbf{B}_S^\top\mathbf{B}_S\right)^{-1}\mathbf{B}_S^\top \frac{\partial P_a}{\partial \mathbf{i}_L}(\mathbf{i}_L) \tag{19}$$

$(\mathbf{i}_L^\star, \mathbf{v}_C^\star)$ is an *asymptotically stable* equilibrium of the closed–loop system with Lyapunov function $P_d(\mathbf{x}) := P_A(\mathbf{x}) + P_a(\mathbf{i}_L)$. Moreover, an estimate of the *domain of attraction* is given by $\Omega_{\bar{c}}$, where

$$\Omega_c := \{\mathbf{x} \in \mathbb{R}^n \mid \tilde{P}_d(\mathbf{x}) \leq c\}$$

are the sub–level sets of $P_d(\mathbf{x})$, and

$$\bar{c} := \sup\{c > P_d(\mathbf{x}^\star) \mid \Omega_c \text{ is bounded}\}.$$

Proof. A sketch of the proof is as follows. We know that the circuit dynamics is described by (9). Now, under condition (14) of Assumption A.5, the control law (19) satisfies $\mathbf{B}_S\mathbf{v}_S = -\frac{\partial}{\partial \mathbf{i}_L}P_a(\mathbf{i}_L)$. This leads to the closed-loop dynamics of the form

$$\mathbf{Q}_A(\mathbf{x})\dot{\mathbf{x}} = \frac{\partial P_d}{\partial \mathbf{x}}(\mathbf{x}), \tag{20}$$

which under Assumption A.6, satisfies

$$\left.\frac{\partial P_d}{\partial \mathbf{x}}(\mathbf{x})\right|_{\mathbf{x}=\bar{\mathbf{x}}} = 0 \Rightarrow \mathbf{x} = \mathbf{x}^\star.$$

Stability of (20) is determined by invoking Proposition 3, where we have now taken $\lambda = -1$ and $\mathbf{M} = \mathrm{diag}(\frac{2}{R_a}\mathbf{I}, 0)$ to ensure that (12) holds. Hence, along the closed-loop dynamics we have

$$\dot{P}_d(\mathbf{x}) \leq -\alpha\left|\frac{\partial P_d}{\partial \mathbf{x}}(\mathbf{x})\right|^2,$$

for some $\alpha > 0$. Asymptotic stability follows immediately from the fact that $|\frac{\partial}{\partial \mathbf{x}}P_d(\mathbf{x})| = 0$ only at the equilibrium. Finally, as the sub–level sets Ω_c are invariant, we conclude invoking La Salle's theorem that any trajectory starting in a bounded sub–level set will converge to the equilibrium. $\triangleleft$

Remark 3. For the sake of simplicity we have chosen constant $\mathbf{M}$ in Theorems 1 and 2. Using state dependent matrices we can relax the conditions of the theorems. For instance, we can relax the strong linearity Assumption A.2 and replace it with

A.2' The characteristic function of the resistors in Σ_C, i.e., $\mathbf{i}_{R_C} = \hat{\mathbf{i}}_{R_C}(\mathbf{v}_{R_C})$, are *strictly increasing*,

which ensures that $\frac{\partial^2 G}{\partial \mathbf{v}_C^2}(\mathbf{v}_C)$ is invertible. Then, take

$$\mathbf{M} = \begin{bmatrix} 0 & 0 \\ 0 & 2\mathbf{\Theta}(\mathbf{x})\left(\frac{\partial^2 J}{\partial \mathbf{v}_C^2}\right)^{-1} \end{bmatrix}, \tag{21}$$

where $\mathbf{\Theta} : \mathbb{R}^{n_C} \to \mathbb{R}$ is a function to be defined.

Example. Consider the nonlinear RLC circuit depicted in Figure 2. We assume that the capacitor is voltage-controlled and the inductor is current-controlled. Suppose that the voltage-controlled resistor is described by a nonlinear function

$$\hat{i}_R(v_R) = R_o^{-1} v_R^3, \quad R_o > 0.$$

Hence, Assumption A.2 is not satisfied. However, Assumption A.2' of Remark 3 holds, and we will prove below that we are still be able to derive a passivity property for the circuit and stabilize the equilibrium points via power shaping.

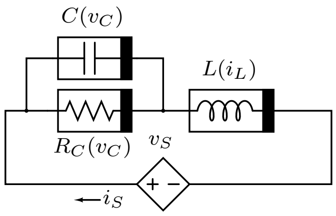

Fig. 2. Nonlinear RLC circuit.

Furthermore, it is easily seen that $\mathbf{\Gamma} = 1$, $F(i_L) = 0$ and $G(v_C) = \frac{1}{R_o}\int_0^{v_C}(v'_C)^3 dv'_C$, and thus the mixed potential function becomes

$$P(i_L, v_C) = i_L v_C - \frac{1}{4R_o} v_C^4. \tag{22}$$

The equilibrium points for this system lie in the set $\{(\bar{i}_L, \bar{v}_C) \mid R_o \bar{i}_L = \bar{v}_C^3, |\bar{i}_L| < \beta\}$, with the equilibrium source voltage $\bar{v}_S = \bar{v}_C$. It is easy to see that for all (non–zero) equilibrium states there is a current flowing through the resistor. Consequently, implication (4) is not satisfied and the circuit is not stabilizable with energy balancing. Let us now derive the power balance inequality. For that, we follow the procedure proposed in Remark 3 and select $\mathbf{\Theta}(v_C) = \frac{3}{R_o} v_C^2$, so that $\mathbf{M} = \mathrm{diag}\{0, 2\}$, and fix $\lambda = 1$, to get

$$\mathbf{Q}_A + \mathbf{Q}_A^\top = \begin{bmatrix} -2L & C - L \\ C - L & 2C\left(1 - \frac{6v_C^2}{R_o}\right) \end{bmatrix},$$

which is negative definite in the neighborhood of the desired equilibrium point of the form $\mathcal{B}_\delta := \{(i_L, v_C) \mid (i_L - i_L^\star)^2 + (v_C - v_C^\star)^2 \leq \delta\}$. We then have that the power balance inequality of Theorem 1 holds for all source voltages v_S that preserve the trajectories of the circuit inside the ball $\mathcal{B}_\delta$. Furthermore, it can be shown that, for sufficiently small δ, P_A is positive semi-definite thus we can also conclude (local) passivity for this circuit. Since $\mathbf{B}_S = 1$, the realizability condition (14) is obviated and we can select any arbitrary function $P_a(i_L)$. For simplicity, we propose $P_a = -v_C^\star i_L + \frac{1}{2}R_a(i_L - i_L^\star)^2$ to obtain, using (19), the control law $\hat{v}_S(i_L) = -R_a(i_L - i_L^\star) + v_C^\star$. As there are no current controlled resistors and P_a is quadratic, Assumption A.7 will hold if R_a satisfies (18), which in this example reduces to $R_a > (L/C)^{\frac{1}{2}}$.

6. CONCLUSION

Our main motivation in this paper was to propose an alternative to the well–known method of energy shaping stabilization of physical systems—which as pointed out in (Ortega *et al.*, 2002; Ortega *et al.*, 2001; Schaft, 2000) is severely stymied by the existence of pervasive damping. In this paper we have, for nonlinear RLC circuits, put forth the paradigm of power shaping and shown that it is not restricted to systems without pervasive dissipation. The starting point for the formulation of the power shaping idea are some new power balancing and passivity properties established for a class of nonlinear RLC circuits with convex energy function and weak electromagnetic coupling. To enlarge the class of circuits that enjoy these properties we have made extensive use of Proposition 3 which provides a procedure to generate alternative circuit topologies that reveal, through the new admissible pairs $(\mathbf{Q}_A, P_A)$, properties of the original circuit that we can exploit in our controller design. Future research includes the extension of our results beyond the realm of RLC circuits, e.g., to mechanical or electromechanical systems. A related question is whether we can find Brayton–Moser like models for this class of systems.

ACKNOWLEDGEMENTS

The first author would like to express his gratitude to Bertram Shi with whom this research started, Arjan van der Schaft for his insightful remarks on Brayton–Moser's paper, and Nikita Barabanov for help with Proposition 3. The second author would like to thank the CTS Control Training Site for making this collaboration possible.

REFERENCES

Brayton, R.K. and J.K. Moser (1964). A theory of nonlinear networks I. *Quart. Appl. Math.* **12**(1), 1–33.

Byrnes, C., A. Isidori and J.C. Willems (1991). Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems. *IEEE Trans Aut. Cont.* **36**(11), 1228–1240.

Desoer, C.A. and E.S. Kuh (1969). *Basic Circuit Theory*. McGraw–Hill, NY.

Jankovic, M., R. Sepulchre and P.V. Kokotović (1996). Constructive Lyapunov stabilization of nonlinear cascade systems. *IEEE Trans. on Aut. Cont.* **41**, 1723–1735.

Maschke, B.M., R. Ortega and A.J. van der Schaft (2000). Energy–based Lyapunov functions for forced Hamiltonian systems with dissipation. *IEEE Trans. on Aut. Cont.* **45**(8), 1498–1502.

Millar, W. (1951). Some general theorems for nonlinear systems possesing resistor. *Phil. Mag.* **42**, 1150–1160.

Ortega, R., A. Loría, P.J. Nicklasson and H. Sira-Ramírez (1998). *Passivity-Based Control of Euler-Lagrange Systems; Mechanical, Electrical and Electromechanical Applications*. Springer-Verlag.

Ortega, R., A.J. van der Schaft, B.M. Maschke and G. Escobar (2002). Interconnection and damping assignment passivity based control of port–controlled Hamiltonian systems. *Automatica*.

Ortega, R., A.J. van der Schaft, I. Mareels and B.M. Maschke (2001). Putting energy back in control. *IEEE Control Syst. Magazine* **21**(2), 18–33.

Ortega, R. and B.E. Shi (2002). A note on passivity of nonlinear RL and RC circuits. *IFAC World Conf. Barcelona, Spain*.

Ortega, R. and M. Spong (1989). Adaptive motion control of rigid robots: A tutorial. *Automatica* **25**(6), 877–888.

Schaft, A.J. van der (2000). *$\mathcal{L}_2$-Gain and Passivity Techniques in Nonlinear Control*. Springer-Verlag.

Youla, D., L. Castriota and H. Carlin (1959). Bounded real scattering matrices and the foundations of linear passive networks. *IRE Trans. Circuit Th.* **4(1)**, 102–124.