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Power Structure over the Grothendieck Ring of Varieties and Generating Series of Hilbert Schemes of Points

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Introduction

The Grothendieck semiring $S_0(\mathcal{V}_{\mathbb{C}})$ of complex quasi-projective varieties is the semigroup generated by isomorphism classes $[X]$ of such varieties modulo the relation $[X] = [X - Y] + [Y]$ for a Zariski closed subvariety $Y \subset X$; the multiplication is defined by the Cartesian product: $[X_1] \cdot [X_2] = [X_1 \times X_2]$. The Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ is the group generated by these classes with the same relation and the same multiplication. Let $\mathbb{L} \in K_0(\mathcal{V}_{\mathbb{C}})$ be the class of the complex affine line, and let $K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}]$ be the localization of Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ with respect to $\mathbb{L}$. A power structure over a (semi)ring R (as in [10]) is a map $(1 + T \cdot R[[T]]) \times R \to 1 + T \cdot R[[T]]$: $(A(T), m) \mapsto (A(T))^m$ $(A(T) = 1 + a_1T + a_2T^2 + \cdots, a_i \in R, m \in R)$ such that all usual properties of the exponential function hold. Over a ring R, a finitely determined (in a natural sense that we shall describe) power structure is defined by a pre-λ-ring structure on R (see [12]). Described in [10] is a power structure over each of the (semi)rings just defined. They are connected with the pre-λ-ring structure on the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ defined by the Kapranov zeta function [6; 11].

The main result of this paper is using the formalism of the power structure to express the generating series of classes (in the Grothendieck (semi)ring of varieties) of Hilbert schemes of zero-dimensional subschemes on a smooth quasi-projective variety of dimension d as an exponent of that for the complex affine space $\mathbb{A}^d$. The conjecture that the generating series of Hilbert schemes of points on a smooth surface can be considered as an exponent was communicated to the authors by D. van Straten. Specializations of this relation give formulas for generating series of certain invariants of the Hilbert schemes (Euler characteristic, Hodge–Deligne polynomial, ...).

We also describe a power structure over the ring $\mathbb{Z}[u_1, \dots, u_r]$ of polynomials in several variables with integer coefficients in such a way that, for $r = 2$, it is the specialization of the power structure over the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ under

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the Hodge–Deligne polynomial homomorphism. This gives the main result of [9] and [4] (in somewhat different terms).

In order to use the described relation to compute the generating series of classes of Hilbert schemes of zero-dimensional subschemes of a smooth variety, one needs to know the corresponding series for the affine space $\mathbb{A}^d$. This series is known only for the plane (i.e., for dimension 2) by a result of Ellingsrud and Strøme [5]. Its computation in the general case is an important and interesting question whose answer is not yet known. For example, in [4] the generating series of Hodge–Deligne polynomials of the discussed Hilbert schemes for a smooth variety is expressed in terms of the corresponding (unknown) series for the affine space $\mathbb{A}^d$.

1. Power Structures

Definition. A *power structure* over a (semi)ring R is a map

$$(1 + T \cdot R[[T]]) \times R \to 1 + T \cdot R[[T]]\colon (A(T), m) \mapsto (A(T))^m$$

that possesses the following properties:

1. $(A(T))^0 = 1$,
2. $(A(T))^1 = A(T)$,
3. $(A(T) \cdot B(T))^m = (A(T))^m \cdot (B(T))^m$,
4. $(A(T))^{m+n} = (A(T))^m \cdot (A(T))^n$,
5. $(A(T))^{mn} = ((A(T))^n)^m$,
6. $(1 + T)^m = 1 + mT +$ terms of higher degree,
7. $(A(T^k))^m = (A(T))^m|_{T \mapsto T^k}$.

Remark. In [10] the properties 6 and 7 were not demanded, though the constructed power structures possessed them.

Definition. A power structure is *finitely determined* if for each $i > 0$ there exists a $j > 0$ such that the i-jet of the series $(A(T))^m$ (i.e., $(A(T))^m \bmod T^{i+1}$) is determined by the j-jet of the series $A(T)$.

One can see that it is possible to take $j = i$.

Definition. A *pre-λ-ring* structure on a commutative ring R is an additive to multiplicative group homomorphism $\lambda_T\colon R \to 1 + T \cdot R[[T]]$; that is, $\lambda_T(m+n) = \lambda_T(m)\lambda_T(n)$ such that $\lambda_T(m) = 1 + mT \pmod{T^2}$. A pre-λ-ring homomorphism is a ring homomorphism between pre-λ-rings that commutes with the pre-λ-ring structures.

We shall use the following general statement.

Proposition 1. *A finitely determined power structure over a ring R is determined by a pre-λ-ring structure on the ring R.* (*In terms of the power structure, the series $\lambda_T(m)$ has the meaning of $(1 - T)^{-m}$ and we shall use that notation from now on.*)

Proof. By properties 6 and 7, each series $A(T) \in 1 + T \cdot R[[T]]$ can be in a unique way written as a product of the form $\prod_{i=1}^{\infty}(1 - T^i)^{-a_i}$ with $a_i \in R$. Then, by properties 3 and 7 (and the finite determinacy of the power structure),

$$(A(T))^m = \prod_{i=1}^{\infty}(1 - T^i)^{-a_i m}. \tag{1}$$

In the other direction, one can easily see that the power structure defined by equation (1) possesses properties 1–7. □

A ring homomorphism $\varphi\colon R_1 \to R_2$ induces the natural homomorphism $R_1[[T]] \to R_2[[T]]$ (also denoted by φ) by $\varphi\big(\sum a_i T^i\big) = \sum \varphi(a_i)T^i$. Proposition 1 yields the following statement.

Proposition 2. *A pre-λ-ring homomorphism $\varphi\colon R_1 \to R_2$ respects the corresponding power structure; that is, $\varphi((A(T))^m) = (\varphi(A(T)))^{\varphi(m)}$.*

On the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ of complex quasi-projective varieties there is a (natural) pre-λ-ring structure defined by $(1 - T)^{-[M]} = \zeta_{[M]}(T)$ for a quasi-projective variety M. Here $\zeta_{[M]}(T)$ is the Kapranov zeta-function of M: $\zeta_{[M]}(T) := 1 + [M] \cdot T + [S^2M] \cdot T^2 + [S^3M] \cdot T^3 + \cdots$, where $S^kM = M^k/S_k$ is the kth symmetric power of the variety M (see [6; 11]).

First, however, the description of the power structure through the pre-λ-ring structure does not permit a definition of the power structure over the Grothendieck semiring $S_0(\mathcal{V}_{\mathbb{C}})$. (And one can say that elements of $S_0(\mathcal{V}_{\mathbb{C}})$ have more geometric meaning: they are represented by "genuine" quasi-projective varieties, not by virtual ones.) Second, the geometric description of the power structure has its own value (e.g., for Theorem 1). Moreover, one can say that the geometric construction of the power structure in [10] defines the coefficients of the series $(A(T))^{[M]}$ more finely and so preserves more structures on them. For instance, if the coefficients of the series $A(T)$ and the exponent $[M]$ are represented by compact spaces, then coefficients of the series $(A(T))^{[M]}$ also can be considered as such. It seems that the geometric construction of the power structure can be adapted for and used in some settings that differ from the Grothendieck ring of varieties. We shall therefore describe the series $(A(T))^{[M]}$, where $A(T) = 1 + \sum_{i=1}^{\infty}[A_i] \cdot T^i$ and where A_i and M are quasi-projective varieties (in words somewhat different from those used in [10]).

It is convenient to describe the power structure on the Grothendieck semiring $S_0(\mathcal{V}_{\mathbb{C}})$ in terms of graded spaces (sets). A *graded space* (with grading from $\mathbb{Z}_{>0}$) is a space A with a function I_A on it with values in $\mathbb{Z}_{>0}$. The number $I_A(a)$ is called the *weight* of the point $a \in A$. To a series $A(T) = 1 + \sum_{i=1}^{\infty}[A_i]T^i$ one associates the graded space $A = \coprod_{i=1}^{\infty} A_i$ with the weight function I_A that sends all points of A_i to i. In the other direction, to a graded space (A, I_A) there corresponds the series $A(T) = 1 + \sum_{i=1}^{\infty}[A_i] \cdot T^i$ with $A_i = I_A^{-1}(i)$. In order to define the series $(A(T))^{[M]}$, we shall describe the corresponding graded space A^M first. The space A^M consists of pairs (K, φ), where K is a finite subset of (the variety)

M and φ is a map from K to the graded space A. The weight function I_{A^M} on A^M is defined by $I_{A^M}(K,\varphi) = \sum_{k\in K} I_A(\varphi(k))$; this gives a set-theoretic description of the series $(A(T))^{[M]}$. To describe the coefficients of this series as elements of the Grothendieck semiring $S_0(\mathcal{V}_{\mathbb{C}})$, one can write it as

$$(A(T))^{[M]} = 1 + \sum_{k=1}^{\infty} \left\{ \sum_{\underline{k}\,:\,\sum ik_i = k} \left[\left(\left(\prod_i M^{k_i} \right) \setminus \Delta \right) \times \prod_i A_i^{k_i} \Big/ \prod_i S_{k_i} \right] \right\} \cdot T^k.$$

Here $\underline{k} = \{k_i : i \in \mathbb{Z}_{>0},\, k_i \in \mathbb{Z}_{\geq 0}\}$ and Δ is the "large diagonal" in $M^{\sum k_i}$ that consists of $\left(\sum k_i\right)$-tuples of points of M with at least two coinciding ones; the permutation group S_{k_i} acts by permuting corresponding k_i factors in $\prod_i M^{k_i} \supset \left(\prod_i M^{k_i}\right) \setminus \Delta$ and the spaces A_i simultaneously (the connection between this formula and the preceding description is clear).

REMARK. This same structure can be constructed over the Grothendieck (semi)-ring of varieties with an action of a finite group.

2. Generating Series of Hilbert Schemes

Let Hilb^n_X, $n \geq 1$, be the Hilbert scheme of zero-dimensional subschemes of length n of a complex quasi-projective variety X; for $x \in X$, let $\mathrm{Hilb}^n_{X,x}$ be the Hilbert scheme of subschemes of X concentrated at the point x. Let

$$\mathbb{H}_X(T) := 1 + \sum_{n=1}^{\infty} [\mathrm{Hilb}^n_X] T^n, \qquad \mathbb{H}_{X,x}(T) := 1 + \sum_{n=1}^{\infty} [\mathrm{Hilb}^n_{X,x}] T^n$$

be the generating series of classes of Hilbert schemes Hilb^n_X and $\mathrm{Hilb}^n_{X,x}$ in the Grothendieck semiring $S_0(\mathcal{V}_{\mathbb{C}})$. Let $\mathbb{A}^d$ be the complex affine space of dimension d.

Computation of invariants of the Hilbert schemes Hilb^n_X for a smooth variety X of dimension d can be made in two steps. The first is computation of the corresponding invariants in the local case (i.e., invariants of the Hilbert schemes $\mathrm{Hilb}^n_{\mathbb{A}^d,0}$) and the second is combining the local results to global ones. The following statement formalizes the second step for invariants of classes in the Grothendieck (semi)ring; it generalizes the one for surfaces obtained in [10] using computations by Göttsche [8] of the class of the Hilbert scheme of points on a surface in the Grothendieck ring of motives.

THEOREM 1. *For a smooth quasi-projective variety X of dimension d, the following identity holds in $S_0(\mathcal{V}_{\mathbb{C}})[[T]]$:*

$$\mathbb{H}_X(T) = (\mathbb{H}_{\mathbb{A}^d,0}(T))^{[X]}. \tag{2}$$

Proof. For a locally closed subvariety $Y \subset X$, let $\mathrm{Hilb}^n_{X,Y}$ be the Hilbert scheme of subschemes of length n of X concentrated at points of Y and let $\mathbb{H}_{X,Y}(T) := 1 + \sum_{n=1}^{\infty} [\mathrm{Hilb}^n_{X,Y}] T^n$ be the corresponding generating series. If Y is a Zariski closed subset of X, then $\mathbb{H}_X(T) = \mathbb{H}_{X,Y}(T) \cdot \mathbb{H}_{X,X\setminus Y}(T)$. It is therefore sufficient to prove that

$$\mathbb{H}_{X,Y}(T) = (\mathbb{H}_{\mathbb{A}^d,0}(T))^{[Y]} \tag{3}$$

for a subvariety Y of X that lies in an affine space $\mathbb{A}^N$ and such that the first d affine coordinates $x_1, \ldots, x_d$ of $\mathbb{A}^N$ define local coordinates on X at each point of Y. For a point $p = (x_1^{(0)}, \ldots, x_d^{(0)}, \ldots, x_N^{(0)}) \in Y$, a zero-dimensional subscheme of length k concentrated at the point p can be defined by equations in $x_1 - x_1^{(0)}, \ldots, x_d - x_d^{(0)}$, and this way the Hilbert scheme $\mathrm{Hilb}^k_{X,\{p\}}$ of zero-dimensional subschemes of length k of X concentrated at the point p can be identified with the Hilbert scheme $\mathrm{Hilb}^k_{\mathbb{A}^d,0}$. Hence a zero-dimensional subscheme of X concentrated at points of Y is defined by a finite subset $K \subset Y$, where to each point x of K there corresponds a zero-dimensional subscheme of the (standard) affine space $\mathbb{A}^d$ concentrated at the origin. The length of this subscheme is equal to the sum of lengths of the corresponding subschemes of $\mathbb{A}^d$. Now (3) follows immediately from the geometric description of the power structure over the Grothendieck semiring of quasi-projective varieties. □

Since $[\mathbb{A}^d] = \mathbb{L}^d$ and therefore $\mathbb{H}_{\mathbb{A}^d}(T) = (\mathbb{H}_{\mathbb{A}^d,0}(T))^{\mathbb{L}^d}$, one has the following statement.

Corollary. *For a smooth quasi-projective variety X of dimension d, the following identity holds in $K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}][[T]]$:*

$$\mathbb{H}_X(T) = (\mathbb{H}_{\mathbb{A}^d}(T))^{\mathbb{L}^{-d}[X]}.$$

Applying homomorphisms of power structures, one can derive specializations of the formula (2). The most-known homomorphisms from the Grothendieck (semi)ring of quasi-projective varieties are the Euler characteristic χ (to the ring of integers $\mathbb{Z}$) and the Hodge–Deligne polynomial (to the ring $\mathbb{Z}[u, v]$ of polynomials in two variables). Over $\mathbb{Z}$ there is the standard power structure: the usual exponentiation. One has $\chi((1-T)^{-[X]}) = (1-T)^{-\chi(X)}$ (see e.g. [2]; this follows immediately from [13])—that is, the Euler characteristic is a homomorphism of power structures. This implies the following statement.

Proposition 3. *For a smooth quasi-projective variety X of dimension d,*

$$\chi(\mathbb{H}_X(T)) = (\chi(\mathbb{H}_{\mathbb{A}^d,0}(T)))^{\chi(X)}.$$

For $d = \dim X = 2$ we obtain, using [5],

$$\chi(\mathbb{H}_X(T)) = \left(\prod_{k \geq 0} \frac{1}{1-T^k}\right)^{\chi(X)}.$$

This is one of the results obtained in [7] using the Weil conjectures.

Applying just the same constructions to the Grothendieck ring of analytic varieties with finite Euler characteristic yields the same formulas in the analytic setting: for the Douady spaces of "n-points" on a complex analytic manifold, proved (in the case of surfaces) by de Cataldo [3].

3. Power Structures on the Ring of Polynomials and Hodge–Deligne Polynomials of Hilbert Schemes

One can define a power structure over the ring $\mathbb{Z}[u_1,\dots,u_r]$ of polynomials in n variables with integer coefficients in the following way. Let $P(u_1,\dots,u_r) = \sum_{\underline{k}\in\mathbb{Z}^r_{\ge 0}} p_{\underline{k}}\underline{u}^{\underline{k}} \in \mathbb{Z}[u_1,\dots,u_r]$, where $\underline{k} = (k_1,\dots,k_r)$, $\underline{u} = (u_1,\dots,u_r)$, and $\underline{u}^{\underline{k}} = u_1^{k_1}\cdots u_r^{k_r}$ with $p_{\underline{k}} \in \mathbb{Z}$. Define

$$(1-T)^{-P(u_1,\dots,u_r)} := \prod_{\underline{k}\in\mathbb{Z}^r_{\ge 0}} (1-\underline{u}^{\underline{k}}T)^{-p_{\underline{k}}},$$

where the power (with an integer exponent $-p_{\underline{k}}$) means the usual one.

It is easily seen that $(1-T)^{-(P_1(\underline{u})+P_2(\underline{u}))} = (1-T)^{-P_1(\underline{u})}(1-T)^{-P_2(\underline{u})}$ and hence by Proposition 1 this defines a power structure over the ring $\mathbb{Z}[u_1,\dots,u_r]$. That is, for polynomials $A_i(\underline{u})$ ($i \ge 0$) and $M(\underline{u})$ there is defined a series $(1 + A_1(\underline{u})T + A_2(\underline{u})T^2 + \cdots)^{M(\underline{u})}$ with coefficients from $\mathbb{Z}[u_1,\dots,u_r]$.

Let $r = 2$ and let $u_1 = u$ and $u_2 = v$. Let $e\colon K_0(\mathcal{V}_{\mathbb{C}}) \to \mathbb{Z}[u,v]$ be the ring homomorphism that sends the class $[X]$ of a quasi-projective variety X to its Hodge–Deligne polynomial $e_X(u,v) = \sum h_X^{ij}(-u)^i(-v)^j$. Given our definition, a well-known fact (see e.g. [1; 4, Prop. 1.2]) about Hodge–Deligne polynomials of the symmetric powers of a variety may be rewritten as follows.

Proposition 4.

$$e((1-T)^{-[X]}) = (1-T)^{-e_X(u,v)},$$

where the powers are according to the power structures in the corresponding rings: $K_0(\mathcal{V}_{\mathbb{C}})$ and $\mathbb{Z}[u,v]$, respectively.

Theorem 1 and Proposition 4 yield the following statement.

Theorem 2. *For a smooth quasi-projective variety X of dimension d,*

$$e(\mathbb{H}_X(T)) = (e(\mathbb{H}_{\mathbb{A}^d,0}(T)))^{e(X)}. \tag{4}$$

This is the main result of [9] and [4] but written (in some sense) in a more invariant way. To write it in a form similar to that used in [4], one would apply to both sides of (4) an isomorphism $L\colon 1 + T\cdot\mathbb{Z}[u,v][[T]] \to \mathbb{Z}[u,v][[T]]$ of abelian groups with multiplication and addition as group operations, respectively. In [4] Cheah used the usual logarithmic map log (and consequently the usual exponential map exp in the other direction). The same is done in a number of papers containing similar computations. These maps are defined only over the field $\mathbb{Q}$ of rational numbers, which forces formulas to be written in $\mathbb{Q}[u,v]$. Just in the same way one can use the isomorphism Log and its inverse Exp, defined by

$$\operatorname{Exp}(P_1(\underline{u})T + P_2(\underline{u})T^2 + \cdots) := \prod_{k\ge 1}(1-T^k)^{-P_k(\underline{u})}$$

(cf. [10]) or other ones; there are infinitely many such pairs. This enables us to write the formula staying in $\mathbb{Z}[u,v]$.

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