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POWERS OF THE CURVATURE OPERATOR OF SPACE FORMS AND GEODESICS OF THE TANGENT BUNDLE

СТЕПЕНІ ОПЕРАТОРА КРИВИЗНИ ПРОСТОРОВИХ ФОРМ І ГЕОДЕЗИЧНІ ДОТИЧНОГО РОЗШАРУВАННЯ

It is well known that if Γ is a geodesic line of the tangent (sphere) bundle with Sasaki metric of a locally symmetric Riemannian manifold, then all geodesic curvatures of the projected curve $\gamma = \pi \circ \Gamma$ are constant. In this paper, we consider the case of tangent (sphere) bundle over the real, complex and quaternionic space forms and give a unified proof of the following property: all geodesic curvatures of projected curve are zero starting from k_3, k_6 and k_{10} for the real, complex and quaternionic space forms, respectively.

Відомо, що якщо Γ — геодезична лінія дотичного (сферичного) розшарування з метрикою Сасакі локально-симетричного ріманова многовиду, то всі геодезичні кривизни спроектованої кривої $\gamma = \pi \circ \Gamma$ є констаитами. У даній статті розглянуто випадок (сферичного) дотичного розшарування над дійсними, комплексними та кватерніонними просторовими формами і наведено уніфіковане доведення наступної властивості: всі геодезичні кривизни спроектованої кривої дорівнюють нулю, починаючи з k_3, k_6 та k_{10} відповідно для дійсної, комплексної та кватерніонної форм.

Introduction. K. Sato [1] and S. Sasaki [2] proved that the projection to the base space of any nonvertical geodesic line on the tangent or the tangent sphere bundle of a real space form $M^n(c)$ is a curve of constant curvatures k_1 and k_2 and zero curvatures $k_3, \ldots, k_{n-1}$. P. Nagy [3] essentially generalized this result. He considered the case of general locally symmetric base manifold and have proved that the geodesic curvatures of projection of any (nonvertical) geodesic line on the tangent sphere bundle are all constant. Nevertheless, it was still interesting to find a clearer description of projections of geodesics for the case of classical symmetric spaces of rank one. The second author made a first step in this direction and proved that the projection to the base space of any nonvertical geodesic line on the tangent or tangent sphere bundle of a complex space form CP^n is a curve of constant curvatures $k_1, \ldots, k_5$ and zero curvatures $k_6, \ldots, k_{n-1}$.

In is this paper, we make a contribution in more clear understanding of geometry of projected geodesics in the case of tangent (sphere) bundle of almost all classical locally symmetric spaces, namely, *spheres, complex and quaternionic projective spaces and their noncompact duals* from a unified viewpoint using the *recurrent properties* of powers of the curvature operator of these spaces. This approach allows to give also a unified proof of the results from [1, 2, 4].

We also use an easily proved result [5] stating that the geodesics of tangent or tangent sphere bundle with Sasaki metric have the same projections to the base manifold.

Remark on notations. Throughout the paper, $\langle \cdot, \cdot \rangle$ and $|\cdot|$ mean the scalar product and the norm of vectors with respect to the corresponding metrics.

1. Summary of main results. Let $(M^n(c), g)$ be a Riemannian manifold of constant curvature c, let $(M^{2n}(c); J; g)$ be a Riemannian manifold with complex structure J of constant holomorphic curvature c and let $(M^{4n}(c); J_1, J_2, J_3; g)$ be a Riemannian manifold with quaternionic structure (J_1, J_2, J_3) of constant quaternionic curvature c. For the sake of brevity, we denote by $\mathcal{M}(c)$ one of these space forms with corresponding standard metrics and will refer to $\mathcal{M}(c)$ as to a space form of constant curvature c. The main result is the following statement:

Theorem 1.1. *Let $\mathcal{M}(c)$ be a space form of constant curvature $c \neq 0$. Let Γ be nonvertical geodesic line on the tangent or tangent sphere bundle over $\mathcal{M}(c)$. Let $\gamma = \pi \circ \Gamma$ be the projection of Γ to $\mathcal{M}(c)$. Then the geodesic curvatures k_1, $k_2, \ldots$ of γ are all constant and*

(a) $k_3 = \ldots = k_{n-1} = 0$ *for the real space form;*

(b) $k_6 = \ldots = k_{2n-1} = 0$ *for the complex space form;*

(c) $k_{10} = \ldots = k_{4n-1} = 0$ *for the quaternionic space form.*

As the referee noted, the result of Theorem 1.1 can be expressed in more clear geometrical terms, namely, the projected curve $\gamma = \pi \circ \Gamma$ *lies in a totally geodesic* S^3 *or* H^3, *in a totally geodesic* CP^3 *or* CH^3, *and in a totally geodesic* QP^3 *or* QH^3 *for the real, complex, and quaternionic space form, respectively.* These assertions can be derived from (6), (10) and (14).

The proof of Theorem 1.1 is based on the recurrent property of powers of curvature operator of spaces under consideration. Let R_{XY} be the curvature operator of $\mathcal{M}(c)$. Define a power of curvature operator R^p_{XY} recurrently in the following way:

$$R^p_{XY}Z = R^{p-1}_{XY}(R_{XY}Z), \quad p > 1.$$

The basic tool for our considerations is a following chain of lemmas:

Lemma 1.1. *Let R_{XY} be the curvature operator of the real space form $(M^n(c), g)$. Then, for any X and Y,*

$$R^p_{XY} = \begin{cases} (-b^2c^2)^{s-1} R_{XY} & \text{for } p = 2s-1; \\ (-b^2c^2)^{s-1} R^2_{XY} & \text{for } p = 2s, \end{cases} \quad s \geq 1,$$

where $b = |X \wedge Y|$ is a norm of bivector $X \wedge Y$.

Lemma 1.2. *Let R_{XY} be the curvature operator of the nonflat complex space form $(M^n(c); J; g)$. Denote by $b = |X \wedge Y|$ the norm of a bivector $X \wedge Y$ and $m = \langle X, JY \rangle$. Then, for any X and Y,*

$$R^p_{XY} = \begin{cases} \mathrm{Lin}\left(J R^2_{XY}, R_{XY}, J\right) & \text{for } p = 2s-1; \\ \mathrm{Lin}\left(R^2_{XY}, JR_{XY}, E\right) & \text{for } p = 2s, \end{cases} \quad s \geq 2,$$

where E is the identity operator and Lin *means a linear combination of corresponding operators with coefficients being polynomials in $1/c$, b, m.*

Lemma 1.3. *Let R_{XY} be the curvature operator of the nonflat quaternionic space form $(M^n(c); J_1, J_2, J_3; g)$. Denote by $b = |X \wedge Y|$ the norm of a bivector $X \wedge Y$. Set $m_1 = \langle X, J_1 Y \rangle$, $m_2 = \langle X, J_2 Y \rangle$, $m_3 = \langle X, J_3 Y \rangle$, $m^2 = m_1^2 + m_2^2 + m_3^2$, $\mathcal{J} = m_1 J_1 + m_2 J_2 + m_3 J_3$. Then, for any X and Y,*

$$R^p_{XY} = \begin{cases} \mathrm{Lin}\left(\mathcal{J} R^4_{XY}, R^3_{XY}, \mathcal{J} R^2_{XY}, R_{XY}, \mathcal{J}\right) & \text{for } p = 2s-1; \\ \mathrm{Lin}\left(R^4_{XY}, \mathcal{J} R^3_{XY}, R^2_{XY}, \mathcal{J} R_{XY}, E\right) & \text{for } p = 2s, \end{cases} \quad s \geq 3,$$

where E is the identity operator and Lin *means a linear combination of corresponding operators with coefficients being polynomials in $1/c$, b, m.*

2. Necessary facts and proof of the main result. Let (M^n, g) be a Riemannian manifold and let TM^n be its tangent bundle. Denote by $(u^1, \ldots, u^n)$ a local coordinate system on M^n. Then, in each tangent space of M^n, the natural coordinate frame $\{\partial/\partial u^1,\dots,\partial/\partial u^n\}$ form a local basis. Let ξ be any tangent vector over the given local chart. Then ξ can be decomposed as

$$\xi = \xi^1\frac{\partial}{\partial u^1} + \dots + \xi^n\frac{\partial}{\partial u^n}.$$

The parameters $(u^1, \dots, u^n; \xi^1, \dots, \xi^n)$ form the so-called *natural induced coordinate system* in TM^n. The *Sasaki metric* line element $d\sigma^2$ with respect to this coordinate system is

$$d\sigma^2 = ds^2 + |D\xi|^2, \tag{1}$$

where ds^2 is a line element of M^n, $D\xi$ is the covariant differential of ξ with respect to Levi – Civita connection on M^n, and $|\cdot|$ means the norm with respect to Riemannian metric on M^n.

The *tangent sphere bundle* T_1M^n can be considered as a hypersurface in the tangent bundle defined by the condition $|\xi| = 1$. We will consider T_1M^n as a submanifold in TM^n with the induced metric.

With respect to the natural coordinate system, each curve Γ on TM^n can be represented as $\Gamma(\sigma) = \{u^1(\sigma),\dots,u^n(\sigma);\xi^1(\sigma),\dots,\xi^n(\sigma)\}$ with respect to the arc-length parameter σ and can be interpreted as the vector field $\xi(\sigma) = = \xi^1(\sigma)\partial/\partial u^1 + \dots + \xi^n(\sigma)\partial/\partial u^n$ along the *projected* curve $\gamma = \pi\circ\Gamma = (u^1(\sigma), \dots$ $\dots, u^n(\sigma))$. If ξ is a *unit* vector field, then Γ lies in T_1M^n and represents an arbitrary curve in T_1M^n.

Denote by $(')$ the covariant derivative along γ with respect to parameter σ. Then Γ is a geodesic line on TM^n or T_1M^n if γ and ξ satisfy, respectively, the system of equations

$$TM^n\colon \begin{cases}\gamma'' = R_{\xi'\xi}\gamma',\\ \xi'' = 0,\end{cases} \qquad T_1M^n\colon \begin{cases}\gamma'' = R_{\xi'\xi}\gamma',\\ \xi'' = -\rho^2\xi,\end{cases} \tag{2}$$

where $\rho^2 = |\xi'|^2$ and $R_{\xi'\xi}$ is the *curvature operator* of M^n based on bivector $\xi'\wedge\xi$.

It follows from (2) that $\rho = \text{const}$ in both cases. Denote by s the arclength parameter on γ. Then it follows from (1) that

$$\frac{ds}{d\sigma} = \sqrt{1-\rho^2}, \tag{3}$$

so that $0 \le \rho \le 1$. According to the latter inequality, the set of geodesics of TM^n and T_1M^n can be splitted naturally into 3 classes, namely,

horizontal geodesics $(\rho = 0)$ generated by parallel (unit) vector fields along the geodesics on the base manifold;

vertical geodesics $(\rho = 1)$ represented by geodesics on a fixed fiber;

umbilical geodesics corresponding to $0 < \rho < 1$.

In what follows, we will consider the properties of projections of umbilical geodesics.

Lemma 2.1 (cf. [2]). *Let (M^n, g) be a locally symmetric Riemannian manifold and let R_{XY} be its curvature operator. Let $\gamma = \pi\circ\Gamma$ be a projection of geodesic* *line on TM^n or T_1M^n to the base space. Then, for the derivatives of γ of order p, we have*

$$\gamma^{(p)} = R^{p-1}_{\xi'\xi}\gamma' = R_{\xi'\xi}\gamma^{(p-1)}$$

and, as a consequence, all geodesic curvatures of γ are constant.

Proof. The equalities follow from parallelism of curvature tensor of M^n and equations (2). Moreover, from the evident identity

$$\langle\gamma^{(p)}, \gamma^{(p-1)}\rangle \equiv 0$$

for all $p > 1$, we conclude that $|\gamma^{(p)}| = \text{const}$ for all $p > 1$ and, therefore, by induction, all the geodesic curvatures of γ are constant.

***Proof of Theorem* 1.1.** *Case* (a). Denote by $e_1, \ldots, e_{n-1}$ the Frenet frame of γ. Using the Frenet formulas for the curve with constant geodesic curvatures and keeping in mind (3), it is easy to see that

$$\begin{aligned}\gamma^{(2s-1)} &= (1-\rho^2)^{s-1/2}k_1k_2\ldots k_{2s-2}e_{2s-1} + \mathrm{Lin}\,\{e_1, e_3, \ldots, e_{2s-3}\},\\ \gamma^{(2s)} &= (1-\rho^2)^{s}k_1k_2\ldots k_{2s-1}e_{2s} + \mathrm{Lin}\,\{e_2, e_4, \ldots, e_{2s-2}\}\end{aligned} \tag{4}$$

for all $s \geq 1$ (with formal setting $k_0 \equiv 1$). Setting $s = 1, 2$ in even derivatives, we see that

$$\begin{aligned}\gamma^{(2)} &= (1-\rho^2)k_1e_2,\\ \gamma^{(4)} &= (1-\rho^2)k_1k_2k_3e_4 + \mathrm{Lin}\,(e_2).\end{aligned} \tag{5}$$

On the other hand, applying Lemma 2.1, Lemma 1.1, and Lemma 2.1 again, we get

$$\gamma^{(4)} = R^3_{\xi'\xi}\gamma' = -b^2c^2R_{\xi'\xi}\gamma' = -b^2c^2\gamma^{(2)}. \tag{6}$$

Using (5), we get

$$(1-\rho^2)k_1k_2k_3e_4 + \mathrm{Lin}\,(e_2) = 0$$

and, therefore, $k_3 = 0$, which completes the proof.

Note that b^2 is constant along γ, since

$$(b^2)' = \left(|\xi'\wedge\xi|^2\right) = \left(\rho^2|\xi|^2 - \langle\xi',\xi\rangle^2\right)' = 2\rho^2\langle\xi',\xi\rangle - 2\langle\xi',\xi\rangle\rho^2 \equiv 0.$$

Case (b). Denote by $e_1, \ldots, e_{2n-1}$ the Frenet frame of γ. Similar to the case (a) considerations, the Frenet formulas give

$$\begin{aligned}\gamma^{(2s-1)} &= (1-\rho^2)^{s-1/2}k_1k_2\ldots k_{2s-2}e_{2s-1} + \mathrm{Lin}\,\{e_1, e_3, \ldots, e_{2s-3}\},\\ \gamma^{(2s)} &= (1-\rho^2)^{s}k_1k_2\ldots k_{2s-1}e_{2s} + \mathrm{Lin}\,\{e_2, e_4, \ldots, e_{2s-2}\}\end{aligned} \tag{7}$$

for all $s \geq 1$. Setting $s = 1, 2, 3, 4$ in odd derivatives, we get

$$\begin{aligned}\gamma' &= (1-\rho^2)^{1/2}e_1,\\ \gamma^{(3)} &= (1-\rho^2)^{3/2}k_1k_2e_3 + \mathrm{Lin}\,(e_1),\\ \gamma^{(5)} &= (1-\rho^2)^{5/2}k_1\ldots k_4e_5 + \mathrm{Lin}\,(e_1, e_3),\\ \gamma^{(7)} &= (1-\rho^2)^{7/2}k_1\ldots k_6e_7 + \mathrm{Lin}\,(e_1, e_3, e_5).\end{aligned} \tag{8}$$

On the other hand, applying Lemma 2.1, Lemma 1.2, and Lemma 2.1 again, we get

$$\gamma^{(5)} = R^4_{\xi'\xi}\gamma' = \mathrm{Lin}\,(R^2_{\xi'\xi}, JR_{\xi'\xi}, E)\gamma' = \mathrm{Lin}\,(\gamma^{(3)}, J\gamma^{(2)}, \gamma'),$$
$$\gamma^{(7)} = R^6_{\xi'\xi}\gamma' = \mathrm{Lin}\,(R^2_{\xi'\xi}, JR_{\xi'\xi}, E)\gamma' = \mathrm{Lin}\,(\gamma^{(3)}, J\gamma^{(2)}, \gamma'). \tag{9}$$

Excluding $J\gamma^{(2)}$ from (9), we arrive at the equation

$$\gamma^{(7)} = \mathrm{Lin}\,(\gamma^{(5)}, J\gamma^{(3)}, \gamma'). \tag{10}$$

Using (8), we get

$$(1-\rho^2)^{7/2}k_1\ldots k_6 e_7 + \mathrm{Lin}\,(e_1, e_3, e_5) = 0$$

and conclude that $k_6 = 0$, which completes the proof.

Note that the coefficients of all linear combinations are constants.

Indeed, by Lemma 1.2, the coefficients are polynomials in $1/c$, $b = |\xi' \wedge \xi|$ and $m = \langle \xi', J\xi\rangle$. The value b is constant along γ by the same reasons as in case (a). The value m is constant along γ, since

$$m' = \langle \xi', J\xi\rangle' = \langle \xi'', J\xi\rangle + \langle \xi', J\xi'\rangle \equiv 0.$$

Case (c). Denote by $e_1, \ldots, e_{4n-1}$ the Frenet frame of γ. As above, the Frenet formulas give

$$\gamma^{(2s-1)} = (1-\rho^2)^{s-1/2}k_1k_2\ldots k_{2s-2}e_{2s-1} + \mathrm{Lin}\,\{e_1, e_3, \ldots, e_{2s-3}\},$$
$$\gamma^{(2s)} = (1-\rho^2)^{s}k_1k_2\ldots k_{2s-1}e_{2s} + \mathrm{Lin}\,\{e_2, e_4, \ldots, e_{2s-2}\} \tag{11}$$

for all $s \geq 1$. Setting $s = 1, 2, 3, 4, 5, 6$ in odd derivatives, we get

$$\gamma' = (1-\rho^2)^{1/2}e_1,$$
$$\gamma^{(3)} = (1-\rho^2)^{3/2}k_1k_2e_3 + \mathrm{Lin}\,(e_1),$$
$$\gamma^{(5)} = (1-\rho^2)^{5/2}k_1\ldots k_4e_5 + \mathrm{Lin}\,(e_1, e_3),$$
$$\gamma^{(7)} = (1-\rho^2)^{7/2}k_1\ldots k_6e_7 + \mathrm{Lin}\,(e_1, e_3, e_5), \tag{12}$$
$$\gamma^{(9)} = (1-\rho^2)^{9/2}k_1\ldots k_8e_9 + \mathrm{Lin}\,(e_1, e_3, e_5, e_7),$$
$$\gamma^{(11)} = (1-\rho^2)^{11/2}k_1\ldots k_{10}e_{11} + \mathrm{Lin}\,(e_1, e_3, e_5, e_7, e_9).$$

Applying again Lemma 2.1, Lemma 1.3, and then Lemma 2.1, we get

$$\gamma^{(7)} = R^6_{\xi'\xi}\gamma' = \mathrm{Lin}\,(R^4_{\xi'\xi}, \mathcal{J}R^3_{\xi'\xi}, R^2_{\xi'\xi}, \mathcal{J}R_{\xi'\xi}, E)\gamma' =$$
$$= \mathrm{Lin}\,(\gamma^{(5)}, \mathcal{J}\gamma^{(4)}, \gamma^{(3)}, \mathcal{J}\gamma^{(2)}, \gamma'),$$
$$\gamma^{(9)} = R^8_{\xi'\xi}\gamma' = \mathrm{Lin}\,(R^4_{\xi'\xi}, \mathcal{J}R^3_{\xi'\xi}, R^2_{\xi'\xi}, \mathcal{J}R_{\xi'\xi}, E)\gamma' =$$
$$= \mathrm{Lin}\,(\gamma^{(5)}, \mathcal{J}\gamma^{(4)}, \gamma^{(3)}, \mathcal{J}\gamma^{(2)}, \gamma'), \tag{13}$$
$$\gamma^{(11)} = R^{10}_{\xi'\xi}\gamma' = \mathrm{Lin}\,(R^4_{\xi'\xi}, \mathcal{J}R^3_{\xi'\xi}, R^2_{\xi'\xi}, \mathcal{J}R_{\xi'\xi}, E)\gamma' =$$
$$= \mathrm{Lin}\,(\gamma^{(5)}, \mathcal{J}\gamma^{(4)}, \gamma^{(3)}, \mathcal{J}\gamma^{(2)}, \gamma').$$

Excluding $\mathcal{J}\gamma^{(2)}$ and $\mathcal{J}\gamma^{(4)}$ from (13), we arrive at the equation

$$\gamma^{(11)} = \mathrm{Lin}\,(\gamma^{(9)}, \gamma^{(7)}, \gamma^{(5)}, \gamma^{(3)}, \gamma'). \tag{14}$$

Using (12), we get

$$(1-\rho^2)^{11/2} k_1 \ldots k_{10} e_{11} + \mathrm{Lin}\,(e_1, e_3, e_5, e_7, e_9) = 0$$

and conclude that $k_{10} = 0$, which completes the proof.

Note that the coefficients of all linear combinations are constants. Indeed, by Lemma 1.3, the coefficients are polynomials in $1/c$, $b = |\xi' \wedge \xi|$ and $m = \sqrt{m_1^2 + m_2^2 + m_3^2}$. The value b is constant along γ by the same reasons as in case (a). The values m_1, m_2, m_3 are all constant along γ, since

$$m_i' = \langle \xi', J_i \xi \rangle' = \langle \xi'', J_i \xi \rangle + \langle \xi', J_i \xi' \rangle \equiv 0$$

for $i = 1, 2, 3$.

3. Proofs of basic lemmas. ***Proof of Lemma* 1.1.** The curvature operator R_{XY} of the real space form $(M^n(c), g)$ has the following expression:

$$R_{XY}Z = c[\langle Y, Z\rangle X - \langle X, Z\rangle Y].$$

Then

$$\begin{aligned}
R_{XY}^2 Z &= c\big[\big\langle Y, R_{XY}Z\big\rangle X - \big\langle X, R_{XY}Z\big\rangle Y\big] = \\
&= c^2[\langle Y, \langle Y, Z\rangle X - \langle X, Z\rangle Y\rangle X - \langle X, \langle Y, Z\rangle X - \langle X, Z\rangle Y\rangle Y] = \\
&= c^2\big[\langle Y, Z\rangle\big(\langle X, Y\rangle X - \langle X, Z\rangle |Y|^2 X - \langle Y, Z\rangle |X|^2 Y\big) + \langle X, Z\rangle\langle X, Y\rangle Y\big] = \\
&= c^2\big[\langle Y, Z\rangle\big(\langle X, Y\rangle X - |X|^2 Y\big) + \langle X, Z\rangle\big(\langle X, Y\rangle Y - |Y|^2 X\big)\big] = \\
&= c\big[\langle Y, Z\rangle R_{XY}X + \langle X, Z\rangle R_{YX}Y\big].
\end{aligned}$$

Therefore,

$$\begin{aligned}
R_{XY}^3 Z &= c\big[\big\langle Y, R_{XY}Z\big\rangle R_{XY}X + \big\langle X, R_{XY}Z\big\rangle R_{YX}Y\big] = \\
&= c^3\big[\big(\langle Y, Z\rangle\langle X, Y\rangle - \langle X, Z\rangle |Y|^2\big)\big(\langle X, Y\rangle X - |X|^2 Y\big) + \\
&+ \big(\langle Y, Z\rangle |X|^2 - \langle X, Z\rangle\langle X, Y\rangle\big)\big(\langle X, Y\rangle Y - |Y|^2 X\big)\big] = \\
&= c^3\big[-\langle Y, Z\rangle X\big(|X|^2|Y|^2 - \langle X, Y\rangle^2\big) + \langle X, Z\rangle Y\big(|X|^2|Y|^2 - \langle X, Y\rangle^2\big)\big] = \\
&= -c^2 b^2 R_{XY}Z,
\end{aligned}$$

where, evidently, $b^2 = |X|^2|Y|^2 - \langle X, Y\rangle^2$ is the square norm of $X \wedge Y$.

We now can find the other powers for R_{XY} inductively.

***Proof of Lemma* 1.2.** The curvature operator R_{XY} of the complex space form $(M^{2n}(c); J; g)$ has the following expression:

$$R_{XY}Z = \frac{c}{4}[\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle JY, Z\rangle JX - \langle JX, Z\rangle JY + 2\langle X, JY\rangle JZ].$$

Introduce the unit sphere-type operator S acting as

$$S(Z) \overset{\mathrm{df}}{=} S_{XY}Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y,$$

and the operator $\hat{S}(Z)$ acting as

$$\hat{S}(Z) \overset{\mathrm{df}}{=} S_{JXJY}Z = \langle JY, Z\rangle JX - \langle JX, Z\rangle JY.$$

Finally, if we denote $m = \langle X, JY\rangle$, then the curvature operator under consideration takes the form

$$R_{XY}Z = \frac{c}{4}\left[S + \hat{S} + 2mJ\right]Z. \tag{15}$$

Since $|X \wedge Y| = |(JX) \wedge (JY)|$, the operators S and $\hat{S}$ satisfy

$$S^3 = -b^2 S, \qquad \hat{S}^3 = -b^2 \hat{S},$$

where $b^2 = |X \wedge Y|^2$.

In what follows, we need a "table of products" for the operators S and $\hat{S}$ (see

Table 1

	S	$\hat{S}$	J
S	S^2	$mJ\hat{S}$	$J\hat{S}$
$\hat{S}$	mJS	$\hat{S}^2$	JS
J	JS	$J\hat{S}$	$-E$

Indeed,

$$(S\hat{S})(Z) = S_{XY}\hat{S}_{JXJY}Z = S_{XY}[\langle JY, Z\rangle JX - \langle JX, Z\rangle JY] =$$
$$= \langle Y, \langle JY, Z\rangle JX - \langle JX, Z\rangle JY\rangle X - \langle X, \langle JY, Z\rangle JX - \langle JX, Z\rangle JY\rangle Y =$$
$$= \langle Y, JX\rangle\langle JY, Z\rangle X + \langle JX, Z\rangle\langle X, JY\rangle Y =$$
$$= mJ[\langle JY, Z\rangle JX - \langle JX, Z\rangle JY] = (mJ\hat{S})(Z),$$
$$(\hat{S}S)(Z) = S_{JXJY}S_{XY}Z = S_{JXJY}[\langle Y, Z\rangle X - \langle X, Z\rangle Y] =$$
$$= \langle JY, \langle Y, Z\rangle X - \langle X, Z\rangle Y\rangle JX - \langle JX, \langle Y, Z\rangle X - \langle X, Z\rangle Y\rangle JY =$$
$$= \langle JY, X\rangle\langle Y, Z\rangle JX + \langle X, Z\rangle\langle JX, Y\rangle JY =$$
$$= mJ[\langle Y, Z\rangle X - \langle X, Z\rangle Y] = (mJS)(Z),$$
$$(SJ)(Z) = S_{XY}JZ = \langle Y, JZ\rangle X - \langle X, JZ\rangle Y =$$
$$= J[\langle JY, Z\rangle JX - \langle JX, Z\rangle JY] = (J\hat{S})(Z),$$
$$(\hat{S}J)(Z) = S_{JXJY}JZ = \langle JY, JZ\rangle JX - \langle JX, JZ\rangle JY =$$
$$= J[\langle Y, Z\rangle X - \langle X, Z\rangle Y] = (JS)(Z),$$

and the other entries of the table can be found in a similar way.

It follows from Table 1 that $J(S + \hat{S}) = (S + \hat{S})J$ and, hence,

$$(S + \hat{S})^2 = S^2 + \hat{S}^2 + S\hat{S} + \hat{S}S = S^2 + \hat{S}^2 + mJ(S + \hat{S}),$$
$$(S + \hat{S})^3 = (S + \hat{S})\left[S^2 + \hat{S}^2 + mJ(S + \hat{S})\right] =$$
$$= S^3 + \hat{S}^3 + \hat{S}S^2 + S\hat{S}^2 + mJ(S + \hat{S})^2 =$$
$$= -b^2 S - b^2 \hat{S} + (\hat{S}S)S + (S\hat{S})\hat{S} + mJ(S + \hat{S})^2 =$$
$$= -b^2(S + \hat{S}) + mJ(S^2 + \hat{S}^2) + mJ(S + \hat{S})^2 =$$

$$= -b^2\left(S+\hat S\right) + mJ\left[\left(S+\hat S\right)^2 - mJ\left(S+\hat S\right)\right] + mJ\left(S+\hat S\right)^2 =$$

$$= (m^2-b^2)\left(S+\hat S\right) + 2mJ\left(S+\hat S\right)^2.$$

Thus,

$$\left(S+\hat S\right)^3 = \operatorname{Lin}\left(S+\hat S, J\left(S+\hat S\right)^2\right). \tag{16}$$

On the other hand, setting for brevity $R_{XY} = R$, we derive the following relations from (15):

$$\begin{gathered} S + \hat S = \frac{4}{c}R - 2mJ = \operatorname{Lin}(R, J), \\ \left(S+\hat S\right)^2 = \operatorname{Lin}(R^2, JR, E). \end{gathered} \tag{17}$$

Comparing (16) and (17), we conclude

$$\left(S+\hat S\right)^3 = \operatorname{Lin}\left[\operatorname{Lin}(R, J), J\operatorname{Lin}(R^2, JR, E)\right] = \operatorname{Lin}(JR^2, R, J).$$

On the other hand, the first relation in (17) implies

$$\left(S+\hat S\right)^3 = \left(\frac{4}{c}\right)^3 R^3 + \operatorname{Lin}(JR^2, R, J).$$

Finally,

$$R^3 = \operatorname{Lin}(JR^2, R, J).$$

It is easy to trace that the coefficients of all linear combinations are polynomials in $1/c$, b, m. To complete the proof, we should note that

$$R^4 = R^3 R = \operatorname{Lin}(JR^2, R, J)R = \operatorname{Lin}(JR^3, R^2, JR) =$$

$$= \operatorname{Lin}\left[J\operatorname{Lin}(JR^2, R, J), R^2, JR\right] = \operatorname{Lin}(R^2, JR, E),$$

which allows to find all powers of R inductively.

***Proof of Lemma* 1.3.** The curvature operator R_{XY} of the quaternionic space form $(M^{4n}(c); J_1, J_2, J_3; g)$ has the following expression:

$$R_{XY}Z = \frac{c}{4}[\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle J_1Y, Z\rangle J_1X - \langle J_1X, Z\rangle J_1Y +$$

$$+ \langle J_2Y, Z\rangle J_2X - \langle J_2X, Z\rangle J_2Y + \langle J_3Y, Z\rangle J_3X - \langle J_3X, Z\rangle J_3Y +$$

$$+ 2\langle X, J_1Y\rangle J_1Z + 2\langle X, J_2Y\rangle J_2Z + 2\langle X, J_3Y\rangle J_3Z],$$

where J_1, J_2, J_3 are operators of quaternionic structure

$$J_1J_2 = J_3, \quad J_2J_3 = J_1, \quad J_3J_1 = J_2,$$

$$J_i^2 = -E, \quad \langle X, J_iY\rangle = -\langle J_iX, Y\rangle, \quad i = \overline{1,3}.$$

Introduce the unit sphere-type operator S acting as

$$S(Z) \overset{\text{df}}{=} S_{XY}Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y,$$

the operators $S_i(Z)$ acting as

$$S_i(Z) \overset{\text{df}}{=} S_{J_iXJ_iY}Z = \langle J_iY, Z\rangle J_iX - \langle J_iX, Z\rangle J_iY, \quad i = \overline{1,3},$$

and the operator $\hat{S}(Z)$ acting as

$$\hat{S}(Z) \equiv S_1(Z) + S_2(Z) + S_3(Z).$$

Finally, denote $m_i = \langle X, J_i Y\rangle$, $i = \overline{1,3}$, $m^2 = m_1^2 + m_2^2 + m_3^2$, $\mathcal{J} = m_1 J_1 + m_2 J_2 + m_3 J_3$. Then the curvature operator under consideration takes the form

$$R_{XY}Z = \frac{c}{4}\left[S + \hat{S} + 2\mathcal{J} \right] Z. \tag{18}$$

Since $|X \wedge Y| = |(J_i X) \wedge (J_i Y)|$, $i = \overline{1,3}$, the operators S and S_i satisfy

$$S^3 = -b^2 S, \qquad S_i^3 = -b^2 S_i, \quad i = \overline{1,3},$$

where $b^2 = |X \wedge Y|^2$.

For the operators S and $\hat{S}$, the table of products is Table 2.

Table 2

	S	S_1	S_2	S_3	J_1	J_2	J_3
S	S^2	$m_1 J_1 S_1$	$m_2 J_2 S_2$	$m_3 J_3 S_3$	$J_1 S_1$	$J_2 S_2$	$J_3 S_3$
S_1	$m_1 J_1 S$	S_1^2	$-m_3 J_3 S_2$	$-m_2 J_2 S_3$	$J_1 S$	$J_2 S_3$	$J_3 S_2$
S_2	$m_2 J_2 S$	$-m_3 J_3 S_1$	S_2^2	$-m_1 J_1 S_3$	$J_1 S_3$	$J_2 S$	$J_3 S_1$
S_3	$m_3 J_3 S$	$-m_2 J_2 S_1$	$-m_1 J_1 S_2$	S_3^2	$J_1 S_2$	$J_2 S_1$	$J_3 S$
J_1	$S_1 J_1$	$S J_1$	$S_3 J_1$	$S_2 J_1$	$-E$	J_3	$-J_2$
J_2	$S_2 J_2$	$S_3 J_2$	$S J_2$	$S_1 J_2$	$-J_3$	$-E$	J_1
J_3	$S_3 J_3$	$S_2 J_3$	$S_1 J_3$	$S J_3$	J_2	$-J_1$	$-E$

One can find the expressions for products SS_i, $S_i S$, SJ_i similarly to Table 1 making formal replacements $\hat{S} \to S_i$ and $J \to J_i$. As for the other entries, we have

$$(S_1 S_2)(Z) = S_{J_1 X J_1 Y} S_{J_2 X J_2 Y} Z = S_{J_1 X J_1 Y}[\langle J_2 Y, Z\rangle J_2 X - \langle J_2 X, Z\rangle J_2 Y] =$$
$$= \langle J_1 Y, \langle J_2 Y, Z\rangle J_2 X - \langle J_2 X, Z\rangle J_2 Y\rangle J_1 X -$$
$$- \langle J_1 X, \langle J_2 Y, Z\rangle J_2 X - \langle J_2 X, Z\rangle J_2 Y\rangle J_1 Y =$$
$$= \langle J_1 Y, J_2 X\rangle\langle J_2 Y, Z\rangle J_1 X + \langle J_1 X, J_2 Y\rangle\langle J_2 X, Z\rangle J_1 Y =$$
$$= J_1[\langle X, J_3 Y\rangle\langle J_2 Y, Z\rangle X - \langle X, J_3 Y\rangle\langle J_2 X, Z\rangle Y] =$$
$$= -J_1 J_2[m_3\langle J_2 Y, Z\rangle J_2 X - m_3\langle J_2 X, Z\rangle J_2 Y] = (-m_3 J_3 S_2)(Z),$$
$$(S_1 J_1)(Z) = S_{J_1 X J_1 Y} J_1 Z = \langle J_1 Y, J_1 Z\rangle J_1 X - \langle J_1 X, J_1 Z\rangle J_1 Y =$$
$$= J_1[\langle Y, Z\rangle X - \langle X, Z\rangle Y] = (J_1 S)(Z),$$
$$(S_1 J_2)(Z) = S_{J_1 X J_1 Y} J_2 Z = \langle J_1 Y, J_2 Z\rangle J_1 X - \langle J_1 X, J_2 Z\rangle J_1 Y =$$
$$= J_1[\langle J_3 Y, Z\rangle X - \langle J_3 X, Z\rangle Y] =$$
$$= -J_1 J_3[\langle J_3 Y, Z\rangle J_3 X - \langle J_3 X, Z\rangle J_3 Y] = (J_2 S_3)(Z)$$

and so on.

We see from Table 2 that

$$\begin{aligned}(S+\hat{S})\mathcal{J} &= (S+S_1+S_2+S_3)(m_1J_1+m_2J_2+m_3J_3) = \\ &= m_1J_1S_1 + m_2J_2S_2 + m_3J_3S_3 + m_1J_1S + m_2J_2S_3 + m_3J_3S_2 + m_1J_1S_3 + \\ &+ m_2J_2S + m_3J_3S_1 + m_1J_1S_2 + m_2J_2S_1 + m_3J_3S = \\ &= (m_1J_1+m_2J_2+m_3J_3)(S+S_1+S_2+S_3) = \mathcal{J}(S+\hat{S}).\end{aligned}$$

Therefore, the operators $(S+\hat{S})\mathcal{J}$ commute and, hence, for the operator $R = c\{(S+\hat{S})+2\mathcal{J}\}/4$, the usual formula for powers can be applied:

$$R^n = \left(\frac{c}{4}\right)^n \sum_{l=0}^{n} \binom{n}{l} (S+\hat{S})^{n-l} 2^l (\mathcal{J})^l.$$

The powers for $\mathcal{J}$ can be found trivially, since

$$\begin{aligned}\mathcal{J}^2 &= m_1^2J_1^2 + m_1m_2(J_1J_2+J_2J_1) + m_1m_3(J_1J_3+J_3J_1) + m_2^2J_2^2 + \\ &+ m_2m_3(J_2J_3+J_3J_2) + m_3^2J_3^2 = -m_1^2E - m_2^2E - m_3^2E = -m^2E,\end{aligned}$$

where $m_1^2 + m_2^2 + m_3^2 = m^2$.

As for the powers of $S+\hat{S}$, the following proposition gives the answer:

Proposition 3.1. *The operator $S+\hat{S}$ possesses the recurrent property*

$$(S+\hat{S})^5 = -2(b^2+m^2)(S+\hat{S})^3 - (b^2-m^2)(S+\hat{S}),$$

where $b^2 = |X \wedge Y|^2$ and $m^2 = m_1^2 + m_2^2 + m_3^2 = \langle X, J_1Y\rangle^2 + \langle X, J_2Y\rangle^2 + \langle X, J_3Y\rangle^2$.

Proof. The proof is technical and, in what follows, we will use some auxiliary operator products. Namely,

$$\begin{gathered} S\hat{S} = S\mathcal{J}, \qquad \hat{S}S = \mathcal{J}S, \\ S\mathcal{J}S = -m^2S, \qquad S\hat{S}\mathcal{J} = -m^2S, \\ S(S_1^2+S_2^2+S_3^2) = S^2\mathcal{J}, \qquad \hat{S}S^2 = \mathcal{J}S^2, \\ \hat{S}S\mathcal{J} = \mathcal{J}S\mathcal{J}, \qquad \hat{S}\mathcal{J}S = \mathcal{J}S^2, \\ \hat{S}^2 = S_1^2 + S_2^2 + S_3^2 - \hat{S}\mathcal{J} + \mathcal{J}S, \\ \hat{S}(S_1^2+S_2^2+S_3^2) = -b^2\hat{S} - (S_1^2+S_2^2+S_3^2)\mathcal{J} + \mathcal{J}S^2. \end{gathered} \tag{19}$$

The proof is straightforward. Applying Table 2, we get

$$\begin{aligned}S\hat{S} &= S(S_1+S_2+S_3) = m_1J_1S_1 + m_2J_2S_2 + m_3J_3S_3 = \\ &= m_1SJ_1 + m_2SJ_2 + m_3SJ_3 = S\mathcal{J}.\end{aligned}$$

In a similar way, we find

$$\hat{S}S = (S_1+S_2+S_3)S = m_1J_1S + m_2J_2S + m_3J_3S = \mathcal{J}S,$$

$$\begin{aligned}\hat{S}^2 &= (S_1+S_2+S_3)(S_1+S_2+S_3) = S_1^2 + S_2^2 + S_3^2 + S_1S_2 + S_1S_3 + S_2S_1 + \\ &+ S_2S_3 + S_3S_1 + S_3S_2 = S_1^2 + S_2^2 + S_3^2 - m_3S_1J_3 - m_2S_1J_2 -\end{aligned}$$

$$- m_3 S_2 J_3 - m_1 S_2 J_2 - m_2 S_3 J_2 - m_1 S_3 J_1 = S_1^2 + S_2^2 + S_3^2 - \hat{S}\mathcal{J} + m_1 S_1 J_1$$

$$+ m_2 S_2 J_2 + m_3 S_3 J_3 = S_1^2 + S_2^2 + S_3^2 - \hat{S}\mathcal{J} + \mathcal{J}S,$$

$$S\mathcal{J}S = S(m_1 J_1 + m_2 J_2 + m_3 J_3)S = (m_1 J_1 S_1 + m_2 J_2 S_2 + m_3 J_3 S_3)S =$$

$$= -m_1^2 S - m_2^2 S - m_3^2 S = -m^2 S,$$

$$S\hat{S}\mathcal{J} = S\mathcal{J}\mathcal{J} = -m^2 S,$$

$$S\left(S_1^2 + S_2^2 + S_3^2\right) = m_1 S J_1 S_1 + m_2 S J_2 S_2 + m_3 S J_3 S_3 =$$

$$= m_1 S^2 J_1 + m_2 S^2 J_2 + m_3 S^2 J_3 = S^2 \mathcal{J},$$

$$\hat{S}\mathcal{J}S = (S_1 + S_2 + S_3)(m_1 J_1 + m_2 J_2 + m_3 J_3)S = (m_1 J_1 S + m_2 J_2 S_3 +$$

$$+ m_3 J_3 S_2 + m_1 J_1 S_3 + m_2 J_2 S + m_3 J_3 S_1 + m_1 J_1 S_2 + m_2 J_2 S_1 + m_3 J_3 S)S =$$

$$= \mathcal{J}S^2 + m_2 J_2 m_3 J_3 S + m_3 J_3 m_2 J_2 S + m_1 J_1 m_3 J_3 S +$$

$$+ m_3 J_3 m_1 J_1 S + m_1 J_1 m_2 J_2 S + m_2 J_2 m_1 J_1 S = \mathcal{J}S^2,$$

$$\hat{S}\left(S_1^2 + S_2^2 + S_3^2\right) = (S_1 + S_2 + S_3)\left(S_1^2 + S_2^2 + S_3^2\right) = S_1^3 + S_2^3 + S_3^3 +$$

$$+ S_1 S_2^2 + S_1 S_3^2 + S_2 S_1^2 + S_2 S_3^2 + S_3 S_1^2 + S_3 S_2^2 =$$

$$= -b^2 \hat{S} - m_3 J_3 S_2^2 - m_2 J_2 S_3^2 - m_3 J_3 S_1^2 - m_1 J_1 S_3^2 -$$

$$- m_2 J_2 S_1^2 - m_1 J_1 S_2^2 =$$

$$= -b^2 \hat{S} - m_3 S_1^2 J_3 - m_2 S_1^2 J_2 - m_3 S_2^2 J_3 - m_1 S_2^2 J_1 - m_2 S_3^2 J_2 - m_1 S_3^2 J_1 =$$

$$= -b^2 \hat{S} - \left(S_1^2 + S_2^2 + S_3^2\right)\mathcal{J} + m_1 S_1^2 J_1 + m_2 S_2^2 J_2 + m_3 S_3^2 J_3 =$$

$$= -b^2 \hat{S} + \left(S_1^2 + S_2^2 + S_3^2\right)\mathcal{J} + m_1 J_1 S^2 + m_2 J_2 S^2 + m_3 J_3 S^2 =$$

$$= -b^2 \hat{S} + \left(S_1^2 + S_2^2 + S_3^2\right)\mathcal{J} + \mathcal{J}S^2.$$

We now are ready to find the powers of $\left(S + \hat{S}\right)$. Using (19), we get

$$\left(S + \hat{S}\right)^2 = S^2 + S\hat{S} + \hat{S}S + \hat{S}^2 = S^2 + S\mathcal{J} + \mathcal{J}S + S_1^2 + S_2^2 + S_3^2 -$$

$$- \hat{S}\mathcal{J} + \mathcal{J}S = S^2 + S\mathcal{J} + 2\mathcal{J}S - \hat{S}\mathcal{J} + S_1^2 + S_2^2 + S_3^2.$$

Multiplying the result by $S + \hat{S}$ and applying again (19), we find

$$\left(S + \hat{S}\right)^3 = \left(S + \hat{S}\right)\left[S^2 + S\mathcal{J} + 2\mathcal{J}S - \hat{S}\mathcal{J} + S_1^2 + S_2^2 + S_3^2\right] =$$

$$= S^3 + S^2\mathcal{J} + 2S\mathcal{J}S - S\hat{S}\mathcal{J} + S\left(S_1^2 + S_2^2 + S_3^2\right) + \hat{S}S^2 + \hat{S}S\mathcal{J} +$$

$$+ 2\hat{S}\mathcal{J}S - \hat{S}^2\mathcal{J} + \hat{S}\left(S_1^2 + S_2^2 + S_3^2\right) =$$

$$= -b^2 S + S^2\mathcal{J} - 2m^2 S + m^2 S + S^2\mathcal{J} + \mathcal{J}S^2 + \mathcal{J}S\mathcal{J} + 2\hat{S}\mathcal{J}S -$$

$$- \left[\left(S_1^2 + S_2^2 + S_3^2\right) - \hat{S}\mathcal{J} + \mathcal{J}S\right]\mathcal{J} + \left[-b^2\hat{S} - \left(S_1^2 + S_2^2 + S_3^2\right)\mathcal{J} + \mathcal{J}S^2\right] =$$

$$= -(b^2 + m^2)S + 2S^2\mathcal{J} + \mathcal{J}S^2 + \mathcal{J}S\mathcal{J} + 2\mathcal{J}S^2 - \left(S_1^2 + S_2^2 + S_3^2\right)\mathcal{J} + \hat{S}\mathcal{J}^2 -$$

$$- \mathcal{J}S\mathcal{J} - b^2\tilde{S} - \left(S_1^2 + S_2^2 + S_3^2\right)\mathcal{J} + \mathcal{J}S^2 =$$

$$= -(b^2 + m^2)S + 2S^2\mathcal{J} + 4\mathcal{J}S^2 - 2\left(S_1^2 + S_2^2 + S_3^2\right)\mathcal{J} - m^2\hat{S} - b^2\hat{S} =$$

$$= -(b^2 + m^2)\left(S + \hat{S}\right) + 2S^2\mathcal{J} + 4\mathcal{J}S^2 - 2\left(S_1^2 + S_2^2 + S_3^2\right)\mathcal{J}.$$

Continue the process

$$\left(S + \hat{S}\right)^4 = \left(S + \hat{S}\right)\left[-(b^2 + m^2)\left(S + \hat{S}\right) + 2S^2\mathcal{J} + 4\mathcal{J}S^2 - 2\left(S_1^2 + S_2^2 + S_3^2\right)\mathcal{J}\right] =$$

$$= -(b^2 + m^2)\left(S + \hat{S}\right)^2 + 2S^3\mathcal{J} + 4S\mathcal{J}S^2 - 2S\left(S_1^2 + S_2^2 + S_3^2\right)\mathcal{J} + 2\hat{S}S^2\mathcal{J} +$$

$$+ 4\hat{S}\mathcal{J}S^2 - 2\hat{S}\left(S_1^2 + S_2^2 + S_3^2\right)\mathcal{J} = -(b^2 + m^2)\left(S + \hat{S}\right)^2 - 2b^2S\mathcal{J} - 4m^2S^2 -$$

$$- 2S^2\mathcal{J}\mathcal{J} + 2\mathcal{J}S^2\mathcal{J} + 4\mathcal{J}S^3 - 2\left[-b^2\hat{S} - \left(S_1^2 + S_2^2 + S_3^2\right)\mathcal{J} + \mathcal{J}S^2\right]\mathcal{J} =$$

$$= -(b^2 + m^2)\left(S + \hat{S}\right)^2 - 2b^2S\mathcal{J} - 4m^2S^2 + 2m^2S^2 + 2\mathcal{J}S^2\mathcal{J} -$$

$$- 4b^2\mathcal{J}S + 2b^2\hat{S}\mathcal{J} + 2S\left(S_1^2 + S_2^2 + S_3^2\right)\mathcal{J}^2 - 2\mathcal{J}S^2\mathcal{J} =$$

$$= -(b^2 + m^2)\left(S + \hat{S}\right)^2 - 2b^2S\mathcal{J} - 2m^2S^2 - 4b^2\mathcal{J}S +$$

$$+ 2b^2\hat{S}\mathcal{J} - 2m^2\left(S_1^2 + S_2^2 + S_3^2\right) = -(b^2 + m^2)\left(S + \hat{S}\right)^2 -$$

$$- 2m^2\left[S^2 + S\mathcal{J} + 2\mathcal{J}S - \hat{S}\mathcal{J} + \left(S_1^2 + S_2^2 + S_3^2\right)\right] +$$

$$+ (2m^2 - 2b^2)(S\mathcal{J} + 2\mathcal{J}S - \hat{S}\mathcal{J}) = -(b^2 + 3m^2)\left(S + \hat{S}\right)^2 +$$

$$+ (2m^2 - 2b^2)(S\mathcal{J} + 2\mathcal{J}S - \hat{S}\mathcal{J}).$$

Finally,

$$\left(S + \hat{S}\right)^5 = \left(S + \hat{S}\right)\left[-(b^2 + 3m^2)\left(S + \hat{S}\right)^2 + (2m^2 - 2b^2)(S\mathcal{J} + 2\mathcal{J}S - \hat{S}\mathcal{J})\right] =$$

$$= -(b^2 + 3m^2)\left(S + \hat{S}\right)^3 +$$

$$+ (2m^2 - 2b^2)\left[S^2\mathcal{J} + 2S\mathcal{J}S - S\hat{S}\mathcal{J} + \hat{S}S\mathcal{J} + 2\hat{S}\mathcal{J}S - \hat{S}^2\mathcal{J}\right] =$$

$$= -(b^2 + 3m^2)\left(S + \hat{S}\right)^3 +$$

$$+ (2m^2 - 2b^2)\left[S^2\mathcal{J} - 2m^2S - S\mathcal{J}^2 + \mathcal{J}S\mathcal{J} + 2\mathcal{J}S^2 - \left(S_1^2 + S_2^2 + S_3^2 - \hat{S}\mathcal{J} + \mathcal{J}S\right)\mathcal{J}\right] =$$

$$= -(b^2 + 3m^2)\left(S + \hat{S}\right)^3 +$$

$$+ (2m^2 - 2b^2)\left[S^2\mathcal{J} - m^2S + \mathcal{J}S\mathcal{J} + 2\mathcal{J}S^2 - \left(S_1^2 + S_2^2 + S_3^2\right)\mathcal{J} + \hat{S}\mathcal{J}^2 - \mathcal{J}S\mathcal{J}\right] =$$

$$= -(b^2 + 3m^2)\left(S + \hat{S}\right)^3 +$$

$$+ (m^2 - b^2)\left[2S^2\mathcal{J} + 4\mathcal{J}S^2 - 2\left(S_1^2 + S_2^2 + S_3^2\right)\mathcal{J} - 2m^2S - 2m^2\hat{S}\right] =$$

$$= -(b^2 + 3m^2)\left(S + \hat{S}\right)^3 + (m^2 - b^2)\times$$

$$\times\left[2S^2\mathcal{J} + 4\mathcal{J}S^2 - 2\left(S_1^2 + S_2^2 + S_3^2\right)\mathcal{J} - (m^2 + b^2)\left(S + \hat{S}\right) + (b^2 - m^2)\left(S + \hat{S}\right)\right] =$$

$$= -(b^2+3m^2)\left(S+\hat{S}\right)^3 + (m^2-b^2)\left[\left(S+\hat{S}\right)^3+(b^2-m^2)\left(S+\hat{S}\right)\right] =$$

$$= -2(b^2+m^2)\left(S+\hat{S}\right)^3 - (b^2-m^2)\left(S+\hat{S}\right)$$

which completes the proof.

Thus,

$$\left(S+\hat{S}\right)^5 = \operatorname{Lin}\left(\left(S+\hat{S}\right)^3, S+\hat{S}\right). \tag{20}$$

On the other hand, setting for brevity $R_{XY}=R$, we derive the following relation from (18):

$$S + \hat{S} = \frac{4}{c}R - 2\mathcal{J} = \operatorname{Lin}(R,\mathcal{J}). \tag{21}$$

Since $\left(S+\hat{S}\right)$ and $\mathcal{J}$ commute, (18) implies the commutation of R and $\mathcal{J}$. Taking this and $\mathcal{J}^2 = -m^2E$ into account, we derive the following relations from (21)

$$\left(S+\hat{S}\right)^3 = \left(\frac{4}{c}\right)^3 R^3 + \operatorname{Lin}(\mathcal{J}R^2, R, \mathcal{J}), \tag{22}$$

$$\left(S+\hat{S}\right)^5 = \left(\frac{4}{c}\right)^5 R^5 + \operatorname{Lin}(\mathcal{J}R^4, R^3, \mathcal{J}R^2, R, \mathcal{J}). \tag{23}$$

It follows from (20), (21), and (22) that

$$\left(S+\hat{S}\right)^5 = \operatorname{Lin}\left[\operatorname{Lin}\left(R^3, \mathcal{J}R^2, R, \mathcal{J}\right), \operatorname{Lin}(R,\mathcal{J})\right] = \operatorname{Lin}(R^3, \mathcal{J}R^2, R, \mathcal{J}).$$

Finally, (23) implies

$$R^5 = \operatorname{Lin}(\mathcal{J}R^4, R^3, \mathcal{J}R^2, R, \mathcal{J}).$$

It is easy to trace that the coefficients of all linear combinations are polynomials in $1/c$, b, m. To complete the proof, we should note that

$$R^6 = R^5R = \operatorname{Lin}(\mathcal{J}R^4, R^3, \mathcal{J}R^2, R, \mathcal{J})R = \operatorname{Lin}(\mathcal{J}R^5, R^4, \mathcal{J}R^3, R^2, \mathcal{J}R) =$$

$$= \operatorname{Lin}\left[\mathcal{J}\operatorname{Lin}\left(\mathcal{J}R^4, R^3, \mathcal{J}R^2, R, \mathcal{J}\right), R^4, \mathcal{J}R^3, R^2, \mathcal{J}R\right] =$$

$$= \operatorname{Lin}(R^4, \mathcal{J}R^3, R^2, \mathcal{J}R, E),$$

which allows to find all powers of R inductively.

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