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Practical decomposition for physically admissible differential Mueller matrices

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Abstract

The differential Mueller matrix expresses the local action of an optical medium on the evolution of a propagating electromagnetic field, including partially coherent and partially polarized waves. Here, we present a derivation of the differential Mueller matrix from the canonical form of Type I Mueller matrices without making use of the exponential generators of uniform media. We demonstrate how to practically obtain this parameterization numerically using an eigenvalue decomposition and find validity criteria to ensure that the matrix satisfies the constraints of a physical system. This provides a convenient tool-set to investigate depolarization effects and extends previous treatments of the differential Mueller matrix formalism.

The Stokes-Mueller formalism governs the propagation of coherent as well as partially coherent and partially polarized paraxial waves. Azzam first introduced the differential formalism for Mueller matrices in 1978 [1], relating the local change of the four-element Stokes vector **S** along the propagation direction to the 4×4 differential, or local, Mueller matrix **m**:

$$\frac{d\mathbf{S}}{dz}=\mathbf{m}\cdot\mathbf{S}=\frac{d\mathbf{M}}{dz}\cdot\mathbf{M}^{-1}\cdot\mathbf{S}. \quad (1)$$

The deterministic case of retardation and diattenuation is well understood [1]. The extension of this formalism to include depolarization effects, however, has attracted recent interest: Ortega-Quijano *et al*. provided a detailed eigen-analysis of the matrices **M** and **m** [2], and later derived **m** with an analogy to group theory [3]. Ossikovski first discussed the constraints that a differential Mueller matrix **m** must fulfill when corresponding to a physically valid Mueller matrix [4], without, however, providing criteria that would be simple to verify. Germer [5] proposed a parameterization for **m**, that was later shown by Devlaminck to not correctly account for the physicality in all situations [6]. Devlaminck proposed an alternative parameterization, restricted to diagonal depolarizing matrices, and

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then extended to a more general situation [7]. The same author recently also provided an interesting model for the physical interpretation of depolarization media [8], and together with Ossikovski, a general criterion for the physical realizability [9]. Still, a simple and practical way to obtain the parameterization of a general **m** has been missing.

Here, we derive the differential Mueller matrix directly from the canonical decomposition of Mueller matrices, introduced by Rao *et al*. [10], and completed by Simon [11]. We confirm the validity criteria of Devlaminck *et al*. [6] and obtain a parameterization that can be easily performed numerically and offers insight into the depolarization mechanism. Restricting our analysis to the case of diagonalizable Mueller matrices (i.e. type I according to Rao [10]), which represent the majority of experimental Mueller matrices, we can express a Mueller matrix **M** in its canonical form

$$\mathbf{M}=\mathbf{L}_2\cdot\mathbf{K}\cdot\mathbf{L}_1, \quad (2)$$

where $\mathbf{L}_{1,2} \in SO(3,1)$ and $\mathbf{K} = \mathrm{diag}(K_0, K_1, K_2, K_3)$ is a diagonal matrix, with $K_0 \geq \max(|K_1|, |K_2|, |K_3|)$ and subject to the constraints

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\cdot[K_0 K_1 K_2 K_3]^T \geq \mathbf{0}. \quad (3)$$

The first element of diagonal matrix **K** is positive and exceeds the norm of all other elements. The remaining elements do not have to be ordered, but it is always possible to find a permutation of $\mathbf{L}_{1,2}$ to do so.

As discussed by Ortega-Quijano [3], an element of SO(3,1) corresponds to a non-depolarizing, i.e. deterministic, Mueller matrix acting as a combination of a retarder and a diattenuator. By definition $\det(\mathbf{L}) = 1$, which excludes any attenuation from **L** and can result in a transmission exceeding unity. **L** further has the important property $\mathbf{L}^{\mathbf{T}}\cdot\mathbf{G}\cdot\mathbf{L} = \mathbf{G}$, where $^{\mathbf{T}}$ denotes the transpose and $\mathbf{G} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski matrix.

Through Eq. (1), the differential Mueller matrix is defined as $\mathbf{m} = (d\mathbf{M}/dz)\cdot\mathbf{M}^{-1}$, and taking the derivative of Eq. (2) one finds

$$\mathbf{m}=\frac{d\mathbf{L}_2}{dz}\cdot\mathbf{L}_2^{-1}+\mathbf{L}_2\cdot\frac{d\mathbf{K}}{dz}\cdot\mathbf{K}^{-1}\cdot\mathbf{L}_2^{-1}+\mathbf{L}_2\cdot\mathbf{K}\cdot\frac{d\mathbf{L}_1}{dz}\cdot\mathbf{M}^{-1}. \quad (4)$$

In the limit $z \to 0$, indicated by a sub-index $_0$, the vanishing layer described by $\mathbf{M}(z)$ has no more effect and one finds $\mathbf{M}_0 = \mathbf{I}$, and accordingly $\mathbf{K}_0 = \mathbf{I}$ as well as $\mathbf{L}_{20} = \mathbf{L}_{10}^{-1}$ and

$$\mathbf{m}\Big|_{z=0}=\underbrace{\frac{d\mathbf{L}_{20}}{dz}\cdot\mathbf{L}_{10}+\mathbf{L}_{20}\cdot\frac{d\mathbf{L}_{10}}{dz}}_{\mathbf{m}_{\det}}+\underbrace{\mathbf{L}_{20}\cdot\frac{d\mathbf{K}_0}{dz}\cdot\mathbf{L}_{20}^{-1}}_{\mathbf{m}_{dep}}. \quad (5)$$

The first two terms are identical to $d\mathbf{L}/dz$ where $\mathbf{L} = \mathbf{L}_2\cdot\mathbf{L}_1$. Although $\mathbf{L}_0 = \mathbf{I}$, the derivative $d\mathbf{L}_0/dz$ is generally not zero and defines the deterministic part of the differential Mueller matrix with the well-known 6 parameters corresponding to retardation (τ_1, τ_2, τ_3) and diattenuation (τ_4, τ_5, τ_6). This can be verified by employing the derivatives of $\mathbf{L}^{\mathbf{T}}\cdot\mathbf{G}\cdot\mathbf{L} = \mathbf{G}$ and using $\mathbf{L}_0 = \mathbf{I}$ to find that $d\mathbf{L}_0/dz = -\mathbf{G}\cdot d\mathbf{L}_0^{\mathbf{T}}/dz\cdot\mathbf{G}$, in accordance with the symmetry properties of $\mathbf{m}_{\mathrm{det}} = -\mathbf{G}\cdot\mathbf{m}_{\mathrm{det}}^{\mathrm{T}}\cdot\mathbf{G}$, which contains the Minkowski symmetric components of $\mathbf{m}$, as discussed by Ossikovski [4].

Because $\mathbf{G}\cdot\mathbf{L}_2 = (\mathbf{L}_2^{\mathbf{T}})^{-1}\cdot\mathbf{G}$, and $\mathbf{G}\cdot d\mathbf{K}/dz\cdot\mathbf{G} = d\mathbf{K}/dz$, the third term defines the depolarizing differential Mueller matrix, with the symmetry property $\mathbf{m}_{\mathrm{dep}} = \mathbf{G}\cdot\mathbf{m}_{\mathrm{dep}}^{\mathrm{T}}\cdot\mathbf{G}$, containing the Minkowski antisymmetric components of $\mathbf{m}$ [4]. $\mathbf{m}_{\mathrm{dep}}$ defines the additional 10 parameters, including the overall attenuation, to complete the familiar symmetric and antisymmetric 16 parameters of the complete differential Mueller matrix:

$$\mathbf{m}_{\mathrm{det}}=\begin{bmatrix} 0 & \tau_4 & \tau_5 & \tau_6 \\ \tau_4 & 0 & -\tau_3 & \tau_2 \\ \tau_5 & \tau_3 & 0 & -\tau_1 \\ \tau_6 & -\tau_2 & \tau_1 & 0 \end{bmatrix}, \mathbf{m}_{dep}=\begin{bmatrix} \kappa_0 & -\zeta_4 & -\zeta_5 & -\zeta_6 \\ \zeta_4 & \kappa_1 & \zeta_3 & \zeta_2 \\ \zeta_5 & \zeta_3 & \kappa_2 & \zeta_1 \\ \zeta_6 & \zeta_2 & \zeta_1 & \kappa_3 \end{bmatrix} \quad (6)$$

The differential Mueller matrix $\mathbf{m}_{\mathrm{dep}}$ is then of the form

$$\mathbf{m}_{dep}=\mathbf{L}\cdot\mathbf{D}\cdot\mathbf{L}^{-1}, \quad (7)$$

where $\mathbf{L} \in$ SO(3,1) and $\mathbf{D} = \mathrm{diag}(D_0, D_1, D_2, D_3)$ is defined through the derivate of the diagonal matrix $\mathbf{K}$. It is always possible to separate a general $\mathbf{M}$ into two consecutive elements and perform the limit $z \to 0$ on the second term, whereas the first term drops out from the computation of $\mathbf{m}$. Taking the limit $z \to 0$ thus does not limit the generality of this formalism. Eq. (7) was further obtained without any assumptions on the uniformity of the medium and without the use of exponential generators. It is a generalization of previous results [6], and is valid as long as $\mathbf{M}$ is a continuous function of z that verifies Eq. (3) at each depth and has defined derivatives.

Next, in order to verify under which conditions $\mathbf{m}$ is a physically valid differential matrix, we recall the original derivation of $\mathbf{m}$ according to Azzam [1]:

$$\mathbf{m}=\lim_{\Delta z\to 0}\frac{\mathbf{M}(\Delta z)-\mathbf{I}}{\Delta z}. \quad (8)$$

Accordingly, the matrix $\mathbf{M}(\Delta z) = \mathbf{I} + \Delta z\,\mathbf{m}$ has to be a physically valid Mueller matrix for sufficiently small Δz. Following Rao [10] and Simon [11], the N-matrix $\mathbf{N} = \mathbf{M}^{\mathrm{T}}\cdot\mathbf{G}\cdot\mathbf{M}$ of a Mueller matrix $\mathbf{M}$ allows to find the positive semi-definite matrix $\mathbf{G}\cdot\mathbf{N} = \mathbf{L}_1\cdot\mathbf{K}\cdot\mathbf{K}\cdot\mathbf{L}_1^{-1}$, with $\mathbf{K}$ subject to Eq. (2) for physically valid $\mathbf{M}$. Applied to $\mathbf{M}(\Delta z)$ and limiting the analysis to the first order in Δz :

$$\mathbf{G}\cdot\mathbf{N}\approx\mathbf{I}+\Delta z\mathbf{m}+\mathbf{G}\cdot\Delta z\mathbf{m}^T\cdot\mathbf{G}=\mathbf{I}+2\Delta z\mathbf{m}_{dep}=\mathbf{L}\cdot(\mathbf{I}+2\Delta z\mathbf{D})\cdot\mathbf{L}^{-1}. \quad (9)$$

The deterministic component $\mathbf{m}_{\mathrm{det}}$ is not subject to any constraints and is eliminated from Eq. (9). But the matrix $\mathbf{D}$ from $\mathbf{m}_{\mathrm{dep}} = \mathbf{L}\cdot\mathbf{D}\cdot\mathbf{L}^{-1}$ has to match $\mathbf{I} + 2\Delta z\mathbf{D} = \mathbf{K}\cdot\mathbf{K}$.

Approximating $(1 + 2\Delta z D_n)^{1/2} \approx 1 + \Delta z D_n$ we retrieve **K**, which is subject to the constraints from Eq. (3).

Because Δz is small, the presence of the identity matrix ensures the first constraint on **K** for any **D**. For the remaining three constraints it is convenient to express $\mathbf{D}=D_0\mathbf{I}+\mathrm{diag}(0, D_1^{'}, D_2^{'}, D_3^{'})$ to obtain the conditions

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \cdot [D'_1 D'_2 D'_3]^T \geq \mathbf{0} \quad (10)$$

This result confirms the criteria evoked by Devlaminck *et al*. [6], but was derived for the general case of an arbitrary differential matrix **m** and is tied directly to the validity criteria of a Mueller matrix **M**. Geometrically speaking, the vector $[D_1^{'}, D_2^{'}, D_3^{'}]^{\mathrm{T}}$, originating from the point $[1, 1, 1]^{\mathrm{T}}$ has to point inside the tetrahedron that forms the validity bound on the elements of **K** [11]. Following Devlaminck *et al*. [6], it is also possible to parameterize $\mathbf{D} = D_0\mathbf{I} - \mathrm{diag}(0, d_2+d_3, d_1+d_3, d_1+d_2)$, in which case the criteria of Eq. (10) translate to $d_{1,2,3} \geq 0$. D_0 defines the attenuation of the medium and for a passive system $D_0 \leq 0$.

In order to practically verify the criteria from Eq. (10), a convenient way to obtain the parameterization of Eq. (7) is needed. A candidate matrix **m** is first separated into its deterministic $\mathbf{m}_{\mathrm{det}} = (\mathbf{m}-\mathbf{G}\cdot\mathbf{m}^{\mathrm{T}}\cdot\mathbf{G})/\mathbf{2}$ and depolarizing part $\mathbf{m}_{\mathrm{dep}} = (\mathbf{m}+\mathbf{G}\cdot\mathbf{m}^{\mathrm{T}}\cdot\mathbf{G})/\mathbf{2}$. It is helpful to decompose the SO(3,1) matrix **L** of Eq. (7) into the sequence of a retarder and a diattenuator $\mathbf{L} = \mathbf{L}_{\mathrm{D}}\cdot\mathbf{L}_{\mathrm{R}}$, with

$$\mathbf{L}_D = \begin{bmatrix} \cosh a & \mathbf{a}^T \sinh a \\ \mathbf{a} \sinh a & \mathbf{a}\cdot\mathbf{a}^T(\cosh a - 1) + \mathbf{I} \end{bmatrix}, \mathbf{L}_R = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}, \quad (11)$$

where **a** is a normalized vector, and a a scalar, related to the diattenuation d of the diattenuator $\mathbf{L}_{\mathrm{D}}$ by $d = \tanh a$. **R** is an SO(3) rotation matrix, defined by a rotation axis **r** and a rotation angle r.

Next, we note that Eq. (7) is simply an eigenvalue decomposition of $\mathbf{m}_{\mathrm{dep}}$, where the eigenvectors correspond to the scaled columns of an element of SO(3,1). It is thus clear that for a valid $\mathbf{m}_{\mathrm{dep}}$ all its eigenvalues have to be real-valued and because of $\mathbf{L}^{\mathbf{T}}\cdot\mathbf{G}\cdot\mathbf{L} = \mathbf{G}$ only one eigenvector **v** features a positive Minkowski-norm $\mathbf{v}^{\mathrm{T}}\cdot\mathbf{G}\cdot\mathbf{v}$. This eigenvector identifies the only physically possible Stokes vector that is mapped onto itself by $\mathbf{m}_{\mathrm{dep}}$. Observing now that

$$\mathbf{m}_{dep} = \mathbf{L}_D \cdot \mathbf{L}_R \cdot \mathbf{D} \cdot \mathbf{L}_R^{-1} \cdot \mathbf{L}_D^{-1} = \mathbf{L}_D \cdot \begin{bmatrix} D_0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{X} \end{bmatrix} \cdot \mathbf{G} \cdot \mathbf{L}_D \cdot \mathbf{G}, \quad (12)$$

where $\mathbf{X} = \mathbf{R}\cdot\mathrm{diag}(D_1, D_2, D_3)\cdot\mathbf{R}^{\mathbf{T}}$, it becomes obvious that this eigenvector is given by the first column-vector of $\mathbf{L}_{\mathrm{D}}$ times D_0:

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$$\mathbf{m}_{dep} \cdot \begin{bmatrix} \cosh a \\ \mathbf{a} sinh a \end{bmatrix} = D_0 \begin{bmatrix} \cosh a \\ \mathbf{a} \sinh a \end{bmatrix}. \quad (13)$$

This vector is completely depolarized by the inverse of $\mathbf{L}_D$ and bypasses the effects of $\mathbf{L}_R$ and $D'_{1,2,3}$ to then be re-polarized by the first column of $\mathbf{L}_D$. Diagonalizing a candidate matrix $\mathbf{m}_{dep} = \mathbf{P} \cdot \Lambda \cdot \mathbf{P}^{-1}$ and dividing the only eigenvector with a positive Minkowski-norm by the square root of this norm provides the first column vector of $\mathbf{L}_D$, from where the entire matrix $\mathbf{L}_D$ is recovered according to Eq. (11). Left-multiplying the (permuted) eigenvector matrix with $\mathbf{L}_D{}^{-1}$ then provides the rotation matrix $\mathbf{L_R} = \mathbf{L}_d{}^{-1} \cdot \mathbf{P}$. To ensure det($\mathbf{L_R}$) = 1, it might be necessary to permute the order of the last three eigenvalues/eigenvectors, which leaves four possible permutations. By convention, the permutation resulting in the smallest retardation of $\mathbf{L}_R$ was chosen. Given a matrix $\mathbf{m}_{dep}$, it is thus possible to obtain its $\mathbf{L} \cdot \mathbf{D} \cdot \mathbf{L}^{-1}$ decomposition, and directly verify the positivity of $d_{1,2,3}$ and hence its validity.

Applied to the experimental values used as an example by Germer (Eq. (19) of [5]), this results in the following decomposition

$$\begin{aligned} D_0 &= -0.009 \\ [\tau_1 \tau_2 \tau_3]^T &= [0.398 - 0.0080]^T \\ [\tau_4 \tau_5 \tau_6]^T &= [-0.150\ 0.006\ 0.004]^T \\ a\mathbf{a} &= [-0.0061 - 0.0019\ 0.0008]^T \\ r\mathbf{r} &= [0.030 - 0.008\ 0.0093]^T \\ [d_1 d_2 d_3]^T &= [1.4151\ 0.9712\ 0.6726]^T. \end{aligned} \quad (14)$$

To obtain more insight into the effect of the involved parameters, it is interesting to express the change of the degree of polarization (DOP) with depth dDOP/dz, caused by a differential depolarizing matrix acting on an input state $\mathbf{S} = I[1, \mathrm{DOP}\ \mathbf{s}^T]$. The input Stokes vector is decomposed into intensity I and normalized Q,U,V components $\mathbf{s}$, scaled by the DOP which is obtained from DOP = $(1-(\mathbf{S}^T \cdot \mathbf{G} \cdot \mathbf{S})/I^2)^{1/2}$. For the derivative:

$$\frac{dDOP}{dz} = -\frac{\mathbf{S} \cdot \mathbf{G} \cdot d\mathbf{S}/dz}{I^2 DOP} + \frac{(1 - DOP^2) dI/dz}{IDOP} = [-DOP\mathbf{s}^T] \cdot \mathbf{m}_{dep} \cdot [1 DOP\mathbf{s}^T]^T. \quad (15)$$

Fig. 1 displays dDOP/dz as well as the angle φ that the Q,U,V components of d$\mathbf{S} = \mathbf{m}_{dep} \cdot \mathbf{S}$ extend with $\mathbf{s}$ as a function of the elevation and azimuth of $\mathbf{s}$ on the Poincare-sphere and for a DOP = 0.42. Figs. 1(a) and 1(b) correspond to a matrix $\mathbf{m}_{dep}$ in the special situation where a = 0 and $\mathbf{L}_D = \mathbf{I}$, without attenuation, and randomly generated $r\mathbf{r}$ = [0.2627; −0.0045; 0.456]^T and $d_{1,2,3}$ = [1.2925, 0.8695, 0.1671]^T. In this case $\mathbf{m}_{dep}$ simplifies to:

$$\mathbf{m}_{dep} = \mathbf{L}_R \cdot \mathbf{D} \cdot \mathbf{L}_R^{-1}, \quad (16)$$

and any input aligned along $\mathbf{s}$ experiences the same amount of depolarization as $-\mathbf{s}$. $D'_{1,2,3}$ define distinct depolarization coefficients along the principle axes defined by the columns of

R. The principle axes are indicated in Fig. (1) by green markers, with a square shape for the positive direction, and a circle for the negative direction. The size of the markers is decreasing in accordance with the effective absolute principle value D'_{n}. In this example all *d*DOP/*dz* values are negative, leading to an increasing depolarization, and the largest depolarization is achieved along the principle value $D'_3 = -d_1 - d_2$. The other two principle axes feature the minimum depolarization and a saddle-point. Along all these axes, the *Q*,*U*,*V* components of d**S** are parallel to the original **S** and besides the depolarizing effect do not alter the polarization state. Any other polarization state changes the orientation of its polarized component **s** upon propagation.

The general case of $a \neq 0$ can be expressed as a superposition of this first case and an additional term:

$$\begin{aligned} \mathbf{m}_{dep} &= \mathbf{L}_R \cdot \mathbf{D} \cdot \mathbf{L}_R^T + \begin{bmatrix} -\gamma^2\eta & \gamma\mathbf{n}^T \\ -\gamma\mathbf{n} & \rho(\mathbf{n}\cdot\mathbf{a}^T + \mathbf{a}\cdot\mathbf{n}^T) - \rho^2\eta\mathbf{a}\cdot\mathbf{a}^T \end{bmatrix} \\ \gamma &= \sinh a, \rho = \cosh a - 1 \\ \eta &= \mathbf{a}^T \cdot \mathbf{R} \cdot diag(D'_1 D'_2 D'_3) \cdot \mathbf{R}^T \cdot \mathbf{a} \\ \mathbf{n} &= \mathbf{R} \cdot diag(D'_1 D'_2 D'_3) \cdot \mathbf{R}^T \cdot \mathbf{a} + \rho\eta\mathbf{a}, \end{aligned} \quad (17)$$

The first term defines the same symmetric depolarization as observed in Figs. 1(a) and 1(b), whereas the second term breaks this symmetry. As previously observed, the first column of $\mathbf{L}_D$ is an eigenvector of the system that represents a physically admissible Stokes vector, oriented along **s** = **a**, with a DOP = tanh *a*. The second term cancels the depolarization of the first term for this particular input polarization state, and it is only subject to the attenuation D_0, without otherwise changing the orientation of **s** or its DOP. The opposite vector −**s**, however, is subject, in general, to both depolarization and re-orientation. The vector **a** thus breaks the symmetry of the first term by defining a privileged polarization direction along which the depolarization cancels out completely for a given input DOP.

Figs. 1(c) and 1(d) display the general effect of $\mathbf{m}_{dep}$ for identical *r***r** and $d_{1,2,3}$ as before, and with an additional *a***a** = $[0.3173; -0.1003; -0.0979]^T$, resulting in an eigenstate DOP = 0.334. The asymmetry is evident, and forces the orientation of the *Q*,*U*,*V* components of d**S** parallel to **s** for the input state **s** = **a**, which is indicated by the gray square. The opposite direction −**a**, marked by the gray circle, does not carry any special feature.

Fig. 2 shows *d*DOP/*dz* along the principle directions defined by the columns of **R** as well as **a**. In the symmetric first case, the depolarization is proportional to the input DOP, identical for negative and positive **s**, and depolarizing for all possible input states. The general case is more involved, and also renders the other axes asymmetric. The maximum and minimum depolarizations occur along directions that do not result in simple analytical expressions. And for DOP ≤ tanh *a*, $\mathbf{m}_{dep}$ acts as a polarizer, for appropriately oriented input states. In the special case when **a** aligns with one of the columns of **R**, then the maximum and minimum depolarization directions become coincident with +**a** and −**a** respectively. In this special case, the vector **n** aligns in parallel to the vector **a** and the expression Eq. (17) simplifies significantly:

$$\mathbf{m}_{dep}=\mathbf{L}_R\cdot\mathbf{D}\cdot\mathbf{L}_R^T+D'_n\begin{bmatrix}-\sinh^2 a & \mathbf{a}^T\sinh a\cosh a\\ -\mathbf{a}\sinh a\cosh a & \mathbf{a}\cdot\mathbf{a}^T\sinh^2 a\end{bmatrix}, \quad (18)$$

where D'_n corresponds to the column of **R** with which **a** aligns.

However, all polarization states except the eigenvector [cosh a, $\mathbf{a}^T$sinh a]^T or any polarization state aligned with the columns of **R** in the trivial case, are subject not only to depolarization but also a change of their polarized component. This precludes a trivial result of the propagation through a uniform medium for all but these polarization preserving state(s). This stresses the importance and effect of the parameter vector **a**.

Returning to the analysis of the experimental data of Eq. (14), a negligible a was found, indicating that the medium principally acts as a symmetric depolarizer. The small value of $r\mathbf{r}$ further demonstrates that the eigenstates of the medium were well aligned with the laboratory frame.

In conclusion, we have presented a parameterization of the differential Mueller matrix following the canonical decomposition of a Mueller matrix of type **I** of Rao *et al*. [10]. This parameterization can be directly obtained numerically from a simple eigenvalue-decomposition and offers an interesting separation into two depolarization mechanisms: one acting symmetrically on polarization states that are opposite to each other on the Poincare sphere, whereas the other effect lacks this symmetry and can cause a polarizing effect on an input state aligned along a specific direction while depolarizing its opposite state. Based on this derivation, we were able to formulate criteria for the physical validity of a differential Mueller matrix in the most general case of a continuous medium, which constrain three of the 16 parameters of a differential Mueller matrix.

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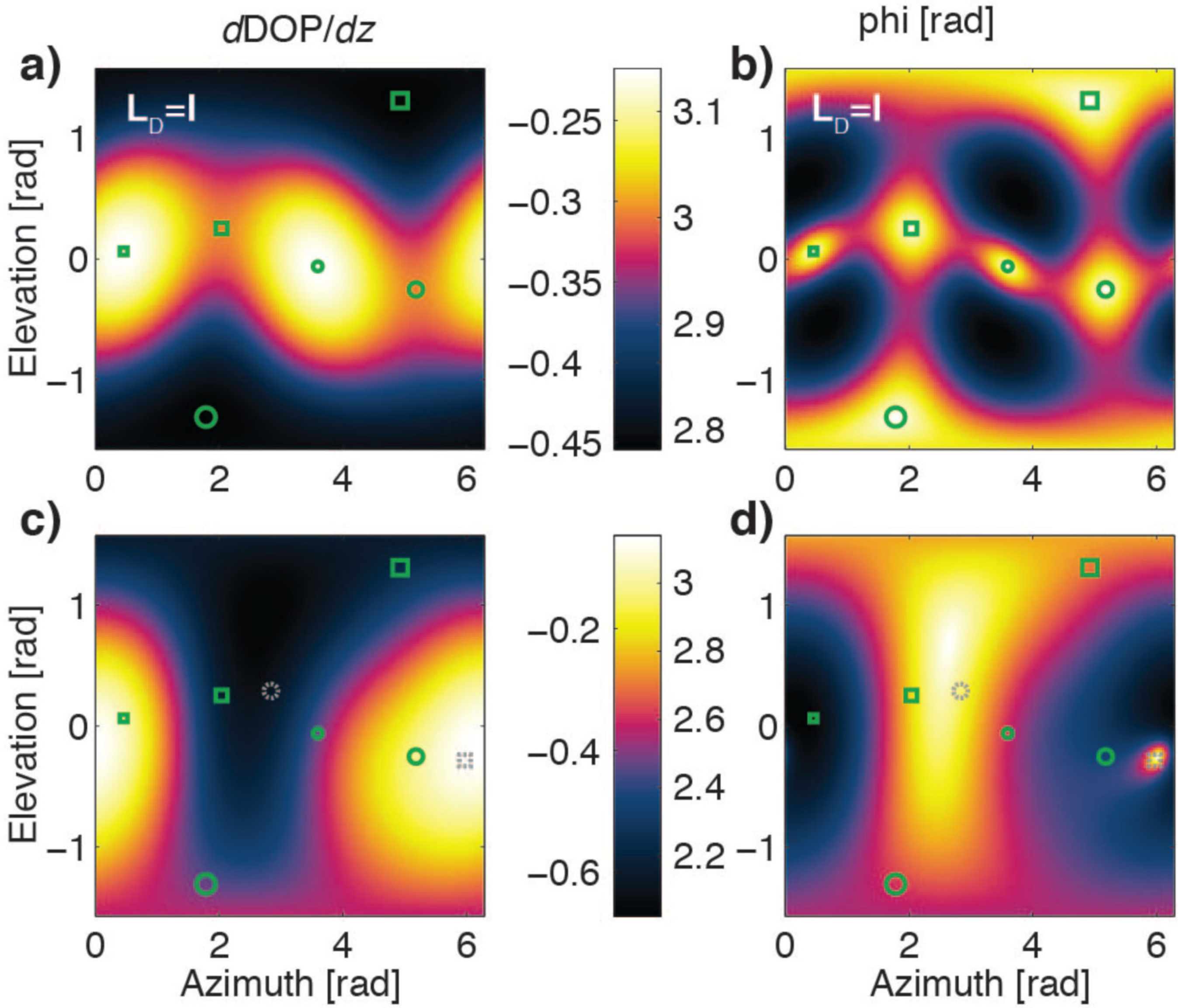

Fig 1.

Variation of the DOP with depth *d*DOP/*dz*, ((a), (c)) and angle ϕ between input and differential Stokes vector ((b), (d)) for a symmetric ((a),(b)) and a general ((c),(d)) $\mathbf{m}_{dep}$. The symmetric case has identical effect on Stokes vectors opposite from each other on the Poincare sphere, whereas the general case displays a strong asymmetry, sparing input polarizations aligned along the specific direction defined by the parametric vector **a**, but depolarizing inputs along −**a** (See text for details).

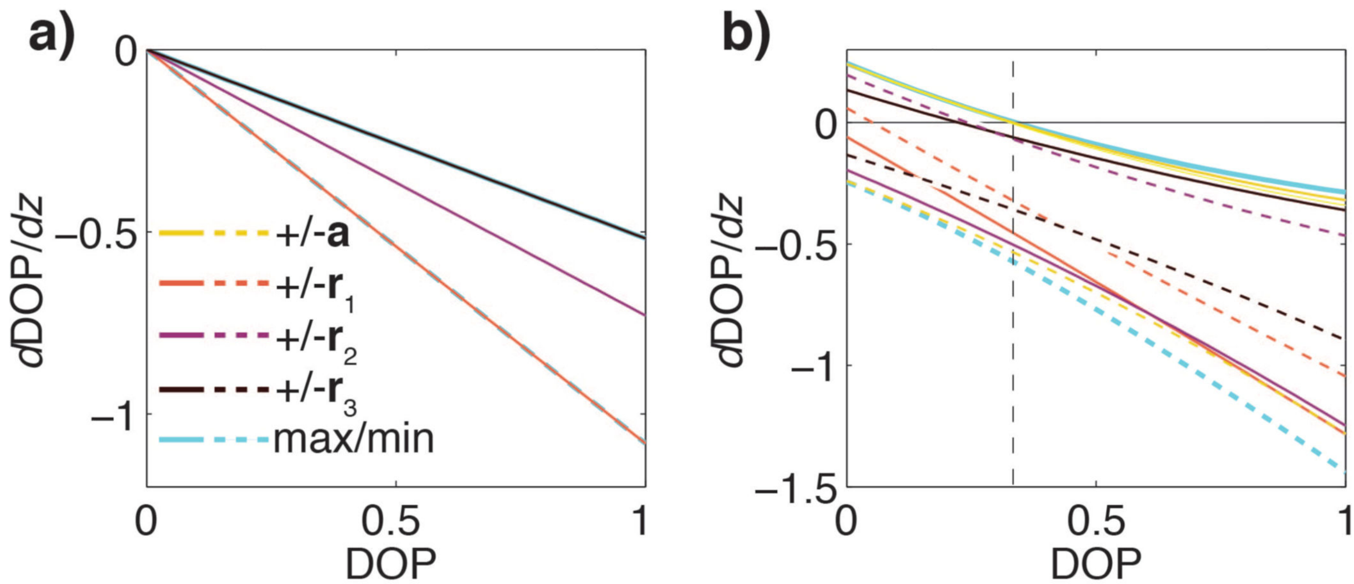

Fig 2.

*d*DOP/*dz* for the symmetric (a) and the general (b) case, evaluated along the principle directions ($\mathbf{r}_n$ indicates the n-th column of the rotation matrix **R**). In the symmetric case, the depolarization depends linearly on the input DOP. The general case polarizes inputs aligned close to **a** for a DOP ≤ tanh *a* and has unequal effect on opposite polarization states.