

# Practical Discrete Unit Disk Cover Using an Exact Line-Separable Algorithm<sup>\*</sup>

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**Abstract.** Given  $m$  unit disks and  $n$  points in the plane, the discrete unit disk cover problem is to select a minimum subset of the disks to cover the points. This problem is NP-hard [11] and the best previous practical solution is a 38-approximation algorithm by Carmi et al. [4]. We first consider the line-separable discrete unit disk cover problem (the set of disk centres can be separated from the set of points by a line) for which we present an  $O(m^2n)$ -time algorithm that finds an exact solution. Combining our line-separable algorithm with techniques from the algorithm of Carmi et al. [4] results in an  $O(m^2n^4)$  time 22-approximate solution to the discrete unit disk cover problem.

## 1 Introduction

Recent interest in specific geometric set cover problems is partly motivated by applications in wireless networking. In particular, when wireless clients and servers are modelled as points in the plane and the range of wireless transmission is assumed to be constant (say one unit), the resulting region of wireless communication is a disk of unit radius centred on the point representing the corresponding wireless transmitting device. Under this model, sender  $a$  successfully transmits a wireless message to receiver  $b$  if and only if point  $b$  is covered by the unit disk centred at point  $a$ . This model applies more generally to a variety of facility location problems for which the Euclidean distance between clients and facilities cannot exceed a given radius, and clients and candidate facility locations are represented by discrete sets of points. Examples include (1) selecting locations for wireless servers (e.g., gateways) from a set of candidate locations to cover a set of wireless clients, (2) positioning a fleet of water bombers at airports such that every active forest fire is within a given maximum distance of a water bomber, (3) selecting a set of weather radar antennae to cover a set of cities, (4)

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selecting locations for anti-ballistic defenses from a set of candidate locations to cover strategic sites. These problems can be modelled by the *discrete unit disk cover problem* (DUDC), whose definition is: Given sets  $P$  (candidate facilities) of  $m$  points and  $Q$  (clients) of  $n$  points in the plane, find a set  $P' \subseteq P$  (facilities) of minimum cardinality such that  $\text{Disk}(P')$  covers  $Q$ , where  $\text{Disk}(A)$  denotes the set of unit disks centred on points in set  $A$ . In this work, we consider the *line-separable discrete unit disk cover* (LSDUDC), where  $P$  and  $Q$  are separated by a line  $l$ .

The DUDC problem is NP-hard [11]. In a recent result, Carmi et al. [4] describe a polynomial-time 38-approximate solution, improving on earlier 108-approximate [6] and 72-approximate solutions [16]. We present an  $O(m^2n)$ -time algorithm that returns an exact solution to the LSDUDC problem, as well as a thorough proof of correctness of the technique. By combining the LSDUDC algorithm with techniques from the algorithm of Carmi et al. [4], we present a 22-approximation algorithm to the DUDC problem, improving on the best previous practical polynomial-time approximation factor of 38.

## 1.1 Related Work

**Line-Separable Discrete Unit Disk Cover.** A solution to the LSDUDC problem was independently discovered and published by [3, Lemma 1], where they propose a dynamic programming algorithm with a time bound of  $O(m^2n)$  but whose correctness is not straightforward nor is it formally argued. This paper presents an alternative algorithm together with a proof of correctness. Both algorithms follow natural approaches, yet a full proof of correctness is not immediate.

**$\epsilon$ -nets for Geometric Hitting Problems.** Using  $\epsilon$ -nets, Mustafa and Ray [15, 14] have recently presented a  $(1 + \epsilon)$ -approximation to the DUDC problem. Their algorithm runs in  $O(m^{2(c/\epsilon)^2+1}n)$  time, where  $c \leq 4\gamma$  [14]. Their  $\gamma$  value can be bounded from above by  $2\sqrt{2}$  [8, 12]. The fastest operation of this algorithm is obtained by setting  $\epsilon = 1$  for a 2-approximation, and this will run in  $O(m^{2 \cdot (8\sqrt{2})^2+1}n) = O(m^{257}n)$  time in the worst case. Clearly, this algorithm will not be practical for large values of  $m$ .

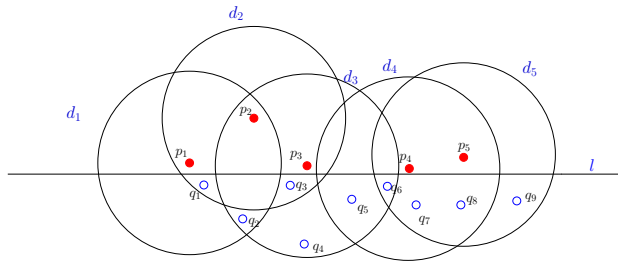
**Minimum Geometric Disk Cover.** In the minimum geometric disk cover problem, the input consists of a set of points in the plane, and the problem is to find a set of unit disks of minimum cardinality whose union covers the points. Unlike our problem, disk centres are not constrained to be selected from a given discrete set, but rather may be centred at arbitrary points in the plane. Again, this problem is NP-hard [7, 17] and has a PTAS solution [9]. Of course the problem can be generalized further: see [5] for a discussion of geometric set cover problems.

**Discrete  $k$ -Centre.** Also related is the discrete Euclidean  $k$ -centre problem: given a set  $P$  of  $m$  points in the plane, a set  $Q$  of  $n$  points in the plane, and an integer  $k$ , find a set of  $k$  disks centred on points in  $P$  whose union covers  $Q$  such that the radius of the largest disk is minimized. Observe that set  $Q$  has a

discrete unit disk cover consisting of  $k$  disks centred on points in  $P$  if and only if  $Q$  has a discrete  $k$ -centre centred on points in  $P$  with radius at most one. This problem is NP-hard if  $k$  is an input variable [2]. When  $k$  is fixed, Hwang et al. [10] give a  $m^{O(\sqrt{k})}$ -time algorithm, and Agarwal and Procopiuc [1] give an  $m^{O(k^{1-1/d})}$ -time algorithm for points in  $\mathbb{R}^d$ .

## 2 Overview of the Algorithm

In this section we describe a polynomial-time algorithm for the line-separable discrete unit disk cover (LSDUDC) problem and prove its correctness. Details of the algorithm and its running time will be discussed in Section 3. Recall that we have two sets  $P = \{p_1, p_2, \dots, p_m\}$  and  $Q = \{q_1, q_2, \dots, q_n\}$  of points in the plane that are separated by a line  $l$ . We want to find a subset  $P' \subseteq P$  of minimum cardinality such that all points of  $Q$  are covered by unit disks centred at the points of  $P'$ . An instance of the problem is shown in Figure 1. Without loss of generality we assume that  $l$  is a horizontal line and points of  $P$  are above  $l$ . Let  $d_i$  denote the unit disk that is centred at  $p_i$ , for  $i \in \{1, 2, \dots, m\}$ , and let  $D$  denote the set of these disks. We use  $p_i$  and  $d_i$  interchangeably, e.g., our solution can be considered both as a set of points (a subset of  $P$ ) and as a set of disks. Further, when we discuss the intersection of a line with a disk, we are referring to the intersection of the line with the boundary of the disk.



**Fig. 1.** An instance of the line-separable discrete unit disk cover problem.

During the execution of our algorithm, it may be determined that a disk  $d \in D$  should be added to the solution or that it is not relevant for the remainder of the computation of the solution set. When this occurs, we remove disk  $d$  from the problem. Similarly, we remove a point  $q \in Q$  if this point is not relevant for the remainder of the computation (i.e., point  $q$  is covered by a disk in the partial solution being constructed). Our algorithm relies on the following three observations:

1. If a disk  $d_1$  covers no points from  $Q$ , we remove it.
2. If a disk  $d_1$  is *dominated* by a disk  $d_2$ , then we can remove  $d_1$  from the problem instance. Disk  $d_2$  dominates  $d_1$  if it covers all points of  $Q$  covered

by  $d_1$ . If two disks cover the same subset of points from  $Q$ , we designate the dominating disk as that whose left intersection with  $l$  is rightmost.

3. If a point  $q_1 \in Q$  is only covered by a disk  $d_1$ , then  $d_1$  must be part of the solution. We also remove  $d_1$  together with all points of  $Q$  covered by  $d_1$ .

These three observations give us three SIMPLIFICATION rules. The idea is to apply these rules to as many disks as possible and simplify the problem. For example, consider the problem instance shown in Figure 1. Initially no disk dominates another, thus we cannot apply the second rule. Disk  $d_3$  is the only disk that covers  $q_4$  and, similarly, disk  $d_5$  is the only disk that covers  $q_9$ . Thus we add  $d_3$  and  $d_5$  to the (initially empty) solution and remove them together with the points that are covered by them, namely  $\{q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9\}$ . Now disk  $d_4$  covers no point and can be removed. There is only one remaining point ( $q_1$ ) and it is covered by the two remaining disks ( $d_1$  and  $d_2$ ). According to our convention,  $d_1$  is dominated by  $d_2$  and is removed. Now  $d_2$  is the only disk covering  $q_1$ . We add  $d_2$  to the solution and remove  $d_2$  and  $q_1$ . No disks or points remain and we are done. Thus the SIMPLIFICATION rules suffice for this instance and give an optimal solution  $\{d_2, d_3, d_5\}$ . This example also illustrates that an optimal solution is not necessarily unique, as  $\{d_1, d_3, d_5\}$  is also an optimal solution. In general, however, these SIMPLIFICATION rules do not suffice to obtain an optimal solution. Referring to Fig. 1, if given only disks  $d_1, d_2$  and  $d_3$  and points  $q_1, q_2$  and  $q_3$ , then no point  $q \in Q$  is covered by only one disk and no disk dominates any other one.

We augment the SIMPLIFICATION rules with a simple greedy step to solve the problem. We rename the disks so that the left intersection of  $d_i$  with  $l$  is to the left of the left intersection of  $d_{i+1}$  with  $l$ . We say that  $d_i$  precedes  $d_{i+1}$  in the ordering (the disks in Figure 1 follow this ordering). This combined algorithm, GREEDY, works by first applying the SIMPLIFICATION rules as many times as possible. Next we find the first remaining disk in the left-to-right order, say  $d_j$ . We add  $d_j$  to our solution and remove  $d_j$  from  $D$  and all points covered by  $d_j$  from  $Q$ . We apply the SIMPLIFICATION rules followed by the greedy step repeatedly until all disks have been removed. Since we remove at least one disk at each greedy step, the algorithm terminates after at most  $m$  iterations. See Algorithm 1 for the corresponding pseudocode.

## 2.1 Correctness of GREEDY

We now prove the correctness of the algorithm by proving that GREEDY gives a minimum LSDUDC solution. Assume for the sake of contradiction that there is an algorithm OPT that gives a cover with fewer disks than GREEDY. Let  $d_1$  be the first disk in the ordering that is selected by GREEDY but not by OPT. Let  $C$  be the set of points in  $Q$  that are covered by  $d_1$  (we consider only the remaining points and disks, i.e., those that have not been removed by the algorithm). First assume that  $C$  is covered by a single disk  $d_0$  in the solution of OPT. Since  $d_1$  is not removed in the SIMPLIFICATION step, it is not dominated by any other disk. Thus the only possibility is that  $d_0$  and  $d_1$  cover exactly the same set of

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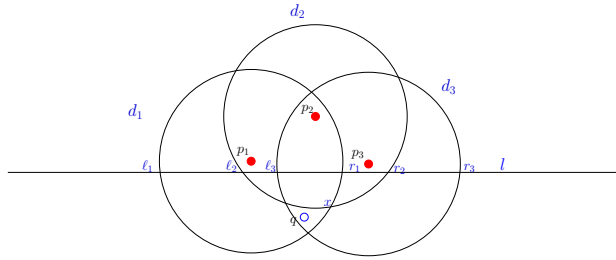
**Algorithm 1** GREEDY ( $D, Q$ )

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 $D \leftarrow \text{sortLeftToRight}(D)$  //sort in increasing order of left intersection with  $l$   
 $S \leftarrow \emptyset$   
while  $D \neq \emptyset$  do do  
  SIMPLIFICATION ( $D, Q, S$ ) //SIMPLIFICATION possibly modifies  $D, Q$  and  $S$   
   $d_\ell \leftarrow$  leftmost disk in  $D$   
   $S \leftarrow S \cup \{d_\ell\}$   
   $D \leftarrow D \setminus d_\ell$   
   $Q' \leftarrow \{q \in Q \mid q \text{ is contained in } d_\ell\}$   
   $Q \leftarrow Q \setminus Q'$   
end while  
return  $S$ 
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(remaining) points (i.e., set  $C$ ) and  $d_0$  precedes  $d_1$  in the ordering. In this case, we replace  $d_0$  with  $d_1$  in OPT, pushing the first difference between the solution of GREEDY and OPT to the right. Otherwise,  $C$  is covered by at least two disks in the solution of OPT. Let  $d_2$  and  $d_3$  be two disks in the solution of OPT such that each of them cover a strict subset of  $C$ . Without loss of generality assume that  $d_2$  precedes  $d_3$  in the ordering. We prove that  $d_1 \cup d_3$  covers all points of  $Q$  covered by  $d_2 \cup d_3$ .



**Fig. 2.** Proof of correctness of GREEDY. If  $d_1$  is the first disk selected by GREEDY and not by OPT, then OPT must have  $d_2$  and  $d_3$  in its solution.

Let  $\ell_i$  and  $r_i$  denote the respective left and right intersection points of the boundary of the unit disk  $d_i$  with the line  $l$ , for  $i \in \{1, 2, 3\}$ . If  $d_2$  precedes  $d_1$  in the ordering,  $d_1$  dominates  $d_2$  (otherwise, GREEDY would select  $d_2$  and not  $d_1$  at this step). In this case we replace  $d_2$  with  $d_1$  in OPT, pushing the difference between the two algorithms to the right. Hence we are left with the case in which  $d_1$  precedes  $d_2$  and  $d_2$  precedes  $d_3$  in the ordering. Thus the points are ordered  $\ell_1, \ell_2, \ell_3, r_1, r_2, r_3$  along line  $l$  (see Figure 2). Note that we cannot have a pair of disks nested below  $l$ , otherwise the nested disk is dominated by the other. Furthermore, we know that  $(d_1 \cap d_3) \setminus d_2 \neq \emptyset$ . Let  $R = d_2 \setminus d_1$ . It suffices to prove that  $R$  is completely contained in  $d_3$ .

**Proposition 1.** *Region  $R$  is contained in disk  $d_3$ .*

*Proof.* Since points  $r_1$  and  $r_2$  both lie between  $\ell_3$  and  $r_3$  on line  $l$ , both points  $r_1$  and  $r_2$  are in disk  $d_3$ . Let  $x$  denote the rightmost point of the intersection of the boundaries of disks  $d_1$  and  $d_2$ . Observe that  $x$  lies on the boundary of region  $(d_1 \cap d_3) \setminus d_2$ . Consequently,  $x \in d_3$ . Since the boundary of  $R$  consists of arcs of unit disks joining the points  $x$ ,  $r_1$ , and  $r_2$ , it follows that  $R$  is contained in the 1-hull of  $\{x, r_1, r_2\}$ , where the 1-hull of  $\{x, r_1, r_2\}$  is the intersection of all unit disks that contain  $\{x, r_1, r_2\}$ , denoted  $1-H(\{x, r_1, r_2\})$ . Since  $\{x, r_1, r_2\} \subseteq d_3$ , it follows that  $R \subseteq 1-H(\{x, r_1, r_2\}) \subseteq d_3$ .  $\square$

Thus by removing  $d_2$  from the solution of OPT and adding  $d_1$  to it we will have a feasible solution with the same number of disks. This pushes the first difference between the solution of GREEDY and OPT to the right. By continuing this argument we can prove that the solution returned by GREEDY uses the same number of disks as OPT and therefore GREEDY is an optimal algorithm.

### 3 Implementation Details and Analysis

We construct a graph  $G = (V, E)$ , where each node  $v_i \in V$  corresponds to disk  $d_i$  for  $i \in \{1, \dots, m\}$  (recall that  $d_i$  is the  $i$ th disk sorted according to its left intersection with  $l$ ). We also associate a counter  $c_{v_i}$  to each node  $v_i$  that stores the number of points in  $Q$  contained in disk  $d_i$  that have not yet been covered by the algorithm. A counter's value refers to points that have not been covered. Similarly, we associate with each edge  $e = (v_{i_1}, v_{i_2})$  a counter  $c_e$  that represents the number of points contained in  $d_{i_1} \cap d_{i_2}$ . This graph can be constructed in  $O(m^2n)$  time by checking which points are contained in the intersection of each pair of disks, adding the corresponding edges, and updating the node and edge counters. The algorithm GREEDY-GRAPH starts by traversing the nodes in order  $v_1, v_2, \dots, v_m$ . At each node  $v_i$ , there are three possible cases: (1) The counter  $c_{v_i}$  is 0; in this case  $d_i$  does not contain any points or is dominated by a set of disks that has already been added to the solution. This disk will not be in the solution set, so we can ignore this node and continue with the next one. This is analogous to the first SIMPLIFICATION rule. (2) There is an edge  $e = (v_i, v_k)$ ,  $k > i$ , such that  $c_e = c_{v_i}$ ; in this case we know that  $d_i$  is dominated by disk  $d_k$ . Again, we ignore this node and continue. This corresponds to an application of the second SIMPLIFICATION rule. (3) Every edge  $e = (v_i, v_k)$ ,  $k > i$ , satisfies  $c_e < c_{v_i}$ ; that means that disk  $d_i$  is not dominated by any disk to its right. In this case we add  $d_i$  to the solution set and we eliminate all remaining points contained by this disk from the graph. We continue with the next node in the graph. Note that this is an application of the third rule of SIMPLIFICATION and the greedy step.

In order to identify the appropriate case above we traverse the adjacency list of each node we visit. This requires  $O(m)$  time in the worst case. When a disk is added to the solution in the third case, all points contained in the disk must be eliminated. Consider the elimination of a point  $p$  in disk  $d_i$ . Let  $N(v_i) = \{v_k \mid c_{(v_i, v_k)} > 0\}$ . For all  $v_k \in N(v_i)$ , we decrease  $c_{v_k}$  and  $c_{(v_i, v_k)}$  by one. In addition, for each pair of elements  $\{v_{k_1}, v_{k_2}\} \subseteq N(v_i)$ , we check whether

the point is contained by both disks, and if this is case we decrease  $c_{(v_{k_1}, v_{k_2})}$  by one. This can take at most  $O(m^2)$  time per point, thus the time required for eliminating all points is bounded by  $O(m^2n)$  time. Since the time required to construct the graph is  $O(m^2n)$ , the overall process takes  $O(m^2n)$  time.

### 3.1 Correctness of GREEDY-GRAPH

We now demonstrate that the GREEDY-GRAPH algorithm is optimal by showing that the set of disks returned by this algorithm has the same cardinality as that returned by the GREEDY algorithm presented in Section 2.

**Lemma 1.** *If  $S$  is the disk cover returned by GREEDY-GRAPH, and  $S'$  is the disk cover returned by GREEDY, then  $|S| = |S'|$ .*

*Proof.* Assume for the sake of contradiction that  $|S| \neq |S'|$ . Recall that GREEDY is optimal, therefore  $|S'|$  and  $|S|$  can only differ if  $|S| > |S'|$ . Let  $d_1$  be the first disk in the left-to-right order that is present in the solution of GREEDY-GRAPH, and not in the solution of GREEDY. At some point during its execution, GREEDY must have decided to discard disk  $d_i$ . The only mechanisms in GREEDY for discarding disks are the first and second SIMPLIFICATION rules. Recall that the first rule removes a disk if it contains no points, and the second rule discards a disk if it is dominated by some other disk. We now show that for any of the following possible events, GREEDY-GRAPH will discard the same disk  $d_1$ .

- **Empty** - Suppose  $d_1$  contains no points. In this case, GREEDY-GRAPH will find that  $c_{v_1} = 0$ . Therefore,  $d_1$  will be discarded by Case 1, in contradiction to our assumption.
- **Dominance (right)** - Now suppose  $d_1$  is dominated by some disk to the right,  $d_r$ . In this case, we will encounter  $d_1$  first during our walk, and we will have that  $c_{v_1} = c_{(v_1, v_r)}$ . Therefore, GREEDY-GRAPH will remove  $d_1$  by Rule 2, in contradiction to our assumption.
- **Dominance (left)** - Suppose  $d_1$  is dominated by some disk to the left,  $d_\ell$ . In this case, we will have encountered  $d_\ell$  first during our walk. There are two possible cases in this scenario:
  - (i) If  $c_{v_\ell} > c_{(v_\ell, v_k)}$  for all  $d_k$ ,  $d_\ell$  is added to  $S$  by Rule 3 of GREEDY-GRAPH. All points covered by  $d_\ell$  are removed, leaving no points covered by  $d_1$ . This is now an instance of the Empty case.
  - (ii) Otherwise,  $c_{v_\ell} = c_{(v_\ell, v_k)}$  for some  $d_k$ . This means that  $d_\ell$  is dominated by  $d_k$ . GREEDY-GRAPH would discard  $d_\ell$  by Rule 3. By transitivity,  $d_k$  also dominates  $d_1$ . If  $d_k$  is to the right of  $d_1$ , then this is now an instance of Dominance (right), and thus we reach a contradiction. If  $d_k$  is to the left of  $d_1$ , then this is again an instance of Dominance (left), so we apply this same argument recursively. The recursion stops either when we reach an instance of Dominance (right) or case (i) of Dominance (left).  $\square$

We have shown that the solution of GREEDY-GRAPH has the same cardinality as the solution of GREEDY, and since GREEDY is optimal, so is GREEDY-GRAPH.

**Theorem 1.** *Given sets  $P$  of  $m$  points and  $Q$  of  $n$  points in the plane, where  $P$  and  $Q$  can be separated by a line  $l$ , LSDUDC can be solved in  $O(m^2n)$  time.*

## 4 Approximate Discrete Unit Disk Cover

We now show that our algorithm for the line-separable discrete unit disk cover (LSDUDC) problem leads to a 22-approximation algorithm for the discrete unit disk cover (DUDC) problem. The approximation algorithm is based on a suitable adaptation of the 38-approximation algorithm of Carmi et al. [4].

For simplicity, we use the notation and assumptions of [4]. In that work, the DUDC problem is supported by a variant of the LSDUDC problem: suppose are given a set of disks  $D = L \cup U$ . The disks in  $U$  are centred above a line  $l$  while the disks in  $L$  are centred below  $l$ . We are also given a set of points  $Q$  covered by  $U$ . The goal is to obtain the subset  $G$  of  $D$  of smallest cardinality such that every point in  $Q$  is covered by a disk in  $G$ .

Note that our line-separable algorithm does not immediately result in a straightforward improvement to the approximation factor of the algorithm of Carmi et al. Their proof of correctness uses the fact that their 2-approximation to the LSDUDC problem consists of disks forming the lower boundary of  $U$ , where the lower boundary is the union of all disk boundary arc segments below  $l$  not contained in other disks. This is not necessarily the case in our solution.

Instead, we first solve the LSDUDC problem optimally using our algorithm on the set of disks  $U$  to obtain a disk set  $H$  and then use the greedy *minimum assisted cover* algorithm (see Carmi et al. [4, §2] for the formal definition) over the sets  $H$  and  $L$  to obtain an improved solution  $E$ . Now we wish to compare the cardinality of  $E$  with that of the global minimum disk cover  $G$ .

Consider the upper and lower components of the solutions  $E$  and  $G$ , i.e.,  $E_U = E \cap U$ ,  $E_L = E \cap L$ ,  $G_U = G \cap U$ , and  $G_L = G \cap L$ . Note that  $|G| \leq |E|$  since  $G$  is the global minimum. Similarly, since  $E$  is the minimum assisted cover based on  $H$ , it follows that  $|E| = |E_U| + |E_L| \leq |H/G_L| + |G_L|$ , where  $H/G_L$  is the smallest subset of  $H$  that forms an assisted cover with  $G_L$ .

Now we will show that  $2|G_U| \geq |H/G_L|$ . Given a disk  $d$  in  $G_U$ , there are two cases: either  $d$  lies above the lower boundary of  $H/G_L$ , i.e.,  $d$  is contained in the union of all the disks in  $H/G_L$ , or  $d$  contains one or more arc segments of the lower boundary of  $H/G_L$ . In the first case, Carmi et al. show that at most two disks in  $H/G_L$  suffice to cover  $d$  and, hence, for every such disk in the global optimum solution  $G$  there are most two disks in  $H/G_L$ . In the second case, let  $V$  denote the subset of disks that have lower boundary segments that are contained in  $d$ . The set of arc segments of the disks in  $V$  consists, from left to right, of a partially-covered arc segment of the lower boundary, zero or more fully-covered arc segments, and a partially-covered arc segment. Let  $W$  denote the disks whose arcs are partially covered together with  $d$ .  $W$  dominates  $V$  and hence there is at most one arc of the lower boundary fully contained in  $d$ ; otherwise replacing  $V$  with  $W$  results in a cover of smaller cardinality, deriving a contradiction, since  $V \subset H$ , and  $H$  is the optimal LSDUDC solution<sup>3</sup>. Furthermore, observe that the partially-covered arc disks must contain points not contained in the fully-covered disk; otherwise they can also be eliminated while reducing the

<sup>3</sup> Recall that all disks in  $V$  and  $U$  are centred above  $l$ , and all points in  $Q$  are below  $l$ .



cardinality of the cover. As those disks contain other points, each of the disks is partially covered by at least one other disk in  $G$ . We arbitrarily associate each disk covered more than once to its leftmost disk in  $G$ . Thus, of the (at most) three disks in  $V$ , at most two are associated to  $d$ . In sum, in either case each disk in  $G_U$  has at most two associated disks in  $H/G_L$  from which it follows that  $2|G_U| \geq |H/G_L|$ . Hence,  $2|G| = 2(|G_U| + |G_L|) \geq 2|G_U| + |G_L| \geq |H/G_L| + |G_L| \geq |E_U| + |E_L| = |E|$ , which gives the approximation factor of two as desired. Carmi et al. [4] prove that any disk can be used in up to eight applications of the assisted LSDUDC algorithm, for which they have a 4-approximation. These operations, followed by a 6-approximation for any remaining disks results in an  $8 \times 4 + 6 = 38$ -approximation for the general DUDC problem. As we have shown that our technique provides a 2-approximation for the assisted LSDUDC problem, we now have an approximation ratio of  $8 \times 2 + 6 = 22$  for DUDC.

#### 4.1 Algorithm Analysis

There are essentially two main components to the algorithm for solving DUDC by Carmi et al. [4]. First, they apply a grid of size  $3/2 \times 3/2$  to the input data. Our LSDUDC algorithm supplemented by their assisting disk technique is run on all grid lines. Note that the number of relevant grid lines is  $O(n)$ . Our technique runs in  $O(m^2n)$ , and the assisting disk operation is easily implementable in  $O(mn)$ , so the running time of the first component is dominated by our step.

The second major component to their technique is finding the 6-approximation for the DUDC of all disk centres and points contained in each of the  $3/2 \times 3/2$  squares of the grid. Their technique is based on the application of a subset of nine properties depending on where the disk centres are located. First, they determine whether a solution exists using one or two centres by brute force, which is easily done in  $O(m^2n)$  time. The determination of which properties may be applied can be done in  $O(m)$  time, and there are only two expensive steps that may be used in any of the procedures, each of which may only be used a constant number of times. First is the assisted LSDUDC technique, whose running time is  $O(m^2n)$ , as we just discussed. The second technique that may be required is to determine the optimal disk cover of a set of points using centres contained in one of the  $1/2 \times 1/2$  squares, which can be solved in  $O(m^2n^4)$  time using the technique presented in [13]. The centre of each disk can only be contained in one square, and so this operation is never performed twice for any given disk. Therefore, the complete DUDC algorithm achieves worst-case performance when all of the disk centres in the plane are confined to a single  $1/2 \times 1/2$  square, so that the  $O(m^2n^4)$  operation is performed over the entire data set.

## 5 Conclusions

This paper presents a polynomial-time algorithm that returns an exact solution to the LSDUDC problem, as well as a proof of correctness of the approach. This algorithm for the line-separable problem allows us to improve the approximation

algorithm of Carmi et al. [4], resulting in a 22-approximate solution to the general DUDC problem, which runs in  $O(m^2n^4)$  time in the worst case.

**Theorem 2.** *Given sets  $P$  of  $m$  points and  $Q$  of  $n$  points in the plane, we can compute a 22-approximation of the DUDC problem in  $O(m^2n^4)$  time in the worst case.*

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