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PP-wave reflection coefficients in weakly anisotropic elastic media

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ABSTRACT

Approximate PP-wave reflection coefficients for weak contrast interfaces separating elastic, weakly transversely isotropic media have been derived recently by several authors. Application of these coefficients is limited because the axis of symmetry of transversely isotropic media must be either perpendicular or parallel to the reflector. In this paper, we remove this limitation by deriving a formula for the PP-wave reflection coefficient for weak contrast interfaces separating two weakly but arbitrarily anisotropic media. The formula is obtained by applying the first-order perturbation theory. The approximate coefficient consists of a sum of the PP-wave reflection coefficient for a weak contrast interface separating two background isotropic half-spaces and a perturbation attributable to the deviation of anisotropic half-spaces from their isotropic backgrounds. The coefficient depends linearly on differences of weak anisotropy parameters across the interface. This simplifies studies of sensitivity of such coefficients to the parameters of the surrounding structure, which represent a basic part of the amplitude-versus-offset (AVO) or amplitude-versus-azimuth (AVA) analysis. The reflection coefficient is reciprocal. In the same way, the formula for the PP-wave transmission coefficient can be derived. The generalization of the procedure presented for the derivation of coefficients of converted waves is also possible although slightly more complicated. Dependence of the reflection coefficient on the angle of incidence is expressed in terms of three factors, as in isotropic media. The first factor alone describes normal incidence reflection. The second yields the low-order angular variations. All three factors describe the coefficient in the whole region, in which the approximate formula is valid. In symmetry planes of weakly anisotropic media of higher symmetry, the approximate formula reduces to the formulas presented by other authors. The accuracy of the approximate formula for the PP reflection coefficient is illustrated on the model with an interface separating an isotropic half-space from a half-space filled by a transversely isotropic material with a horizontal axis of symmetry. The results show a very good fit with results of the exact formula, even in cases of strong anisotropy and strong velocity contrast.

INTRODUCTION

A basic part of amplitude-versus-offset (AVO) or amplitude-versus-azimuth (AVA) analysis is a study of the effects of parameters of a medium on the reflection (R) and, possibly, transmission (T) coefficients of waves generated by an incidence of a wave at an interface separating two media. The coefficients are given by relatively complicated formulas even in the case of isotropic media. It is difficult to understand the dependence of the coefficients on the medium parameters and on angles of incidence from these formulas. In anisotropic media, the R/T coefficients are available in the explicit form only if symmetry planes of anisotropic media of a higher symmetry are especially oriented with respect to an interface, [e.g., Daley and Hron (1977) and Keith and Crampin (1977)]. In the case of general anisotropy, the coefficients must be determined by solving numerically the system of equations resulting from the boundary conditions [e.g., Gajewski and Pšenčík (1987)]. The latter approach does not allow any physical insight into the dependence of the coefficients on the parameters of the media surrounding the interface and on the incidence angles.

Since in many practical cases anisotropy is weak and/or the contrast across the interface is weak, solving the problem of reflection/transmission can be substantially simplified taking these facts into account. Using assumption of weak contrast and weak anisotropy, Thomsen (1993) extended Banik's (1987) work and derived a PP-reflection coefficient for a weak contrast interface separating two weakly transversely isotropic

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media with axes of symmetry perpendicular to the interface [see also Tsvankin (1996) for a discussion of this formula]. Rueger (1996) corrected and generalized Thomsen's results for PP-reflections in planes containing symmetry axes of transversely isotropic and orthorhombic media so that media with symmetry axes parallel to the interface could be considered. Haugen and Ursin (1996) derived PP reflection coefficients in the symmetry planes of a model containing an interface separating a transversely isotropic medium with axis of symmetry perpendicular to the interface from a transversely isotropic medium with axis of symmetry parallel to the interface. Rueger (1997) derived formulas for PP reflection coefficients in transversely isotropic media with axes of symmetry perpendicular and parallel to the interface.

These authors mostly concentrated on solving reflection/transmission problems in symmetry planes or in anisotropic media of higher symmetry. In practice, measurement profiles may not be situated in symmetry planes, symmetry planes need not be perpendicular or parallel to interfaces, and media surrounding interfaces may be of lower symmetry. It is therefore desirable to find universal formulas that apply in most general cases. An important step in this respect has been made by Ursin and Haugen (1996), who derive approximate R/T coefficients for weak contrast interfaces in anisotropic media of arbitrary strength, and by Zillmer et al. (1997), who derive approximate R/T coefficients for strong contrast interfaces separating two weakly but generally anisotropic media. Their rather complicated formulas could be simplified considerably if, instead of strong anisotropy (strong contrast interface), weak anisotropy (weak contrast interface) is considered (see Zillmer et al., 1998). According to Thomsen (1993), "... at most reflecting interfaces, the contrast in elastic properties is small" and thus the assumption of a weak contrast is appropriate.

Here, we give an outline of an approach that allows us to derive R/T coefficients for weak contrast interfaces separating arbitrary weak anisotropic media. Our approach is a generalization of the approach used by Thomsen (1993). We concentrate on the case of incident P-wave and present a simple formula for the PP reflection coefficient. The coefficient consists of a sum of the PP-wave reflection coefficient for a weak contrast interface separating two background isotropic half-spaces and a perturbation attributable to the deviation of anisotropic half-spaces from their isotropic backgrounds. Because of the use of the perturbation theory, the resulting approximate formula is applicable only in regions in which the reflection coefficient is a small quantity. The coefficient depends on weak anisotropy parameters (see Pšenčík and Gajewski, 1998), on parameters of the background isotropic medium, and on azimuth and angle of incidence. The dependence on the weak anisotropy parameters is linear, which is important for the studies of sensitivity of the reflection coefficient to these parameters.

We test accuracy of the derived formula on models consisting of two homogeneous half-spaces separated by a plane interface. The half-space in which the incident wave propagates is isotropic. The other half-space is transversely isotropic with a horizontal axis of symmetry.

In the same way as for the PP reflection coefficient, the approximate formula for the PP transmission coefficient can be derived. The proposed approach can also be generalized for the derivation of formulas for the coefficients of converted waves.

In the following, the perturbations are denoted systematically by the symbol δ. The contrast, i.e., the difference of a parameter across an interface, is denoted by Δ. Component notation of vectors and matrices is used throughout the paper. The Roman lowercase indices attain values 1, 2, and 3; uppercase Roman indices attain only values 1 and 2. The Greek indices run from 1 to 6. Einstein summation convention is used for the repeated indices. Voigt notation $A_{\alpha\beta}$ for density normalized elastic parameters, with α,β running from 1 to 6, is used in parallel with the tensor notation a_{ijkl}. Quantities related to the background unperturbed medium are denoted by the superscript 0. Since we do not use the power 0, we hope this notation does not lead to misinterpretations.

REFLECTION/TRANSMISSION OF PLANE WAVES IN ANISOTROPIC MEDIA

Let us consider two homogeneous anisotropic half-spaces separated by a plane interface Σ specified by a normal ν_i. One half-space is characterized by the density $\rho^{(1)}$ and the density-normalized elastic parameters $a^{(1)}_{ijkl}$. The same parameters in the other half-space are denoted by $\rho^{(2)}$ and $a^{(2)}_{ijkl}$. A harmonic plane wave incident at the interface generates six plane harmonic waves: reflected and transmitted $S1$, $S2$, and P-waves. The displacement vector of any of the mentioned waves can be expressed in the following way:

$$u_i^{(N)}(x_m, t) = U^{(N)} g_i^{(N)} \exp\left[-i\omega\left(t - p_k^{(N)} x_k\right)\right]. \qquad (1)$$

Here, the superscript $N = 0$ corresponds to the incident wave; $N = 1, 2$, and 3 correspond to $S1$, $S2$, and P reflected waves; and $N = 4, 5$, and 6 correspond to $S1$, $S2$, and P transmitted waves, respectively. The symbol $U^{(N)}$ denotes scalar amplitude, $g_i^{(N)}$ is unit polarization vector, $p_i^{(N)}$ is slowness vector, ω is circular frequency, and t is time. The corresponding tractions can be written as follows:

$$\begin{aligned} T_i^{(N)}(x_m, t) &= \rho^{(1)} a^{(1)}_{ijkl} \nu_j u^{(N)}_{k,l}, \qquad N = 1, 2, 3, \\ T_i^{(N)}(x_m, t) &= \rho^{(2)} a^{(2)}_{ijkl} \nu_j u^{(N)}_{k,l}, \qquad N = 4, 5, 6. \end{aligned} \qquad (2)$$

The traction corresponding to the incident wave ($N = 0$) has the form of one of the above relations, depending in which half-space the incident wave propagates. The incident and generated waves satisfy the boundary conditions at the interface: continuity of the displacement and the traction vectors.

One of the consequences of the boundary conditions is the following equation,

$$p_k^{(N)} x_k = p_k^{(0)} x_k, \qquad (3)$$

which holds for any N along Σ. Equation (3) implies the equality of tangent components of slowness vectors of incident and generated waves to the interface Σ, i.e., Snell's law. The slowness vector of any generated wave can thus be written as

$$p_i^{(N)} = b_i + \xi^{(N)} \nu_i, \qquad (4)$$

where

$$b_i = p_i^{(0)} - \left(p_k^{(0)} \nu_k\right) \nu_i. \qquad (5)$$

The symbol $\xi^{(N)}$ is a projection of the slowness vector $p_i^{(N)}$ into the normal ν_i to the interface, which can be determined from

$$det[a_{ijkl}(b_j + \xi\nu_j)(b_l + \xi\nu_l) - \delta_{ik}] = 0. \qquad (6)$$

Equation (6) is the condition of solvability of the Christoffel equation,

$$\left(a_{ijkl}p_i^{(N)}p_l^{(N)} - \delta_{jk}\right)g_j^{(N)} = 0. \qquad (7)$$

The elastic parameters $a_{ijkl}^{(1)}$ must be considered for reflected waves and $a_{ijkl}^{(2)}$ for transmitted waves in equations (6) and (7). Equation (6) is a polynomial equation of the sixth order with real coefficients. Thus, its six roots are real or form complex conjugate pairs. Since the number of roots exceeds the number of generated waves on each side of the interface, a selection of the roots must be made (see Gajewski and Pšenčík, 1987).

From the boundary conditions we also obtain the following set of equations:

$$\begin{aligned} &U^{(1)}g_i^{(1)} + U^{(2)}g_i^{(2)} + U^{(3)}g_i^{(3)} - U^{(4)}g_i^{(4)} - U^{(5)}g_i^{(5)} \\ &\quad - U^{(6)}g_i^{(6)} = -U^{(0)}g_i^{(0)}, \\ &U^{(1)}X_i^{(1)} + U^{(2)}X_i^{(2)} + U^{(3)}X_i^{(3)} - U^{(4)}X_i^{(4)} - U^{(5)}X_i^{(5)} \\ &\quad - U^{(6)}X_i^{(6)} = -U^{(0)}X_i^{(0)}, \end{aligned} \qquad (8)$$

where

$$\begin{aligned} X_i^{(N)} &= \rho^{(1)}a_{ijkl}^{(1)}\nu_j g_k^{(N)}p_l^{(N)}, \qquad N = 1,2,3, \\ X_i^{(N)} &= \rho^{(2)}a_{ijkl}^{(2)}\nu_j g_k^{(N)}p_l^{(N)}, \qquad N = 4,5,6. \end{aligned} \qquad (9)$$

We refer to the vectors $X_i^{(N)}$ as to the amplitude-normalized traction vectors. The vector $X_i^{(0)}$, corresponding to the incident wave, has the form of one of the above expressions, depending on the half-space in which the incident wave propagates. The polarization vectors appearing in equation (9) can be determined from the Christoffel equation (7), in which appropriate elastic parameters are considered.

Equations (8) can be expressed in a more compact way in the matrix form

$$C_{\alpha\beta}U_\beta = B_\alpha, \qquad (10)$$

where

$$C_{\alpha\beta} \equiv \begin{pmatrix} g_1^{(1)} & g_1^{(2)} & g_1^{(3)} & -g_1^{(4)} & -g_1^{(5)} & -g_1^{(6)} \\ g_2^{(1)} & g_2^{(2)} & g_2^{(3)} & -g_2^{(4)} & -g_2^{(5)} & -g_2^{(6)} \\ g_3^{(1)} & g_3^{(2)} & g_3^{(3)} & -g_3^{(4)} & -g_3^{(5)} & -g_3^{(6)} \\ X_1^{(1)} & X_1^{(2)} & X_1^{(3)} & -X_1^{(4)} & -X_1^{(5)} & -X_1^{(6)} \\ X_2^{(1)} & X_2^{(2)} & X_2^{(3)} & -X_2^{(4)} & -X_2^{(5)} & -X_2^{(6)} \\ X_3^{(1)} & X_3^{(2)} & X_3^{(3)} & -X_3^{(4)} & -X_3^{(5)} & -X_3^{(6)} \end{pmatrix},$$

$$U_\alpha \equiv (R_{S1}, R_{S2}, R_P, T_{S1}, T_{S2}, T_P)^T, \qquad (11)$$

$$B_\alpha \equiv -\left(g_1^{(0)}, g_2^{(0)}, g_3^{(0)}, X_1^{(0)}, X_2^{(0)}, X_3^{(0)}\right)^T.$$

Here $C_{\alpha\beta}$ is a 6×6 displacement-stress matrix for generated waves. The column vector B_α is a displacement-stress vector for the incident wave. The column vector U_α is the vector of reflection and transmission coefficients.

REFLECTION/TRANSMISSION OF PLANE WAVES IN WEAKLY ANISOTROPIC MEDIA

We consider now each of the half-spaces filled with a weakly anisotropic material, i.e., with material whose density-normalized elastic parameters (hereafter reffered to as elastic parameters) and the density can be expressed as

$$a_{ijkl}^{(I)} = a_{ijkl}^0 + \delta a_{ijkl}^{(I)}, \quad \rho^{(I)} = \rho^0 + \delta\rho^{(I)}, \quad I = 1,2. \quad (12)$$

In equation (12), a_{ijkl}^0 and ρ^0 denote the elastic parameters and the density of the background isotropic medium, which is the same for both half-spaces. The parameters a_{ijkl}^0 are given by the formula

$$a_{ijkl}^0 = (\alpha^2 - 2\beta^2)\delta_{ij}\delta_{kl} + \beta^2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \qquad (13)$$

The symbols α and β denote the P- and S- wave velocities of the background isotropic medium. The quantities $\delta a_{ijkl}^{(I)}$ and $\delta\rho^{(I)}$ in equation (12) denote small deviations of elastic parameters and the density of the weakly anisotropic media in both half-spaces from their values in the background medium. They are assumed to satisfy the conditions

$$\left|\delta a_{ijkl}^{(I)}\right| \ll \left\|a_{ijkl}^0\right\|, \qquad \left|\delta\rho^{(I)}\right| \ll \rho^0, \qquad I = 1,2, \quad (14)$$

where the norm $\|\cdot\|$ can be defined, for example, as $\|a_{ijkl}^0\| = max|a_{ijkl}^0|$. Since we consider the background of both half-spaces to be the same homogeneous isotropic medium with parameters a_{ijkl}^0 and ρ^0, equations (12) and inequalities (14) yield automatically the conditions of weak contrast across the interface

$$|\Delta a_{ijkl}| \ll \left\|a_{ijkl}^0\right\|, \qquad |\Delta\rho| \ll \rho^0, \qquad I = 1,2. \quad (15)$$

In weakly anisotropic media, equation (10) can be linearized to obtain

$$\left(C_{\alpha\beta}^0 + \delta C_{\alpha\beta}\right)\left(U_\beta^0 + \delta U_\beta\right) = B_\alpha^0 + \delta B_\alpha. \qquad (16)$$

Here the quantities $\delta C_{\alpha\beta}$, δU_β, and δB_α in equation (16) represent perturbations of $C_{\alpha\beta}^0$, U_β^0, and B_α^0. The symbols $C_{\alpha\beta}^0$, U_β^0, and B_α^0 denote the matrix $C_{\alpha\beta}$ and the vectors U_β and B_α [see equation (10)] specified for the background isotropic medium. Since the background isotropic medium is homogeneous without any interface, the matrix $C_{\alpha\beta}^0$ can be understood as a displacement-stress matrix for a fictitious (nonexistent) interface. It has a form

$$C_{\alpha\beta}^0 \equiv \begin{pmatrix} g_1^{0(1)} & g_1^{0(2)} & g_1^{0(3)} & -g_1^{0(4)} & -g_1^{0(5)} & -g_1^{0(6)} \\ g_2^{0(1)} & g_2^{0(2)} & g_2^{0(3)} & -g_2^{0(4)} & -g_2^{0(5)} & -g_2^{0(6)} \\ g_3^{0(1)} & g_3^{0(2)} & g_3^{0(3)} & -g_3^{0(4)} & -g_3^{0(5)} & -g_3^{0(6)} \\ X_1^{0(1)} & X_1^{0(2)} & X_1^{0(3)} & -X_1^{0(4)} & -X_1^{0(5)} & -X_1^{0(6)} \\ X_2^{0(1)} & X_2^{0(2)} & X_2^{0(3)} & -X_2^{0(4)} & -X_2^{0(5)} & -X_2^{0(6)} \\ X_3^{0(1)} & X_3^{0(2)} & X_3^{0(3)} & -X_3^{0(4)} & -X_3^{0(5)} & -X_3^{0(6)} \end{pmatrix}. \qquad (17)$$

Vectors $g_i^{0(N)}$ and $X_i^{0(N)}$, $N = 1, 2, \ldots, 6$ in equation (17) are the polarization and amplitude-normalized traction vectors specified in the background isotropic medium. The column vector U_α^0 contains the R/T coefficients at the fictitious interface.

Taking into account that $C_{\alpha\beta}^0$, U_β^0, and B_α^0 satisfy equation (10) and neglecting perturbations of the second order, equation (16) yields the following important result:

$$\delta U_\alpha = (\mathbf{C}^0)^{-1}_{\alpha\beta}\left(\delta B_\beta - \delta C_{\beta\gamma} U^0_\gamma\right), \tag{18}$$

in which

$$\delta C_{\alpha\beta} \equiv \begin{pmatrix} \delta g_1^{(1)} & \delta g_1^{(2)} & \delta g_1^{(3)} & -\delta g_1^{(4)} & -\delta g_1^{(5)} & -\delta g_1^{(6)} \\ \delta g_2^{(1)} & \delta g_2^{(2)} & \delta g_2^{(3)} & -\delta g_2^{(4)} & -\delta g_2^{(5)} & -\delta g_2^{(6)} \\ \delta g_3^{(1)} & \delta g_3^{(2)} & \delta g_3^{(3)} & -\delta g_3^{(4)} & -\delta g_3^{(5)} & -\delta g_3^{(6)} \\ \delta X_1^{(1)} & \delta X_1^{(2)} & \delta X_1^{(3)} & -\delta X_1^{(4)} & -\delta X_1^{(5)} & -\delta X_1^{(6)} \\ \delta X_2^{(1)} & \delta X_2^{(2)} & \delta X_2^{(3)} & -\delta X_2^{(4)} & -\delta X_2^{(5)} & -\delta X_2^{(6)} \\ \delta X_3^{(1)} & \delta X_3^{(2)} & \delta X_3^{(3)} & -\delta X_3^{(4)} & -\delta X_3^{(5)} & -\delta X_3^{(6)} \end{pmatrix}, \tag{19}$$

$$\delta B_\alpha \equiv -\left(\delta g_1^{(0)}, \delta g_2^{(0)}, \delta g_3^{(0)}, \delta X_1^{(0)}, \delta X_2^{(0)}, \delta X_3^{(0)}\right)^T.$$

The vector δU_α contains perturbations of three reflection and three transmission coefficients from their values U^0_α in the background medium.

DISPLACEMENT-STRESS MATRIX FOR A FICTITIOUS INTERFACE AND ITS INVERSION

To determine the displacement-stress matrix $C^0_{\alpha\beta}$, we need to know the slowness vectors, polarization vectors, and amplitude-normalized traction vectors of all generated waves in the background isotropic medium [see equation (17)].

We introduce a Cartesian coordinate system so that its x- and y-axis are situated in the interface Σ and the positive z-axis points upward. The normal ν_i to Σ also points upward: $\nu_i = (0, 0, 1)^T$. The incident wave is assumed to impinge on the interface from above, against the direction of the normal ν_i. To simplify the following considerations we assume the incidence plane coincides with the (x, z) plane. Later, we shall generalize obtained results for an arbitrary orientation of the incidence plane.

The slowness vectors of the incident, unconverted transmitted and all the remaining fictitiously generated waves in the background isotropic medium can then be expressed in the following way:

$$p_k^{0(0)} \equiv p_k^{0(6)} \equiv \left(p_1^0, 0, p_3^{0P}\right)^T, \qquad p_k^{0(3)} \equiv \left(p_1^0, 0, -p_3^{0P}\right)^T,$$

$$p_k^{0(1)} \equiv p_k^{0(2)} \equiv \left(p_1^0, 0, -p_3^{0S}\right)^T, \tag{20}$$

$$p_k^{0(4)} \equiv p_k^{0(5)} \equiv \left(p_1^0, 0, p_3^{0S}\right)^T.$$

Components p_1^0 and p_3^{0P} satisfy the eikonal equation for P-waves: $p_1^0 = \alpha^{-1}\sin\theta_P$ and $p_3^{0P} = \alpha^{-1}\cos\theta_P$. Components p_1^0 and p_3^{0S} satisfy the eikonal equation for S-waves: $p_1^0 = \beta^{-1}\sin\theta_S = \alpha^{-1}\sin\theta_P$ and $p_3^{0S} = \beta^{-1}\cos\theta_S$. Here, θ_P and θ_S are acute angles made by the corresponding slowness vector and the normal ν_i.

The polarization vectors of the incident and generated P-waves are parallel to the corresponding slowness vectors in an isotropic medium. We can thus write them as

$$g_k^{0(0)} \equiv g_k^{0(6)} \equiv \alpha\left(p_1^0, 0, p_3^{0P}\right)^T, \quad g_k^{0(3)} \equiv \alpha\left(p_1^0, 0, -p_3^{0P}\right)^T. \tag{21}$$

The polarization vectors of an S-wave in an isotropic medium are two mutually perpendicular unit vectors situated in the plane perpendicular to the slowness vector of the considered S-wave. In the isotropic medium, which represents a background of a weakly anisotropic medium, the orientation of the polarization vectors of the S-wave is not arbitrary. Their orientation must be such that a transition from isotropy to weak anisotropy causes only a small perturbation of the polarization vectors. In the plane perpendicular to the slowness vector of the transmitted S-wave, the polarization vectors $g_k^{0(4)}$ and $g_k^{0(5)}$ of the transmitted S-wave can be expressed as

$$g_k^{0(4)} \equiv \left(\beta p_3^{0S}\cos\Phi, \sin\Phi, -\beta p_1^0\cos\Phi\right)^T,$$
$$g_k^{0(5)} \equiv \left(-\beta p_3^{0S}\sin\Phi, \cos\Phi, \beta p_1^0\sin\Phi\right)^T. \tag{22}$$

In equation (22), the angle Φ must be chosen so it guarantees a small perturbation of polarization vectors from the isotropic to weakly anisotropic medium. The angle Φ thus depends on the deviations of elastic parameters of the weakly anisotropic medium from their values in the background isotropic medium. Similarly, for the reflected S-wave we get

$$g_k^{0(1)} \equiv \left(-\beta p_3^{0S}\cos\Psi, \sin\Psi, -\beta p_1^0\cos\Psi\right)^T,$$
$$g_k^{0(2)} \equiv \left(\beta p_3^{0S}\sin\Psi, \cos\Psi, \beta p_1^0\sin\Psi\right)^T, \tag{23}$$

where the angle Ψ plays the same role as Φ in equation (22).

Specifying parameters in equation (9) for the background isotropic medium and using equations (13) and (20)–(23), we can determine the quantities $X_i^{0(N)}$ appearing in the interface matrix $C^0_{\alpha\beta}$. In this way, the interface matrix $C^0_{\alpha\beta}$ for the fictitious interface attains the following form:

$$\begin{pmatrix} -\beta p_3^{0S}\cos\Psi & \beta p_3^{0S}\sin\Psi & \alpha p_1^0 & -\beta p_3^{0S}\cos\Phi & \beta p_3^{0S}\sin\Phi & -\alpha p_1^0 \\ \sin\Psi & \cos\Psi & 0 & -\sin\Phi & -\cos\Phi & 0 \\ -\beta p_1^0\cos\Psi & \beta p_1^0\sin\Psi & -\alpha p_3^{0P} & \beta p_1^0\cos\Phi & -\beta p_1^0\sin\Phi & -\alpha p_3^{0P} \\ \beta Y\cos\Psi & -\beta Y\sin\Psi & -Z_P & -\beta Y\cos\Phi & \beta Y\sin\Phi & -Z_P \\ -\rho^0\beta^2 p_3^{0S}\sin\Psi & -\rho^0\beta^2 p_3^{0S}\cos\Psi & 0 & -\rho^0\beta^2 p_3^{0S}\sin\Phi & -\rho^0\beta^2 p_3^{0S}\cos\Phi & 0 \\ Z_S\cos\Psi & -Z_S\sin\Psi & \alpha Y & Z_S\cos\Phi & -Z_S\sin\Phi & -\alpha Y \end{pmatrix}, \tag{24}$$

where

$$Y = \rho^0\left(1 - 2\beta^2\left(p_1^0\right)^2\right), \qquad Z_P = 2\alpha\rho^0\beta^2 p_1^0 p_3^{0P},$$
$$Z_S = 2\rho^0\beta^3 p_1^0 p_3^{0S}. \tag{25}$$

For the inversion of the matrix $C^0_{\alpha\beta}$, the symbolic manipulation software Reduce (Hearn, 1991) was used. The inverted matrix has the form

$$(\mathbf{C}^0)^{-1}_{\alpha\beta} \equiv \begin{pmatrix} -\dfrac{\beta^2 p_1^0 Y\cos\Psi}{Z_S} & \dfrac{\sin\Psi}{2} & -\beta p_1^0\cos\Psi & \dfrac{\cos\Psi}{2\rho^0\beta} & -\dfrac{\beta p_1^0\sin\Psi}{Z_S} & \dfrac{\beta^2\left(p_1^0\right)^2\cos\Psi}{Z_S} \\ \dfrac{\beta^2 p_1^0 Y\sin\Psi}{Z_S} & \dfrac{\cos\Psi}{2} & \beta p_1^0\sin\Psi & -\dfrac{\sin\Psi}{2\rho^0\beta} & -\dfrac{\beta p_1^0\cos\Psi}{Z_S} & -\dfrac{\beta^2\left(p_1^0\right)^2\sin\Psi}{Z_S} \\ \dfrac{\beta^2 p_1^0}{\alpha} & 0 & -\dfrac{p_1^0\beta^2 Y}{Z_P} & -\dfrac{\beta^2\left(p_1^0\right)^2}{Z_P} & 0 & \dfrac{1}{2\rho^0\alpha} \\ -\dfrac{\beta^2 p_1^0 Y\cos\Phi}{Z_S} & -\dfrac{\sin\Phi}{2} & \beta p_1^0\cos\Phi & -\dfrac{\cos\Phi}{2\rho^0\beta} & -\dfrac{\beta p_1^0\sin\Phi}{Z_S} & \dfrac{\beta^2\left(p_1^0\right)^2\cos\Phi}{Z_S} \\ \dfrac{\beta^2 p_1^0 Y\sin\Phi}{Z_S} & -\dfrac{\cos\Phi}{2} & -\beta p_1^0\sin\Phi & \dfrac{\sin\Phi}{2\rho^0\beta} & -\dfrac{\beta p_1^0\cos\Phi}{Z_S} & -\dfrac{\beta^2\left(p_1^0\right)^2\sin\Phi}{Z_S} \\ -\dfrac{\beta^2 p_1^0}{\alpha} & 0 & -\dfrac{p_1^0\beta^2 Y}{Z_P} & -\dfrac{\beta^2\left(p_1^0\right)^2}{Z_P} & 0 & -\dfrac{1}{2\rho^0\alpha} \end{pmatrix}. \tag{26}$$

P-WAVE INCIDENCE ON AN INTERFACE BETWEEN WEAKLY ANISOTROPIC MEDIA

Because there are no reflected and/or converted transmitted waves at the fictitious interface, the vector U^0_α attains the form

$$U^0_\alpha \equiv (0, 0, 0, 0, 0, 1)^T. \tag{27}$$

Because of the form of vector U^0_α, equation (18) reduces to

$$\delta U_\alpha \equiv (\mathbf{C}^0)^{-1}_{\alpha\beta}\big(g_1^{(6)} - g_1^{(0)}, g_2^{(6)} - g_2^{(0)}, g_3^{(6)} - g_3^{(0)},$$
$$X_1^{(6)} - X_1^{(0)}, X_2^{(6)} - X_2^{(0)}, X_3^{(6)} - X_3^{(0)}\big)^T. \tag{28}$$

In deriving equation (28), we took into account that the vectors $g_i^{0(0)}$ and $g_i^{0(6)}$ and the vectors $X_i^{0(0)}$ and $X_i^{0(6)}$ in the background medium are identical. From inspection of equation (28), we can conclude that for the determination of R/T coefficients of waves generated by the incidence of a P-wave at the interface Σ separating two weakly anisotropic media, it is sufficient to know, in addition to the inverse of the interface matrix $C^0_{\alpha\beta}$, only the displacement-stress vectors of the incident ($N = 0$) and transmitted ($N = 6$) P-waves in weakly anisotropic media surrounding the interface Σ. Knowledge of the displacement-stress vectors of the reflected waves or any of converted transmitted waves is not needed. For the evaluation of the displacement-stress vectors of the incident and transmitted P-waves, knowledge of the vectors $p_i^{(N)}$ and $g_i^{(N)}$ for $N = 0, 6$ is necessary. The determination of these vectors is described in the Appendix.

PP-WAVE REFLECTION COEFFICIENTS IN THE (*X*, *Z*) PLANE

Let us insert the quantities specified in the Appendix into formula (28), which yields the PP reflection (transmission) coefficient in the (x, z) plane. Since only the third line (or sixth line in the case of a PP transmitted wave) of matrix (26) enters the final formula for the coefficient, the PP-wave reflection coefficient (and similarly transmission coefficent) does not depend on angles Ψ and Φ. This considerably simplifies further considerations.

In the following, we concentrate on the case of a reflected P-wave. The formula for the reflection coefficient R_{PP} reads

$$R_{PP}(\theta_P) = \frac{\Delta A_{33}}{4\alpha^2}\cos^{-2}\theta_P - 2\frac{\Delta A_{55}}{\alpha^2}\sin^2\theta_P + \frac{1}{2}\frac{\Delta\rho}{\rho}\left(1 - 4\frac{\beta^2}{\alpha^2}\sin^2\theta_P\right) + \frac{1}{2}\Delta\delta^*\sin^2\theta_P + \frac{1}{2}\Delta\epsilon^*\sin^2\theta_P\tan^2\theta_P.$$

We rewrite it into the form used by Thomsen (1993):

$$R_{PP}(\theta_P) = \frac{\rho\Delta A_{33} + 2\alpha^2\Delta\rho}{4\rho\alpha^2} + \frac{1}{2}\left[\frac{\Delta A_{33}}{2\alpha^2} - \frac{4\left(\rho\Delta A_{55} + \beta^2\Delta\rho\right)}{\rho\alpha^2} + \Delta\delta^*\right]\sin^2\theta_P + \frac{1}{2}\left(\frac{\Delta A_{33}}{2\alpha^2} + \Delta\epsilon^*\right)\sin^2\theta_P\tan^2\theta_P. \tag{29}$$

Note that Δ denotes a difference of values of a parameter across the interface Σ,

$$\Delta w = w^{(2)} - w^{(1)}. \tag{30}$$

The symbols $\delta^{*(I)}$ and $\epsilon^{*(I)}$ denote

$$\delta^{*(I)} = \frac{A_{13}^{(I)} + 2A_{55}^{(I)} - A_{33}^{(I)}}{\alpha^2}, \quad \epsilon^{*(I)} = \frac{A_{11}^{(I)} - A_{33}^{(I)}}{2\alpha^2}. \tag{31}$$

Voigt notation $A_{\alpha\beta}^{(I)}$ for density-normalized elastic parameters is used above instead of the tensor notation $a_{ijkl}^{(I)}$. The symbol θ_P in equation (29) denotes the angle of incidence of the P-wave at the interface Σ. The parameters α, β, and ρ of the background isotropic medium can be chosen arbitrarily but so the weakly anisotropic media on both sides of the interface Σ do not deviate much from the background isotropic medium. The dependence of the reflection coefficient on the choice of parameters α, β, and ρ offers an opportunity to control the precision of formula (29) in the vicinity of a selected angle of incidence. Here we choose the parameters as (e.g., Banik, 1987; Thomsen, 1993) $\alpha=\bar{\alpha}$, $\beta=\bar{\beta}$, $\rho=\bar{\rho}$, where the bar denotes averaging of the values of a parameter w on both sides of the interface

$$\bar{w} = \tfrac{1}{2}\left(w^{(1)} + w^{(2)}\right). \tag{32}$$

As the abovementioned authors did, we define

$$\left(\alpha^{(I)}\right)^2 = A_{33}^{(I)}, \qquad \left(\beta^{(I)}\right)^2 = A_{55}^{(I)}. \tag{33}$$

With specification (33), the parameters $\delta^{*(I)}$ and $\epsilon^{*(I)}$ in equation (31) transform into

$$\delta^{(I)} = \frac{A_{13}^{(I)} + 2A_{55}^{(I)} - A_{33}^{(I)}}{A_{33}^{(I)}}, \quad \epsilon^{(I)} = \frac{A_{11}^{(I)} - A_{33}^{(I)}}{2A_{33}^{(I)}}. \tag{34}$$

Note that $\delta^{(I)}$ and $\epsilon^{(I)}$ are linearized versions of Thomsen's (1986) parameters. We call them weak anisotropy parameters (see Pšenčík and Gajewski, 1998). If we introduce P-wave impedance $Z^{(I)}$ and shear modulus $G^{(I)}$,

$$Z^{(I)} = \rho^{(I)}\alpha^{(I)}, \qquad G^{(I)} = \rho^{(I)}\left(\beta^{(I)}\right)^2, \tag{35}$$

formula (29) for the reflection coefficient can be rewritten into

$$R_{PP}(\theta_P) = R_{PP}^{iso}(\theta_P) + \tfrac{1}{2}\left(\Delta\delta + \Delta\epsilon\tan^2\theta_P\right)\sin^2\theta_P, \tag{36}$$

where

$$R_{PP}^{iso}(\theta_P) = \frac{1}{2}\frac{\Delta Z}{\bar{Z}} + \frac{1}{2}\frac{\Delta\alpha}{\bar{\alpha}}\tan^2\theta_P - 2\left(\frac{\bar{\beta}}{\bar{\alpha}}\right)^2\frac{\Delta G}{\bar{G}}\sin^2\theta_P. \tag{37}$$

The coefficient $R_{PP}^{iso}(\theta_P)$ is the well-known PP-wave reflection coefficient for a weak contrast interface between two slightly different isotropic media (see, e.g., Aki and Richards, 1980). Equation (36) thus gives the PP-wave reflection coefficient for an interface between two slightly different weakly anisotropic media as a sum of the PP-wave reflection coefficient for a weak contrast interface separating two background isotropic half-spaces and a perturbation is attributable to the deviation of anisotropic half-spaces from their isotropic backgrounds. The perturbation is controlled by the weak anisotropy parameters δ and ϵ. Although formula (36) holds for arbitrary weakly anisotropic media, it is identical with the formula of Rueger (1996), which was derived for an interface separating two weakly transversely isotropic media with axis of symmetry perpendicular to the interface.

The contrasts of parameters δ and ϵ affect the reflection coefficient only for nonzero offsets. As already observed by previous investigators, the contrast of parameter δ has a decisive influence on R_{PP} for nearly normal incidence, i.e., for small offsets. If the contrast $\Delta\delta$ is small, then the deviation of the reflection coefficient from $R_{PP}^{iso}(\theta_P)$ will be negligible even if anisotropy of half-spaces surrounding the interface is nonnegligible. If one of the half-spaces is isotropic, the contrasts of the weak anisotropy parameters in equation (36) reduce to values of the weak anisotropy parameters of the anisotropic half-space. If the half-space, in which incident wave propagates, is isotropic, then the parameters appear in the reduced equation with the same signs as their contrasts. If the other half-space is isotropic, the parameters appear in the reduced equation with opposite signs than corresponding contrasts in (36). For the weak anisotropy parameters, δ and ϵ equal zero in both half-spaces, i.e., for the case of an interface separating two isotropic half-spaces, reflection coefficient (36) reduces to $R_{PP}^{iso}(\theta_P)$. Formula (36) is applicable only if the term $\Delta\epsilon\tan^2\theta_P$ is small. Generally for large angles of incidence, formula (36) diverges.

Another important fact which follows directly from equations (36) and (37) is that coefficient $R_{PP}(\theta_P)$ is symmetrical with respect to the normal incidence. This means coefficient $R_{PP}(\theta_P)$ is reciprocal.

PP-WAVE REFLECTION COEFFICIENTS FOR A GENERAL INCIDENCE

Formula (36) holds only in the plane (x, z). It can be generalized for any vertical plane of incidence making an arbitrary angle ϕ with the (x, z) plane. The following transformation relations for the involved elastic parameters must be taken into account:

$$\begin{aligned}
A_{11} &= A'_{11}\cos^4\phi + 4A'_{16}\cos^3\phi\sin\phi + 2(A'_{12} + 2A'_{66}) \\
&\quad \times\cos^2\phi\sin^2\phi + 4A'_{26}\sin^3\phi\cos\phi + A'_{22}\sin^4\phi, \\
A_{13} &= A'_{13}\cos^2\phi + 2A'_{36}\cos\phi\sin\phi + A'_{23}\sin^2\phi, \\
A_{33} &= A'_{33}, \\
A_{55} &= A'_{55}\cos^2\phi + 2A'_{45}\cos\phi\sin\phi + A'_{44}\sin^2\phi.
\end{aligned} \tag{38}$$

Inserting equation (38) into equation (29) and using again $A_{\alpha\beta}$ instead of $A'_{\alpha\beta}$, we arrive at the expression for the PP reflection coefficient $R_{PP}(\phi, \theta_P)$ in an arbitrary plane of incidence at an interface separating two weakly anisotropic media:

$$\begin{aligned}
R_{PP}(\phi, \theta_P) &= \frac{\rho\Delta A_{33} + 2\alpha^2\Delta\rho}{4\rho\alpha^2} \\
&+ \frac{1}{2}\left[\frac{\Delta(A_{13} + 2A_{55} - A_{33})}{\alpha^2}\cos^2\phi\right. \\
&+ \frac{\Delta(A_{23} + 2A_{44} - A_{33})}{\alpha^2}\sin^2\phi \\
&+ 2\frac{\Delta(A_{36} + 2A_{45})}{\alpha^2}\cos\phi\sin\phi - 4\frac{\Delta A_{55}}{\alpha^2}\cos^2\phi \\
&- 8\frac{\Delta A_{45}}{\alpha^2}\cos\phi\sin\phi - 4\frac{\Delta A_{44}}{\alpha^2}\sin^2\phi \\
&\left. - 4\frac{\beta^2\Delta\rho}{\rho\alpha^2} + \frac{\Delta A_{33}}{2\alpha^2}\right]\sin^2\theta_P + \frac{1}{2}\left[\frac{\Delta A_{33}}{2\alpha^2}\right. \\
&+ \frac{\Delta(A_{11} - A_{33})}{2\alpha^2}\cos^4\phi + \frac{\Delta(A_{22} - A_{33})}{2\alpha^2}\sin^4\phi \\
&+ \frac{\Delta(A_{12} + 2A_{66} - A_{33})}{\alpha^2}\cos^2\phi\sin^2\phi \\
&\left. + 2\frac{\Delta A_{16}\cos^2\phi + \Delta A_{26}\sin^2\phi}{\alpha^2}\sin\phi\cos\phi\right] \\
&\times\sin^2\theta_P\tan^2\theta_P.
\end{aligned} \tag{39}$$

The final form of the formula for the reflection coefficient depends again on the choice of the parameters of the background isotropic medium. If we choose α, β, Z, and G in the same way as in equations (32), (33), and (35), equation (39) can be rewritten

$$\begin{aligned} R_{PP}(\phi,\theta_P) = R^{iso}_{PP}(\theta_P) \\ &+\frac{1}{2}\Bigg[\Delta\left(\frac{A_{13}+2A_{55}-A_{33}}{A_{33}}\right)\cos^2\phi \\ &+\Bigg(\Delta\left(\frac{A_{23}+2A_{44}-A_{33}}{A_{33}}\right) \\ &-8\Delta\left(\frac{A_{44}-A_{55}}{2A_{33}}\right)\Bigg)\sin^2\phi \\ &+2\Bigg[\Delta\left(\frac{A_{36}+2A_{45}}{A_{33}}\right) \\ &-4\Delta\left(\frac{A_{45}}{A_{33}}\right)\Bigg]\cos\phi\sin\phi\Bigg]\sin^2\theta_P \\ &+\frac{1}{2}\Bigg[\Delta\left(\frac{A_{11}-A_{33}}{2A_{33}}\right)\cos^4\phi \\ &+\Delta\left(\frac{A_{22}-A_{33}}{2A_{33}}\right)\sin^4\phi \\ &+\Delta\left(\frac{A_{12}+2A_{66}-A_{33}}{A_{33}}\right)\cos^2\phi\sin^2\phi \\ &+2\Delta\left(\frac{A_{16}}{A_{33}}\right)\cos^3\phi\sin\phi \\ &+2\Delta\left(\frac{A_{26}}{A_{33}}\right)\sin^3\phi\cos\phi\Bigg]\sin^2\theta_P\tan^2\theta_P. \end{aligned} \tag{40}$$

The symbol $R^{iso}_{PP}(\theta_P)$ denotes again the weak contrast reflection coefficient at an interface separating two isotropic media [see equation (37)]. Equation (40) represents the PP reflection coefficient for a weak contrast interface separating two arbitrary weakly anisotropic media. We can see from equation (40) that at normal incidence, the reflection coefficient is not affected by anisotropy. This is a generalization of Tsvankin's (1996) observation concerning transverse isotropy. In equation (40), we can identify contrasts of 8 of 14 P-wave weak anisotropy parameters. Equation (40) also depends on the contrast of two additional parameters, $(A_{44}-A_{55})/A_{33}$ and A_{45}/A_{33}. Thus, if the contrasts of the eight P-wave weak anisotropy parameters were known from some independent information, formula (40) could be used to determine the latter parameters, i.e., parameters related to the S-wave propagation in weakly anisotropic media.

PP-WAVE REFLECTION COEFFICIENTS FOR TRANSVERSELY ISOTROPIC MEDIA WITH A HORIZONTAL AXIS OF SYMMETRY ALONG THE *X*-AXIS

Let us consider a transversely isotropic medium with a horizontal axis of symmetry. This kind of anisotropy is very important because it can be understood as caused by a system of parallel vertical cracks. For this case, the nonzero density-normalized elastic parameters satisfy the following relations:

$$A_{33}=A_{22},\quad A_{66}=A_{55},\quad A_{13}=A_{12},\quad A_{23}=A_{33}-2A_{44}. \tag{41}$$

For such a specification, formula (40) reduces to

$$\begin{aligned} R_{PP}(\phi,\theta_P) = R^{iso}_{PP}(\theta_P) \\ &+\frac{1}{2}\left[\Delta\delta\cos^2\phi-8\left(\frac{\bar\beta}{\bar\alpha}\right)^2\Delta\gamma\sin^2\phi\right]\sin^2\theta_P \\ &+\frac{1}{2}(\Delta\epsilon\cos^4\phi+\Delta\delta\cos^2\phi\sin^2\phi)\sin^2\theta_P\tan^2\theta_P, \end{aligned} \tag{42}$$

where δ and ϵ are given by equation (34) and $\gamma=(A_{44}-A_{55})/2A_{55}$. Here, γ is a parameter related to the S-wave splitting parameter introduced by Thomsen (1986) (see also Rueger, 1996). Equation (42) indicates again that the PP reflection coefficient can be used for retrieving the parameter γ.

If we specify formula (42) for $\phi=0$, i.e., for a profile along the axis of symmetry, we get a formula equivalent to the formula obtained earlier by Rueger (1996). The apparent differences between the formulas are caused because we do not introduce anisotropy parameters of the equivalent VTI model as Rueger (1996) did. For $\phi=\pi/2$, i.e., in the isotropy plane of the transversely isotropic medium, formula (42) reduces to

$$R_{PP}(\phi,\theta_P)=R^{iso}_{PP}(\phi,\theta_P)-4\left(\frac{\bar\beta}{\bar\alpha}\right)^2\Delta\gamma\sin^2\theta_P. \tag{43}$$

In this case, the deviation of the reflection coefficient from R^{iso}_{PP} is controlled solely by the contrast in parameter γ.

TEST EXAMPLE

We consider three models consisting of two homogeneous half-spaces. The half-space in which the incident wave propagates is isotropic; the other half-space is transversely isotropic with a horizontal axis of symmetry. For these models we calculate values of the reflection coefficient $R_{PP}(\phi,\theta_P)$ using numerical solution of equations (8) and compare them with values calculated from the approximate formula resulting from equation (42) for the above specification of the model:

$$\begin{aligned} R_{PP}(\phi,\theta_P) = R^{iso}_{PP}(\theta_P) \\ &+\frac{1}{2}\left[\delta^{(2)}\cos^2\phi-8\left(\frac{\bar\beta}{\bar\alpha}\right)^2\gamma^{(2)}\sin^2\phi\right]\sin^2\theta_P \\ &+\frac{1}{2}\left(\epsilon^{(2)}\cos^4\phi+\delta^{(2)}\cos^2\phi\sin^2\phi\right)\sin^2\theta_P\tan^2\theta_P. \end{aligned} \tag{44}$$

Note again that for models with isotropic overburden, the reflection coefficient depends directly on weak anisotropy parameters of the anisotropic half-space.

The P- and S-wave velocities in the isotropic half-space for case A are $\alpha=4.0$ km/s, $\beta=2.31$ km/s, and $\rho=2.65$ g/cm^3; for case B they are $\alpha=3.0$ km/s, $\beta=1.73$ km/s, and $\rho=2.2$ g/cm^3. Anisotropy of the lower half-space is assumed to be caused by a system of vertical parallel dry cracks (see Hudson, 1981). The P- and S-wave velocities of the host rock are 4.0 km/s and 2.31 km/s, and the density is 2.6 g/cm^3. The aspect ratio 10^{-4} and the crack density of 0.05 for case C and 0.1 for case D are considered. The corresponding matrices of the density-normalized elastic parameters, in (km/s)^2, for the axis of symmetry along the x-axis in case C have the form

$$\begin{pmatrix} 11.96 & 3.99 & 3.99 & 0.00 & 0.00 & 0.00 \\ & 15.55 & 4.88 & 0.00 & 0.00 & 0.00 \\ & & 15.55 & 0.00 & 0.00 & 0.00 \\ & & & 5.33 & 0.00 & 0.00 \\ & & & & 4.76 & 0.00 \\ & & & & & 4.76 \end{pmatrix}.$$

In case D they have the form

$$\begin{pmatrix} 9.43 & 3.14 & 3.14 & 0.00 & 0.00 & 0.00 \\ & 15.27 & 4.60 & 0.00 & 0.00 & 0.00 \\ & & 15.27 & 0.00 & 0.00 & 0.00 \\ & & & 5.33 & 0.00 & 0.00 \\ & & & & 4.25 & 0.00 \\ & & & & & 4.25 \end{pmatrix}.$$

Vertical sections of the phase velocity surfaces containing the axes of symmetry for cases C and D are shown in Figure 1.

We consider three models: A/C, A/D, and B/D. In models A/C and A/D, the phase velocity of the overburden is for all azimuths greater than the phase velocity in the reflecting half-space. In model B/D, the relation is opposite. In all cases, the values of reflection coefficients start to deviate from zero considerably for greater values of angles of incidence and the approximate formulas become inapplicable. For this reason, we consider the angles of incidence in only a limited interval.

The values of the velocities and the density of the background isotropic medium were determined by averaging [see equations (32) and (33)]. For specific values, see the figure captions. The results are displayed in the form of four plots shown in Figures 2–4. In all the plots, the horizontal axis corresponds to the angle of incidence θ_P, measured in degrees. The vertical axis corresponds to the azimuth ϕ, also in degrees. As mentioned above, the angles of incidence are considered only in the limited interval (0°, 42°). Azimuth $\phi = 0°$ corresponds to the profile along the axis of symmetry; azimuth $\phi = 90°$ corresponds to the profile in the plane perpendicular to the axis of

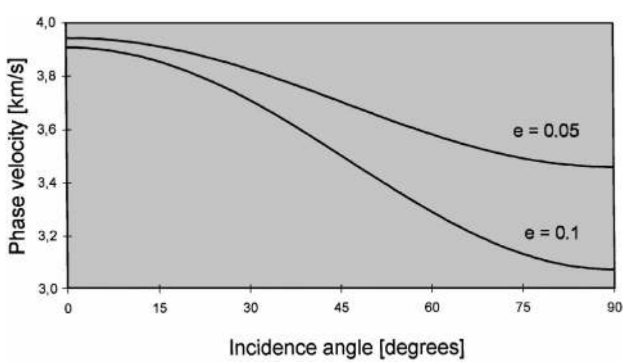

FIG. 1. Vertical sections of the P-wave phase velocity surfaces containing the axes of symmetry for two dry crack models [with crack density $e = 0.05$ (C) and $e = 0.1$ (D)].

symmetry, i.e., in the isotropy plane. The upper plots show exact and approximate R_{PP} reflection coefficients; the lower plots show absolute and relative errors of the approximate formula.

From Figure 1, we can see that anisotropy in model A/C is not too strong (about 12%). The contrast of velocities across the reflector is always <10%, so we can expect a good performance of the approximate formula for this model. This is confirmed by Figure 2. The relative error of the approximately determined coefficient is always <3% for angles of incidence <20°. For larger azimuths, this accuracy is guaranteed even for the broader range of angles of incidence.

In model A/D (see Figure 3) the velocity contrast is slightly higher than in model A/C, but it never exceeds 12%. Anisotropy of the reflecting half-space is, however, much stronger than in model A/C: it reaches nearly 20%. The relative errors of the approximately determined coefficient are slightly higher; still, its fit with the exact coefficient is very good.

Figure 4 shows results for model B/D. In addition to strong anisotropy (nearly 20% as in Figure 3), we also deal with strong velocity contrast (nearly 25%). Because the contrast is positive, a critical angle exists in this case. For the azimuth of 90°, i.e., in the plane of isotropy, the critical angle is about 50°. For other azimuths it will be slightly higher. This means the approximate formula for the reflection coefficient can be used only for angles <50°. We can see in Figure 4 that the coefficients practically do not vary with azimuth for angles of incidence <25° because of the dominant role of the isotropic contrast for these angles. For angles of incidence >25°, effects of anisotropy become observable. In this region greater deviations of the approximate coefficients from the exact ones can be observed. The deviations are larger for greater azimuths. This is because critical angles for larger azimuths (close to the isotropy plane) appear for lower values of the angles of incidence. The approximate formula works best for azimuths around 30°. For lower azimuths, it yields larger values than exact; for greater azimuths, it yields smaller values.

DISCUSSION AND CONCLUSIONS

A formula for PP reflection coefficient for weak contrast interfaces separating two arbitrary weakly anisotropic media was derived. It was shown that the PP reflection coefficient depends, among other things, on differences of 8 of the 14 weak anisotropy parameters characterizing the phase velocity and polarization vector of a P-wave in a weakly anisotropic medium (see Pšenčík and Gajewski, 1998). In addition to these parameters, the formula also depends on contrasts of parameters characterizing vertical propagation of S-waves. The reflection coefficient does not depend at all on the contrasts of elastic parameters A_{14}, A_{15}, A_{24}, A_{25}, A_{34}, A_{35}, A_{46}, and A_{56} or their combinations. Thus, these parameters can never be recovered from the study of a PP reflection coefficient. Since the dependence of the reflection coefficient on the differences of weak anisotropy parameters across an interface is linear, it offers automatically an easy way to construct sensitivity operators to determine these parameters in AVO (or AVA) studies from the measured reflection coefficients.

The derived formulas do not depend only on differences of elastic parameters across an interface but also depend on the choice of the background medium. For this article, traditional averaging was used to determine the parameters of the background medium. Other choices, however, are possible. For example, for a forward modeling of reflections for small angles of incidence, it might be useful to choose the parameters in such a way that the normal incidence term equals the value of the exact normal incidence reflection coefficient.

The derived formula is applicable in regions in which the reflection coefficient is relatively small. For small incidence angles, the first normal incidence term and the lowest order angular correction term will describe the coefficient with sufficient precision. The deviations of the coefficient from the isotropic coefficient are, in such a situation, controlled by the differences of three P-wave weak anisotropy parameters,

$$\frac{A_{13}+2A_{55}-A_{33}}{A_{33}},\quad \frac{A_{23}+2A_{44}-A_{33}}{A_{33}},\quad \frac{A_{36}+2A_{45}}{A_{33}}$$

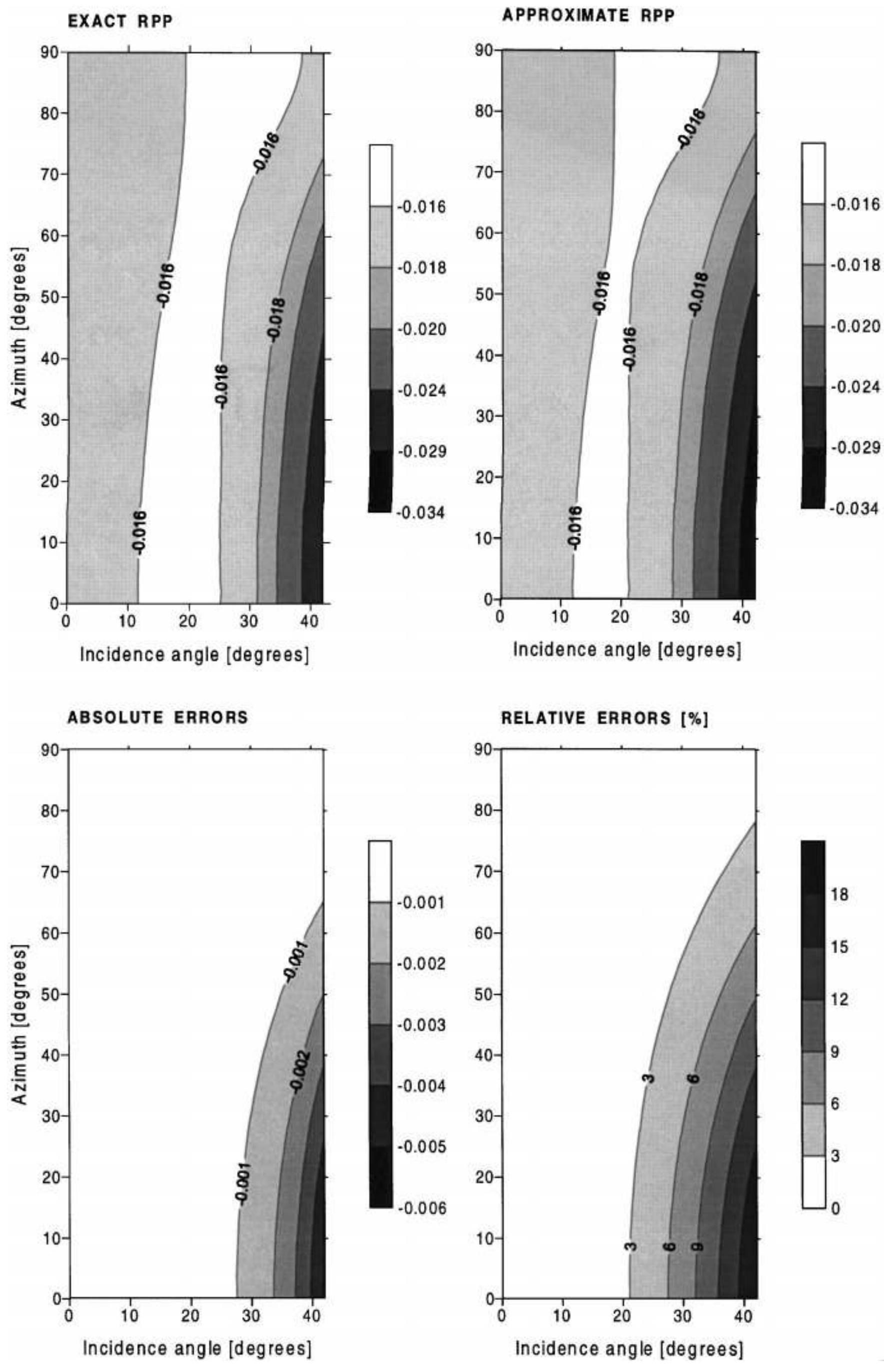

FIG. 2. The maps of exact and approximate PP reflection coefficients (upper pictures) and absolute and relative errors of the approximate coefficient (bottom pictures) at the interface between isotropic half-space ($\alpha = 4.00$ km/s, $\beta = 2.31$ km/s, $\rho = 2.65$ g/cm^3) and transversely isotropic half-space (axis of symmetry along x-axis; host rock: $\alpha = 4.00$ km/s, $\beta = 2.31$ km/s, $\rho = 2.60$ g/cm^3; dry cracks: aspect ratio = 0.0001, crack density = 0.05). Isotropic background is $\alpha = 3.97$ km/s, $\beta = 2.25$ km/s, $\rho = 2.63$ g/cm^3.

and two additional parameters,

$$\frac{A_{44} - A_{55}}{A_{33}}, \qquad \frac{A_{45}}{A_{33}}.$$

Comparison with formulas for the P-wave phase velocity and polarization (see Pšenčík and Gajewski, 1998) show that small-angle P-wave reflections are affected by the same weak anisotropy parameters. A similar conclusion holds for larger angles, for which the higher order angular correction term must also be considered.

We tested the accuracy of the derived formula on models containing an interface separating an isotropic half-space, in which the incident wave propagates, from the transversely isotropic half-space with the horizontal axis of symmetry. We calculated the reflection coefficient for all azimuths and a wide range of angles of incidence and compared the results with exactly calculated coefficients. Even if the velocity contrast

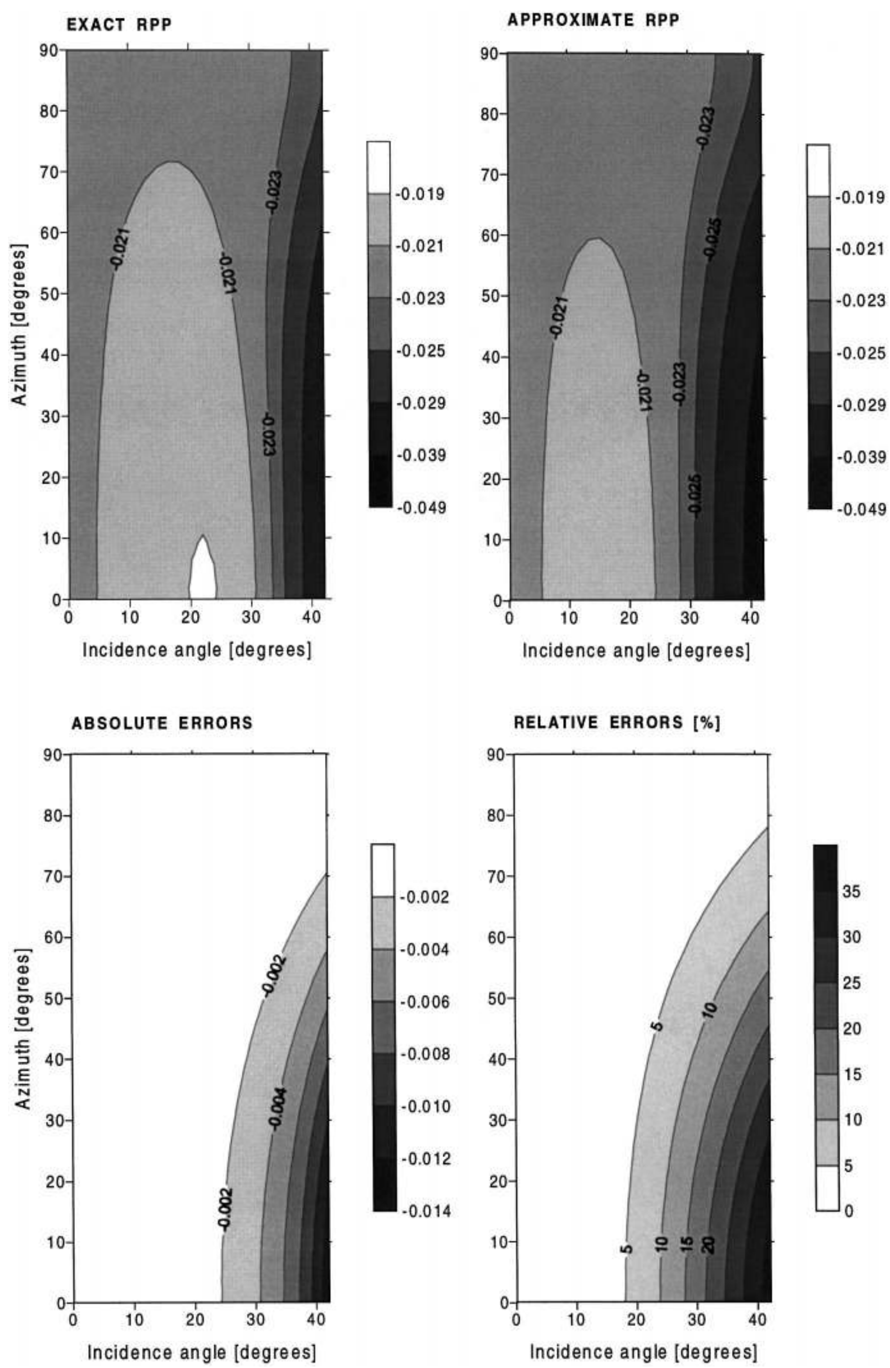

FIG. 3. The same as in Figure 2, with isotropic half-space ($\alpha = 4.00$ km/s, $\beta = 2.31$ km/s, $\rho = 2.65$ g/cm^3) and transversely isotropic half-space (axis of symmetry along x-axis; host rock: $\alpha = 4.00$ km/s, $\beta = 2.31$ km/s, $\rho = 2.60$ g/cm^3; dry cracks: aspect ratio = 0.0001, crack density = 0.1). Isotropic background is $\alpha = 3.95$ km/s, $\beta = 2.19$ km/s, $\rho = 2.63$ g/cm^3.

between the two half-spaces was rather strong (nearly 25%) and anisotropy of the reflecting half-space was also rather strong (nearly 20%), the approximate formula yielded very good results.

To keep the paper reasonably short, we presented only the formula for the PP reflection coefficient. Derivation of the formula for the PP transmission coefficient is straightforward from the presented equations (Pšenčík and Vavryčuk, 1998). Derivation of R/T coefficients for converted waves is more complicated because of the appearance of angles Φ and Ψ specifying the polarization vectors of S-waves in background isotropic medium.

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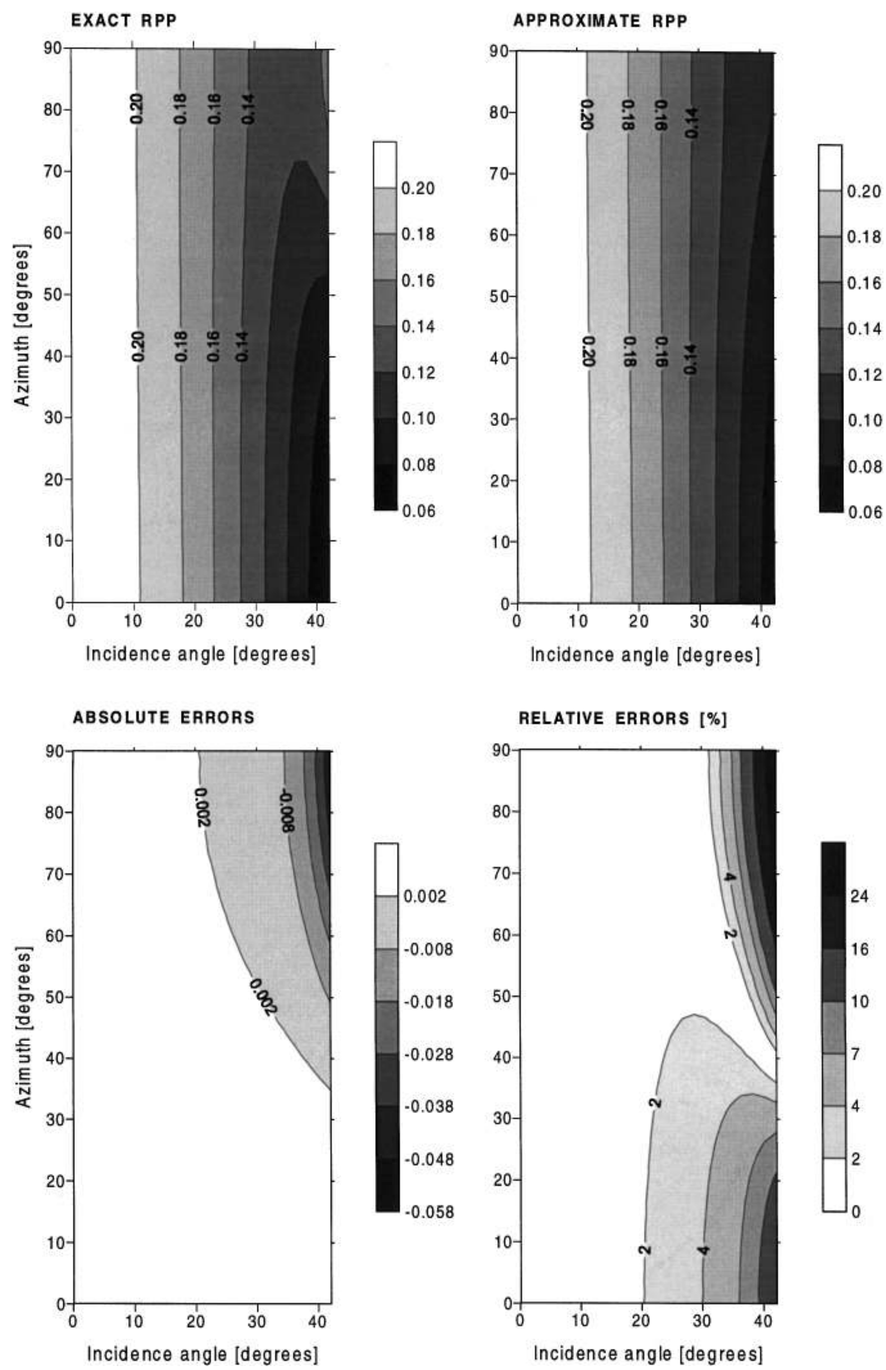

FIG. 4. The same as Figure 2, with isotropic half-space ($\alpha = 3.00$ km/s, $\beta = 1.73$ km/s, $\rho = 2.20$ g/cm^3) and transversely isotropic half-space (axis of symmetry along x-axis; host rock: $\alpha = 4.00$ km/s, $\beta = 2.31$ km/s, $\rho = 2.60$ g/cm^3; dry cracks: aspect ratio = 0.0001, crack density = 0.1). Isotropic background is $\alpha = 3.45$ km/s, $\beta = 1.90$ km/s, $\rho = 2.40$ g/cm^3.

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REFERENCES

Aki, K., and Richards, P. G., 1980, Quantitative seismology: W.H. Freeman & Co.
Backus, G. E., 1965, Possible forms of seismic anisotropy of the uppermost mantle under oceans: J. Geophys. Res., **70**, 3429–3439.
Banik, N. C., 1987, An effective parameter in transversely isotropic media: Geophysics, **52**, 1654–1664.
Červený, V., 1982, Direct and inverse kinematic problems for inhomogeneous anisotropic media—Linearization approach: Contr. Geophys. Inst. Slov. Acad. Sci., **13**, 127–133.
Daley, P. F., and Hron, F., 1977, Reflection and transmission coefficients for transversely isotropic media: Bull. Seismol. Soc. Am., **67**, 661–675.
Gajewski, D., and Pšenčík, I., 1987, Computation of high-frequency seismic wavefields in 3-D laterally inhomogeneous anisotropic media: Geophys. J. R. Astr. Soc., **91**, 383–411.
Haugen, G. U., and Ursin, B., 1996, AVO-A analysis of vertically fractured reservoir underlaying shale: Presented at the 66th Ann. Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 1826–1829.
Hearn, A. C., 1991, Reduce 3.4. User's manual: Rand Corp.,
Hudson, J. A., 1981, Wave speeds and attenuation of elastic waves in material containing cracks: Geophys. J. R. Astr. Soc., **64**, 133–150.
Jech, J., and Pšenčík, I., 1989, First-order perturbation method for anisotropic media: Geophys. J. Internat., **99**, 369–376.
Keith, C. M., and Crampin, S., 1977, Seismic body waves in anisotropic media: Reflection and refraction at a plane interface: Geophys. J. Roy. Astr. Soc. **49**, 181–208.
Pšenčík, I., and Gajewski, D., 1998, Polarization, phase velocity and NMO velocity of qP waves in arbitrary weakly anisotropic media: Geophysics, **63**, 1754–1766.
Pšenčík, I., and Vavryčuk, V., 1998, Weak contrast PP wave displacement R/T coefficients in weakly anisotropic elastic media: Pageoph, **151**, 699–718.
Rueger, A., 1996, Analytic description of reflection coefficients in anisotropic media: 58th Ann. Mtg., Eur. Assn. Geoscientists and Engineers, Extended Abstracts, **1**, P026.
——— 1997, P-wave reflection coefficients for transversely isotropic models with vertical and horizontal axis of symmetry: Geophysics, **62**, 713–722.
Thomsen, L., 1986, Weak elastic anisotropy: Geophysics, **51**, 1954–1966.
———1993, Weak anisotropic reflections, *in* Castagna, J. P., and Backus, M. M., Eds., Offset-dependent reflectivity—Theory and practice of AVO analysis: Soc. Expl. Geophys., 103–111.
Tsvankin, I., 1996, P-wave signatures and parameterization of transversely isotropic media: An overview: Geophysics, **61**, 467–483.
Ursin, B., and Haugen, G. V., 1996, Weak-contrast approximation of the elastic scattering matrix in anisotropic media: Pageoph, **148**, 685–714.
Zillmer, M., Gajewski, D., and Kashtan, B. M., 1997, Reflection coefficients for weak anisotropic media: Geophys. J. Internat., **129**, 389–398.
———1998, Anisotropic reflection coefficients for a weak-contrast interface: Geophys. J. Internat., **132**, 159–166.

APPENDIX

SLOWNESS AND POLARIZATION VECTORS OF INCIDENT AND UNCONVERTED TRANSMITTED *P*-WAVES IN WEAKLY ANISOTROPIC MEDIA

The slowness vector $p_i^{(0)}$ of the incident P-wave propagating in a weakly anisotropic medium whose direction coincides with the direction of the slowness vector $p_i^{0(0)}$ in the background medium can be written as

$$p_i^{(0)} = p_i^{0(0)} + \delta p_i^{(0)} = p_i^{0(0)}\big(1 - \alpha^{-1}\delta c^{(0)}\big). \tag{A-1}$$

Here $\delta c^{(0)}$ denotes a deviation of the phase velocity of the incident wave from the phase velocity in the background medium. The deviations $\delta c^{(0)}$ and similarly $\delta c^{(6)}$ for the transmitted P-wave, which will be needed later, are given by the well-known formula (Backus, 1965; Červený, 1982; Jech and Pšenčík, 1989),

$$\begin{aligned}\delta c^{(0)} &= \tfrac{1}{2}\alpha\delta a_{ijkl}^{(1)} p_i^{0(0)} g_j^{0(0)} g_k^{0(0)} p_l^{0(0)},\\ \delta c^{(6)} &= \tfrac{1}{2}\alpha\delta a_{ijkl}^{(2)} p_i^{0(0)} g_j^{0(0)} g_k^{0(0)} p_l^{0(0)}.\end{aligned} \tag{A-2}$$

For the determination of the slowness vector $p_i^{(6)}$ of the transmitted P-wave, we use equation (4). The tangential projection b_i of the slowness vector into the interface Σ can be determined from equations (5) and (A-1) as

$$\begin{aligned}b_i &= b_i^0 + \delta b_i\\ &= \big[p_i^{0(0)} - \big(p_k^{0(0)}\nu_k\big)\nu_i\big] - \alpha^{-1}\delta c^{(0)}\big[p_i^{0(0)} - \big(p_k^{0(0)}\nu_k\big)\nu_i\big].\end{aligned} \tag{A-3}$$

The projection $\xi^{(6)}$ of the slowness vector into the normal ν_i to the interface Σ can be determined from the eikonal equation,

$$a_{ijkl}^{(2)} p_i^{(6)} p_l^{(6)} g_j^{(6)} g_k^{(6)} = 1, \tag{A-4}$$

which simply follows from the Christoffel equation (7). In a weakly anisotropic medium, equation (A-4) can be expanded as

$$\begin{aligned}&a_{ijkl}^{0} p_i^{0(0)} p_l^{0(0)} g_j^{0(0)} g_k^{0(0)} + \delta a_{ijkl}^{(2)} p_i^{0(0)} p_l^{0(0)} g_j^{0(0)} g_k^{0(0)}\\ &+ 2a_{ijkl}^{0}\delta b_i p_l^{0(0)} g_j^{0(0)} g_k^{0(0)} + 2a_{ijkl}^{0}\delta\xi^{(6)}\nu_i p_l^{0(0)} g_j^{0(0)} g_k^{0(0)} = 1.\end{aligned} \tag{A-5}$$

Here δb_i and $\delta\xi^{(6)}\nu_i$ are the deviations of the tangent and normal components of the slowness vector of the transmitted wave in the weakly anisotropic medium from the same components in the background isotropic medium. The deviation δb_i given in equation (A-3); $\delta\xi^{(6)}\nu_i$ can be determined from equation (A-5). Using equations (13), (20), (21), (A-2) and (A-3) in equation (A-5), we get, after some manipulation,

$$\delta\xi^{(6)} = \frac{\delta c^{(0)}\big[1 - \alpha^2\big(\nu_k p_k^{0(0)}\big)^2\big] - \delta c^{(6)}}{\alpha^3\big(\nu_k p_k^{0(0)}\big)}. \tag{A-6}$$

Using equations (A-3) and (A-6), we can write the final formula for the slowness vector of the transmitted wave in a weakly anisotropic medium as

$$p_i^{(6)} = p_i^{0(0)} + \delta p_i^{(6)}$$
$$= p_i^{0(0)} - \left[\alpha^{-1} p_i^{0(0)} \delta c^{(0)} + \frac{\left(\delta c^{(6)} - \delta c^{(0)}\right)}{\alpha^3 \left(\nu_k p_k^{0(0)}\right)} \nu_i\right]. \quad \text{(A-7)}$$

The polarization vector of the incident or transmitted P-wave propagating in the weakly anisotropic medium can be determined from formulas presented by Jech and Pšenčík (1989). For a given slowness vector, the polarization vector of the P-wave is given as the sum of a unit vector in the direction of the slowness vector and a perturbation. We can write

$$g_i^{(0)} = g_i^{0(0)} + \delta g_i^{(0)}, \qquad g_i^{(6)} = g_i^{0(6)} + \delta g_i^{(6)}, \quad \text{(A-8)}$$

where $g_i^{0(0)}$ is given in equation (21) and $g_i^{0(6)}$ can be obtained by normalization of the slowness vector $p_i^{(6)}$ given in equation (A-7),

$$g_i^{0(6)} = \alpha p_i^{0(0)} + \left(\delta c^{(6)} - \delta c^{(0)}\right)\left(p_i^{0(0)} - \frac{\nu_i}{\alpha^2 \left(\nu_k p_k^{0(0)}\right)}\right). \quad \text{(A-9)}$$

The perturbations $\delta g_i^{(N)}$, $N = 0, 6$, in equation (A-8) are (Jech and Pšenčík, 1989)

$$\delta g_m^{(0)} = \frac{\alpha}{\alpha^2 - \beta^2} \delta a_{ijkl}^{(1)} p_i^{0(0)} g_j^{0(0)} g_k^{0(0)} \left(\delta_{lm} - g_l^{0(0)} g_m^{0(0)}\right), \quad \text{(A-10)}$$
$$\delta g_m^{(6)} = \frac{\alpha}{\alpha^2 - \beta^2} \delta a_{ijkl}^{(2)} p_i^{0(0)} g_j^{0(0)} g_k^{0(0)} \left(\delta_{lm} - g_l^{0(0)} g_m^{0(0)}\right).$$

The determination of the amplitude-normalized tractions in weakly anisotropic media is straightforward. It can be obtained by inserting expressions (13), (A-1), (A-7), and (A-8) into equation (9).